3.4 The Cauchy-Schwarz inequality and a new triangle inequality

- Recall the triangle inequality on $\mathbb{R}$:

$$|x+y| \leq |x| + |y| \qquad \text{for all } x, y \in \mathbb{R}.$$

- How would this generalize to $\mathbb{R}^2$?
- Let's view points of $\mathbb{R}^2$ as vectors: $\vec{x} = (x_1, x_2), \vec{y} = (y_1, y_2)$ be vectors in $\mathbb{R}^2$. We define their "norms" as

$$\|\vec{x}\| := \sqrt{x_1^2 + x_2^2}, \quad \|\vec{y}\| := \sqrt{y_1^2 + y_2^2}.$$

- The norm of the vector measures the length of the arrow representing the vector.
- Then the triangle inequality says:

$$\|(x_1, x_2) + (y_1, y_2)\| \leq \|(x_1, x_2)\| + \|(y_1, y_2)\|.$$

Exercise.

Explain by means of a sketch why you should believe the triangle inequality is true, and also explain where the name "triangle inequality" comes from.

So $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ really says that in a triangle, the length of any side is less than the sum of the lengths of the other two sides.

- We need a proof of the triangle inequality that works in $\mathbb{R}^2$. The proof we used on $\mathbb{R}$ at the beginning of the course doesn't generalize.

- Want to show

$$\|\vec{x}+\vec{y}\|^2 \le \left(\|\vec{x}\|+\|\vec{y}\|\right)^2$$

i.e. $(x_1+y_1)^2+(x_2+y_2)^2 \le \|\vec{x}\|^2+\|\vec{y}\|^2 + 2\|\vec{x}\|\,\|\vec{y}\|$

i.e. $(x_1+y_1)^2+(x_2+y_2)^2 \le x_1^2+x_2^2+y_1^2+y_2^2 + 2\sqrt{x_1^2+x_2^2}\sqrt{y_1^2+y_2^2}$

$\Leftrightarrow$ $2x_1y_1+2x_2y_2 \le 2\sqrt{x_1^2+x_2^2}\,\sqrt{y_1^2+y_2^2}$

i.e. $\vec{x}\cdot\vec{y} \le \|\vec{x}\|\,\|\vec{y}\|$

- If we could prove this we could prove the triangle inequality.
- This is known as the <u>Cauchy-Schwarz inequality</u>.

- Our proof will work in an infinite dimensional vector space called $L^2[a,b]$.
- Here, a vector is a function $f:[a,b]\to\mathbb{R}$.
- The <u>norm</u> of f is $\|f\|_2 = \sqrt{\int_a^b (f(x))^2\,dx}$

and the dot product $\langle f,g\rangle$ is $\int_a^b f(x)g(x)\,dx$.

(So <u>sums</u> in $\mathbb{R}^2$ have been replaced by integrals).

Cauchy-Schwarz and Triangle Inequalities for $f, g : [a, b] \to \mathbb{R}$

Theorem

Let $f, g : [a, b] \to \mathbb{R}$. Suppose that $f, g \in \mathscr{R}[a, b]$. Then the following are true.

(i) We necessarily have that fg is also Riemann integrable on $[a, b]$.

(ii) (Cauchy-Schwarz Inequality) $| < f, g > | \leq \|f\|_2 \|g\|_2$.

(iii) (Triangle Inequality for $L^2[a, b]$) $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$.

Exercise.

a) Let $a, b, c \in \mathbb{R}$ with $a > 0$. Say we know that for all $t \in \mathbb{R}$ we have

$$at^2 + bt + c \geq 0.$$

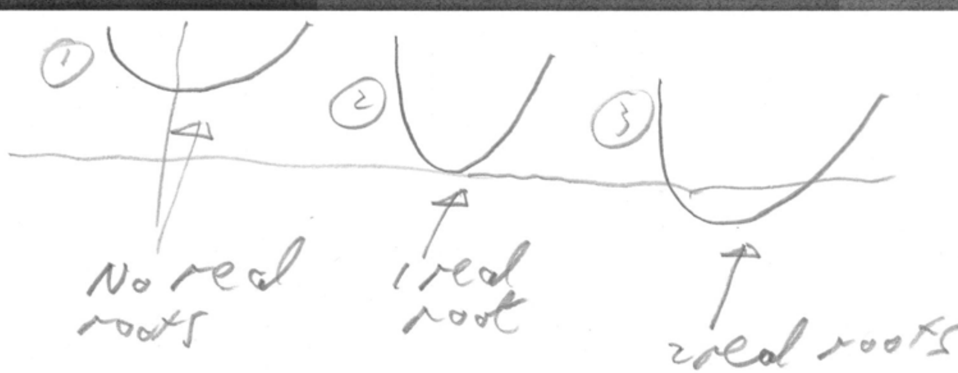
What can we say about a, b and c?

b) Write a proof of the theorem. You will find the first part of this exercise useful in doing the proof.

c) Use the ideas of this proof to write a proof of the triangle inequality in $\mathbb{R}^n$.

(a) • The function $t \mapsto at^2+bt+c$ is a parabola that looks like $\cup$.

• The location of the coordinate axes tells us if at^2+bt+c has any real roots.



- So $at^2+bt+c \geq 0$ says cases ① or ② occur, so there are <u>not</u> two real roots.
- quadratic formula says the roots are

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

So <u>not</u> two real roots $\Longleftrightarrow b^2-4ac \leq 0$,

i.e. $\Longleftrightarrow \boxed{b^2 \leq 4ac}$

So to recap:

$$\boxed{\begin{gathered} at^2+bt+c \geq 0 \text{ for all } t \\ \Longleftrightarrow \quad b^2 \leq 4ac. \end{gathered}}$$

- a more formal proof can be done by completing the square.

