Properties of the integral

Theorem (Gluing theorem)

Let $a < b < c$. Let $f : [a, c] \to \mathbb{R}$. Suppose that $f \in \mathscr{R}[a, b]$ and $f \in \mathscr{R}[b, c]$. Then $f \in \mathscr{R}[a, c]$ and $\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$.

Exercise.

Write the proof of the gluing theorem.

- We'll do a proof of this in Homework 14.

Properties of the integral- Extension of the Gluing Theorem

- We've only defined $\int_a^b f(x)\,dx$ in case $a < b$.
- But we want to define $\int_a^b f(x)\,dx$ even if $a > b$, and furthermore we want the above Gluing Theorem to be true, regardless of the order of the numbers a, b and c. This motivates us to make the following definition:

Definition

Let $a > b$. Then we define $\int_a^b f(x)\,dx$ as follows: $\int_a^b f(x)\,dx := -\int_b^a f(x)\,dx$.

Exercise.

Assuming we want the gluing theorem to hold regardless of the order of a, b, c, why were we forced to make the above definition?

If Gluing Thm true, then we would have

$$\int_a^b f(x)dx + \int_b^a f(x)dx = \int_a^a f(x)dx.$$

But we can prove $\int_a^a f(x)dx = 0$. So

$$\int_a^b + \int_b^a = \int_a^a = 0$$

$$\Longrightarrow \int_a^b = -\int_b^a$$

So this is forced on us if we want the Gluing Theorem to be true for any order of a, b and c.

Theorem

For any three different real numbers a, b, c and any f which is Riemann integrable on any of the three closed intervals we can form from these three numbers, we have

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Exercise.

Make use of the definition on the previous slide (and whatever else is needed) to prove this theorem.

Check $\int_a^c = -\int_c^a = -\left[\int_b^a - \int_b^c\right]$

Gluing Thm.

$= \int_a^b + \int_b^c$ ✓

- It's of interest for us to know lots of examples of bounded functions which are Riemann integrable.
- The following theorem guarantees that the set of Riemann integrable functions includes at least the set of continuous functions.

Theorem (Integrability of continuous functions)

Let f be continuous on $[a, b]$. Then $f \in \mathscr{R}[a, b]$.

Hints on the proof:

- The proof follows if we make the right use of
  - the uniform continuity of f (how do you know f is uniformly continuous?)
  - and the Partition Characterization of Integrability.

Exercise.

Write the proof.

Theorem (Integrability of continuous functions) Let f be continuous on $[a,b]$. Then $f \in \mathcal{R}[a,b]$.

Proof We prove the theorem using the partition characterization of integrability. Let $\varepsilon > 0$.

Since f is continuous and $[a,b]$ is compact, it follows f is uniformly continuous. Thus there exists $\delta > 0$ such that for all $x, y \in [a,b]$, if $|x-y| < \delta$, then $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$.

Choose $n \in \mathbb{N}$ with $n > \frac{b-a}{\delta}$. Let $P = \{x_0, x_1, \dots, x_n\}$ be the partition of $[a,b]$ given by

$$x_0 = a,\ x_1 = a + \frac{b-a}{n},\ x_2 = a + \frac{2(b-a)}{n},\ \dots,\ x_n = a + \frac{n(b-a)}{n} = b.$$

For each i, if $x, y \in [x_{i-1}, x_i]$, then $|x-y| \le x_i - x_{i-1} = \frac{b-a}{n} < \delta$, so $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$. By the Extreme Value Theorem, for each i, we can choose x to be the point of max and y the point of min of f on $[x_{i-1}, x_i]$. Thus $M_i - m_i = f(x) - f(y) < \frac{\varepsilon}{b-a}$, so

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (x_i - x_{i-1}) \\ &= \frac{\varepsilon}{b-a} \cdot (b-a) \\ &= \varepsilon. \end{aligned}$$

Thus $f \in \mathcal{R}[a,b]$. $\square$

- Another general class of Riemann integrable functions is given by the next theorem.

