3.3 The Riemann Integral and Convergence

- In this section we study the following questions:

  Let f_n be a sequence of functions all in $\mathscr{R}[a,b]$. Suppose we know that $f_n \to f$ in some specified type of convergence.

  (i) Is it necessarily true that the limit function f is in $\mathscr{R}[a,b]$?

  (ii) If so, is it true that $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$ as a sequence of numbers?

- If $f_n \to f$ pointwise:
  - We show by example that (i) above need not be true.
  - Another example will illustrate that even if (i) holds, (ii) need not hold.
- If $f_n \to f$ uniformly: We will prove that necessarily (i) and (ii) both hold.

Exercise 1.

We know that $\mathbb{Q}$ is a countable set. Since $\mathbb{Q} \cap [0,1]$ is an infinite subset of $\mathbb{Q}$, it is also a countable set. Let $\{r_m\}_{m=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [0,1]$. For each $n \in \mathbb{N}$ define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \ldots, r_n\} \\ 0 & \text{if } x \notin \{r_1, r_2, \ldots, r_n\} \end{cases}$$

a) To what function f does the sequence $\{f_n\}_{n=1}^{\infty}$ converge pointwise?

b) Is it true that f is Riemann integrable?

c) Is it true that each of f_n are Riemann integrable on $[0,1]$? If so, how much is $\displaystyle\int_0^1 f_n(x)\,dx$?

d) So what does this exercise show?

(a) $f_n \to f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \text{ irrational} \end{cases}$

[If $x \in \mathbb{Q} \cap [0,1]$ then $f_n(x) = 1$ for all n large enough
If x is irrational, $f_n(x) = 0$ for all n,
so $f(x) = 0$]

(b) No f is not integrable.

(c) Yes by Homework 14 any function which is 0 except at finitely many places is integrable and its integral is 0.

So $\int_0^1 f_n(x)\,dx = 0.$

(d) The pointwise limit of integrable functions need not be integrable.

Pointwise Convergence and the Riemann integral

Exercise 2.

For each $n \in \mathbb{N}$ define $f_n : [0,1] \to \mathbb{R}$ by

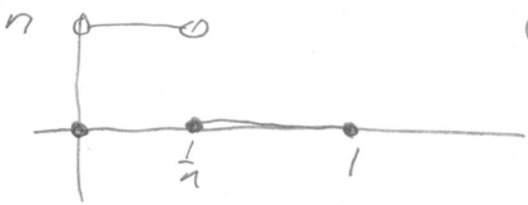
$$f_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ n & \text{if } 0 < x \le 1/n \\ 0 & \text{if } 1/n < x \le 1 \end{cases}$$

a) To what function f does the sequence $\{f_n\}_{n=1}^{\infty}$ converge pointwise?

b) Is f Riemann integrable? If so, how much is $\int_0^1 f(x)\,dx$?

c) Is it true that each of f_n are Riemann integrable on $[0,1]$? If so, how much is $\int_0^1 f_n(x)\,dx$?

d) So what does this exercise show?



(a) For each $x \in (0,1]$, for $n > \frac{1}{x}$ we have $f_n(x) = 0$, so $\lim_{n\to\infty} f_n(x) = 0$. For all n, $f_n(0) = 0$, so $\lim_{n\to\infty} f_n(x) = 0$. Thus $\{f_n\}$ converges pointwise to 0.

(b) Yes, the 0-function is integrable and its integral

is 0.

(c) Yes, since each f_n is monotone. Also $\int_0^1 f_n(x)\,dx = (n)(\frac{1}{n}) = 1$.

(d) A sequence $\{f_n\}$ of integrable functions can converge pointwise to a function f, such that f is integrable, but $\int_0^1 f_n\,dx \not\to \int_0^1 f\,dx$.

- After the above two negative examples, we give the following positive result.

Theorem

Let f_n be a sequence of functions in $\mathscr{R}[a, b]$. Suppose the sequence converges uniformly on $[a, b]$ to the function f. Then $f \in \mathscr{R}[a, b]$ and $\int_a^b f_n(x)\, dx \to \int_a^b f(x)\, dx$.

Idea of the proof:

- Note that there is no assumption of continuity on the f_n's, we only get to use the weaker hypothesis of Riemann integrability.
- Given an $\varepsilon > 0$ we can find f_N such that $\|f_N - f\|_\infty < \varepsilon$. We could hope to be able to use this to make similar statements about f which we could make about f_N.
- We should make use of the Partition Characterization of integrability. First write what this implies about f_N (namely that it implies the existence of a certain partition P of $[a, b]$ with nice properties) and then use the fact that $\|f_N - f\|_\infty < \varepsilon$ to see that what this says about $U(f, P)$ and $L(f, P)$.

Exercise.

Write a proof of the theorem.

Theorem Let $\{f_n\}_n$ be a sequence of functions in $\mathcal{R}[a,b]$. If the sequence converges uniformly to a function f, then $f \in \mathcal{R}[a,b]$ and $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$.

Proof We first show that $f \in \mathcal{R}[a,b]$ using the Partition Characterization of Integrability.

Let $\varepsilon > 0$. By the uniform convergence, there exists $N \in \mathbb{N}$ such that $\|f_N - f\|_\infty < \varepsilon$. Since $f_N \in \mathcal{R}[a,b]$, there exists a partition P of $[a,b]$ such that $U(f_N, P) - L(f_N, P) < \varepsilon$.

Note that for any interval $I_i = [x_{i-1}, x_i]$ of P,

$$-\varepsilon + \inf_{x \in I_i} f_N(x) \le \inf_{x \in I_i} f(x) \le \sup_{x \in I_i} f(x) \le \sup_{x \in I_i} f_N(x) + \varepsilon.$$

It follows that

$$\begin{aligned} U(f,P) - L(f,P) &\le U(f_N, P) - L(f_n, P) + 2\varepsilon(b-a) \\ &< \varepsilon + 2\varepsilon(b-a) = (1 + 2(b-a))\,\varepsilon. \end{aligned}$$

This proves $f \in \mathcal{R}[a,b]$.

Finally, for $n \in \mathbb{N}$,

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \le \int_a^b |f_n(x) - f(x)|\,dx \le \|f_n - f\|_\infty (b-a) \to 0 \text{ as } n \to \infty$$

By the uniform convergence. $\square$