Definition

Let f be a real-valued function with domain D_f. Let x_0 be a cluster point of D_f. We define the number $f'(x_0)$ to be

$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided the limit exists. If it does exist, we say that f is differentiable at x_0, and we call $f'(x_0)$ the derivative of f at x_0.

- Note that in the above definition, we only consider h for which $x_0 + h \in D_f$ since otherwise $f(x_0 + h)$ is not defined. This is always to be assumed; we won't mention it again.

Theorem (Linear Transformation Characterization of Differentiability)

Let f be a real-valued function with domain D_f. Let x_0 be a cluster point of D_f.

1. If f is differentiable at x_0, then there exists a function ε of h such that $f(x_0+h)-f(x_0)$ can be written in the form
$$f(x_0+h)-f(x_0)=f'(x_0)h+\varepsilon h$$
where $\varepsilon \to 0$ as $h \to 0$.
2. If there exists a real number m such that $f(x_0+h)-f(x_0)$ can be written in the form
$$f(x_0+h)-f(x_0)=mh+\varepsilon h$$
where ε is a function of h such that $\varepsilon \to 0$ as $h \to 0$, then f is differentiable at x_0 and $m=f'(x_0)$.

Exercise.

Write the proof of this theorem.

Note: - The reason we want the above theorem is that we no longer are dividing by h (that

creates problems in proving the chain rule) and also we remove "limit" from the formulation and turn it into an algebraic statement.

- We demonstrate the method with all the remaining theorems of the section.

Theorem (Basic Properties of Differentiability)

Let f, g be functions with common domain D, let x_0 be a cluster point of D. Let c be a fixed real number. Suppose that f and g are differentiable at x_0.

b) (Linearity) Then $f + g$ and $c\, f$ are both differentiable at x_0, and we have the formulas

$$
\begin{aligned}
(f+g)'(x_0) &= f'(x_0) + g'(x_0)\\
(cf)'(x_0) &= cf'(x_0).
\end{aligned}
$$

Exercise.

Write the proof using the linear transformation characterization of differentiability.

Theorem (Basic properties of Differentiability)

Let f, g be functions with common domain D. Let x_0 be a cluster point of D. Let c be a fixed real number. Suppose f and g are differentiable at x_0. Then all of the following hold.

(a) (Differentiability implies continuity) f is continuous at x_0.

(b) (Linearity) $f+g$ and cf are both differentiable at x_0 and

$$(f+g)'(x_0) = f'(x_0) + g'(x_0)$$

$$(cf)'(x_0) = c f'(x_0).$$

(c) (Product Rule) $f \cdot g$ is differentiable at x_0 and

$$(f \cdot g)'(x_0) = f'(x_0) g(x_0) + f(x_0) \cdot g'(x_0).$$

(d) (Reciprocal Rule) If in addition $g'(x_0) \neq 0$ then $\frac{1}{g}$ is differentiable at x_0 and

$$\left(\frac{1}{f}\right)'(x_0) = \frac{-f'(x_0)}{(f(x_0))^2}$$

(e) (Quotient Rule) If in addition $g'(x_0) \neq 0$, then $\frac{f}{g}$ is differentiable at x_0 and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.$$

We will prove the theorem using the linear transformation characterization of differentiation. We just show the proofs of (a), the first part of (a) and (b). The other parts are similar.

(a) We are given that there exists a function $\varepsilon(h)$ such that

$$f(x_0+h)-f(x_0)=f'(x_0)h+\varepsilon(h)\cdot h,$$

where $\varepsilon(h)\to 0$ as $h\to 0$. It follows that

$$\lim_{h\to 0} f(x_0+h)-f(x_0)=0,$$

and this is exactly what continuity of f at x_0 means. $\square$

Proof of the first formula in (Q): We are given that there exist functions $\varepsilon_f(h)$ and $\varepsilon_g(h)$ such that

$$f(x_0+h)-f(x_0) = f'(x_0)h + \varepsilon_f(h)\cdot h$$

$$g(x_0+h)-g(x_0) = g'(x_0)h + \varepsilon_g(h)\cdot h,$$

where ε_f and $\varepsilon_g \to 0$ as $h \to 0$. Adding, we get

$$\big(f(x_0+h)+g(x_0+h)\big)-\big(f(x_0)+g(x_0)\big) = \big(f'(x_0)+g'(x_0)\big)h + \big(\varepsilon_f(h)+\varepsilon_g(h)\big)h.$$

Since $\varepsilon_f(h)+\varepsilon_g(h) \to 0$ as $h \to 0$, it follows $f+g$ is differentiable at x_0 and $(f+g)'(x_0) = f'(x_0)+g'(x_0)$. $\square$

© <u>Proof of the Product Rule</u> We are given that there exist functions $\varepsilon_f(h)$ and $\varepsilon_g(h)$ such that

$$f(x_0+h) = f(x_0) + f'(x_0)h + \varepsilon_f(h)\cdot h$$

$$g(x_0+h) = g(x_0) + g'(x_0)h + \varepsilon_g(h)\cdot h,$$

where $\varepsilon_f(h)$ and $\varepsilon_g(h) \to 0$ as $h \to 0$. Multiplying and collecting terms gives

$$f(x_0+h)g(x_0+h) - f(x_0)g(x_0) = \big(f'(x_0)g(x_0) + f(x_0)g'(x_0)\big)h$$
$$+ h\big[f(x_0)\varepsilon_g(h) + f'(x_0)g'(x_0)h + f'(x_0)\varepsilon_g(h)h + g(x_0)\varepsilon_f(h) + \varepsilon_f(h)g'(x_0)\cdot h + \varepsilon_f(h)\varepsilon_g(h)\cdot h\big]$$

The result follows from the fact that the circled quantity goes to 0 as $h \to 0$. $\square$