Proving (i): If $f, g \in \mathcal{R}[a,b]$ then $f \cdot g \in \mathcal{R}[a,b]$

- The formula $f \cdot g = \dfrac{(f+g)^2 - (f-g)^2}{4}$

  and the linearity theorem show it is enough to prove the square of an integrable function is integrable.
- But note $f^2 = |f|^2$ and we showed f integrable implies $|f|$ integrable.
- So it is enough to prove it for squares of nonnegative functions.
- So if $f \geq 0$ then for any interval I_i of a partition

$$\sup\{(f(x))^2 : x \in I_i\} = \left(\sup\{f(x) : x \in I_i\}\right)^2,$$

  with a similar formula for inf. This trick is the key to the proof.

Theorem If $f, g \in \mathcal{R}[a,b]$, then $f \cdot g \in \mathcal{R}[a,b]$.

Proof Obrerve that $f \cdot g = \dfrac{(f+g)^2 - (f-g)^2}{4}$, so by linearity of integrability, the result would follow if we could prove that the square of an integrable function is integrable.

Since f integrable implies $|f|$ integrable and $f^2 = |f|^2$, it is enough for us to prove that the square of a non-negative integrable function is integrable.

So let $f \in \mathcal{R}[a,b]$, $f \geq 0$. We will use the Partition Characterization of Integrabilitity to prove $f^2 \in \mathcal{R}[a,b]$.

Let $\varepsilon > 0$. Since $f \in \mathcal{R}[a,b]$, there exits a partition $P = \{x_0, \cdots, x_n\}$ such that

$$\mathcal{U}(f,P) - \mathcal{L}(f,P) \leq \frac{\varepsilon}{2\|f\|_\infty}.$$

For each interval I_i of P, since $f \geq 0$,

$$M_{f,i}^2 := \left(\sup\{f(x) : x \in I_i\}\right)^2 = \sup\{(f(x))^2 : x \in I_i\} := M_{f^2,i},$$

and with similar notation,

$$m_{f,i}^2 = m_{f^2,i}.$$

Thus

$$U(f^2,P) - L(f^2,P) = \sum_{i=1}^{n} \left(M_{f^2,i} - m_{f^2,i}\right)\Delta_i = \sum_i \left(M_{f,i}^2 - m_{f,i}^2\right)\Delta_i$$

$$= \sum_i \left(M_{f,i} - m_{f,i}\right)\left(M_{f,i} + m_{f,i}\right)\Delta_i$$

$$\leq 2\|f\|_\infty \sum_i \left(M_{f,i} - m_{f,i}\right)\Delta_i$$

$$= 2\|f\|_\infty \left(U(f,P) - L(f,P)\right) < \varepsilon.$$

Thus $f^2 \in \mathcal{R}[a,b]$.

$\square$

Theorem. Let $f, g \in \mathcal{R}[a,b]$. Then the following are true.

(i) (Cauchy-Schwarz inequality) $|\langle f, g\rangle| \leq \|f\|_2 \, \|g\|_2$,

where $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$ and $\|f\|_2 = \left(\int_a^b (f(x))^2\,dx\right)^{1/2}$.

(ii) (Triangle Inequality for $L^2[a,b]$) $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$.

Proof (i) Say A, B, C satisfy $A > 0$ and $At^2 + Bt + C \geq 0$ for all $t \in \mathbb{R}$. Then completing the square gives

$$0 \leq At^2 + Bt + C = A\left(t^2 + \frac{B}{A}t + \frac{B^2}{4A^2} - \frac{B^2}{4A^2} + \frac{C}{A}\right) = A\left(\left(t + \frac{B}{2A}\right)^2 + \frac{4AC - B^2}{4A^2}\right).$$

It follows by taking $t = \frac{-B}{2A}$ that $B^2 \leq 4AC$.

We have for each $t \in \mathbb{R}$,

$$0 \leq \int_a^b (tf(x) + g(x))^2\,dx = \Big(\underset{\substack{\uparrow \\ A}}{\int_a^b (f(x))^2\,dx}\Big)t^2 + \Big(\underset{\substack{\uparrow \\ B}}{\int_a^b 2f(x)g(x)\,dx}\Big)t + \underset{\substack{\uparrow \\ C}}{\int_a^b (g(x))^2\,dx}$$

so applying the above result,

$$\left(\int_a^b 2f(x)g(x)\,dx\right)^2 \leq 4\left(\int_a^b (f(x))^2\,dx\right)\left(\int_a^b (g(x))^2\,dx,\right.$$

i.e. $|\langle f, g\rangle| \leq \|f\|_2 \, \|g\|_2$.

(ii) By the Cauchy-Schwarz Inequality and properties of the inner product,

$$\|f+g\|_2^2 = \langle f+g, f+g\rangle = \langle f,f\rangle + 2\langle f,g\rangle + \langle g,g\rangle$$
$$= \|f\|_2^2 + 2\langle f,g\rangle + \|g\|_2^2$$
$$\le \|f_2\|_2^2 + 2\|f\|_2\|g\|_2 + \|g\|_2^2$$
$$= (\|f_2\|_2 + \|g\|_2)^2,$$

so the triangle inequality follows by taking the square root of both sides. $\square$