Theorem (Integrability of monotone functions)

Let f be a monotone function on $[a, b]$. Then $f \in \mathscr{R}[a, b]$.

Hints on the proof:

- Monotone functions need not be continuous, so cannot use ideas of continuity.
- Do you see why it's sufficient to just prove it for f monotone increasing?
- For any partition of $[a, b]$ and any associated interval $[x_{i-1}, x_i]$ of that partition, how much is the supremum of $f(x)$ as x varies over that interval? How much is the infimum of $f(x)$ as x varies over that interval?
- For any such partition, how much is $U(f, P) - L(f, P)$? (Do you know what is a telescoping sum?)
- Recall the Partition Characterization of Integrability.

Exercise.

Write the proof.

Theorem (Integrability of monotone functions)

Let $f:[a,b]\to\mathbb{R}$ be monotone. Then $f\in\mathcal{R}[a,b]$.

Proof We prove this with the assumption f is increasing. Otherwise we could replace f by $-f$.

We prove the result using the partition characterization of integrability. Let $\varepsilon>0$. Let P be any partition of $[a,b]$ such that $\|P\|<\frac{\varepsilon}{f(b)-f(a)}$.

For each subinterval $[x_{i-1},x_i]$, $m_i=f(x_{i-1})$ and $M_i=f(x_i)$, so

$$
\begin{aligned}
U(f,P)-L(f,P) &= \sum_i (M_i-m_i)(x_i-x_{i-1}) \\
&= \sum_i \big(f(x_i)-f(x_{i-1})\big)(x_i-x_{i-1}) \\
&\le \|P\| \sum_i \big(f(x_i)-f(x_{i-1})\big) \\
&= \|P\|\cdot\big(f(b)-f(a)\big) < \frac{\varepsilon}{f(b)-f(a)}\cdot\big(f(b)-f(a)\big) \\
&= \varepsilon.
\end{aligned}
$$

Thus $f\in\mathcal{R}[a,b]$.

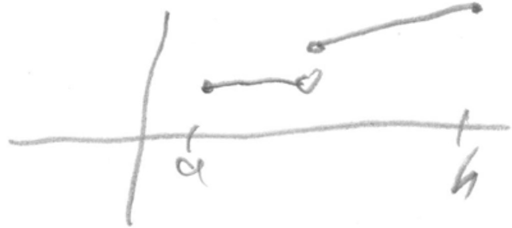
$\square$

Exercise

a) Write down an example of a function on some interval $[a, b]$ (or just the sketch of a function) which is Riemann integrable but not continuous. How do you know your example is Riemann integrable?

b) Let $f : [a, b] \to \mathbb{R}$ be given by $f = g - h$, where $g, h : [a, b] \to \mathbb{R}$ are both monotone increasing. Is f Riemann integrable? Why or why not?

(a) Any monotone function which is not continuous would work, for example



By the theorem on slide 13, it is integrable.

Exercise.

(a) Let $f : [a, b] \to \mathbb{R}$ be bounded and let $c \in (a, b)$. Suppose that f is continuous everywhere on $[a, b]$ except at the point c. Prove that f is Riemann integrable on $[a, b]$.

(b) What generalization of this result can be proved with a simple modification to the proof in (a).

(a) This follows from homework 14 and the Gluing Theorem.

(b) we can add any <u>finite</u> # of discontinuities to the statement.

Exercise.

Define $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$.

a) Where is f discontinuous?

b) Explain why $f \notin \mathscr{R}[0,1]$.

(a) Everywhere.

(b) We've seen before that <u>every</u> lower sum is 0 and every upper sum is 1 (since $\mathbb{Q}$ and $\mathbb{Q}^c$ are both dense). Thus $\overline{\int_0^1} f(x)\,dx = 1$

and $\underline{\int_0^1} f(x)\,dx = 0$.

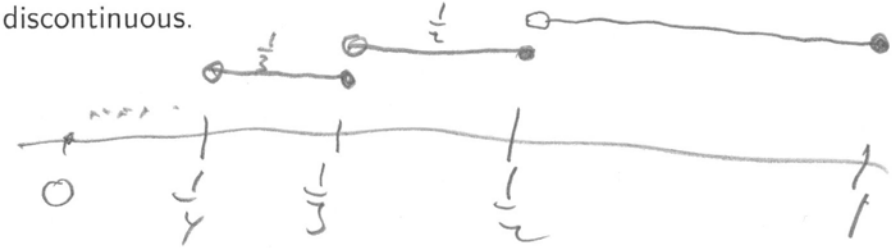
Exercise

Consider the following function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } \frac{1}{n+1} < x \le \frac{1}{n} \end{cases}$.

a) Where are the discontinuities of f? So how many discontinuities does f have?

b) Prove that $f \in \mathscr{R}[0,1]$.

- The above example shows that a function can have lots of discontinuities and still be Riemann integrable.
- An interesting question is to characterize the set of points on which a Riemann integrable function can be discontinuous.



$f \in R[0,1]$ since it is monotone, yet f has countably many discontinuities.

Thm $f \in R[a,b] \iff$ the set of discontinuities of f is a set of "measure 0".

This is a technical term.

Additional Elementary Properties of functions in $\mathscr{R}[a, b]$

Theorem

Let $f, g : [a, b] \to \mathbb{R}$.

a) Suppose that $f \in \mathscr{R}[a, b]$. If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(x)\, dx \geq 0$.

b) Suppose that f and g are in $\mathscr{R}[a, b]$. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$.

c) If $f \in \mathscr{R}[a, b]$, then $|f| \in \mathscr{R}[a, b]$ and furthermore $\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\, dx$. However, the converse is false; there exists f such that $|f| \in \mathscr{R}[a, b]$, but $f \notin \mathscr{R}[a, b]$.

Exercise.

Write the proof.

(a) This is obvious since by assumption $f \geq 0$ we get that every lower sum is ≥ 0.

(b) This follows from (a) and linearity.

Since $g - f \geq 0$, by (a) and linearity $g - f \in \mathscr{R}[a, b]$ and $\int_a^b (g-f) \geq 0$. Apply linearity again to get $\int_a^b g - \int_a^b f \geq 0$.

(c) If we knew $|f| \in \mathscr{R}[a, b]$ then the rest would follow from (b) since $f \leq |f|$ and $-f \leq |f|$.

Theorem If $f \in R[a,b]$, then $|f| \in R[a,b]$.

Proof We make use of the Partition Characterization of Integrability. Let $\varepsilon > 0$. Then there exists a partition $P = \{x_0, x_1, \ldots x_n\}$ of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

For each subinterval $I_i = [x_{i-1}, x_i]$ of I, let

$$M_{f,i} = \sup\{f(x) : x \in I\} \text{ and } m_{f,i} = \inf\{f(x) : x \in I_i\},$$

with similar definitions for $M_{|f|,i}$ and $m_{|f|,i}$.

For any $x, y \in I_i$,

$$\big||f(x)| - |f(y)|\big| \le |f(x) - f(y)| \le \sup\{|f(s) - f(t)| : s,t \in I_i\}$$
$$= M_{f,i} - m_{f,i}.$$

Since the right side is independent of x and y, it follows

$$\sup\{\big||f(x)| - |f(y)|\big| : x,y \in I_i\} \le M_{f,i} - m_{f,i},$$

and since the left side is $M_{|f|,i} - m_{|f|,i}$, we get after multiplication by Δ_i and summing on i,

$$U(|f|,P)-L(|f|,P)\leq U(f,P)-L(f,P)<\varepsilon.$$

Thus, by the Partition Characterization of Integrability, $|f|\in\mathcal{R}[a,b]$. $\square$