- We develop things differently from our text, combining the material from sections 3.1 and 3.2.

Definition: Partition of $[a, b]$ and associated terminology

(i) By a partition P of the interval $[a, b]$ we mean a finite set of points $P = \{x_0, x_1, \dots, x_n\}$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

(ii) The above partition divides $[a, b]$ into n intervals $I_i := [x_{i-1}, x_i]$, $i = 1, 2, \dots, n$. Let's refer to each of these intervals as "the intervals of the partition". We denote the length of the *ith* such subinterval by Δ_i, that is $\Delta_i := x_i - x_{i-1}$.

(iii) (Norm of a partition) For the above partition, we define the norm $\|P\|$ of it to be the length of the longest of the partitioning subintervals, that is

$$\|P\| := \max\{\Delta_i : i = 1, \dots, n\}.$$

(iv) (Refinement of a partition) Let P, P' be any two partitions of $[a, b]$. We say that P' is a refinement of P if $P \subseteq P'$.

(v) (Common refinement of two partitions) Let P', P'' be two partitions of $[a, b]$. The common refinement of P' and P'' is the partition $P' \cup P''$.

Theorem (Refinement Theorem)

Let P be a partition of $[a, b]$ and let P' be any refinement of P. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P).$$

Consequently,

$$|U(f, P') - L(f, P')| \leq |U(f, P) - L(f, P)|.$$

Hints on the proof:

- Do you see that $L(f, P') \leq U(f, P')$ is obvious?
- Do you see that $L(-f, P) = -U(f, P)$ and $U(-f, P) = -L(f, P)$?
- Do you see how the previous bullet allows us to reduce the proof to showing that $U(f, P') \leq U(f, P)$?

Exercise.

Write the proof of the theorem.

Theorem (Refinement Theorem)

Let P be a partition of $[a,b]$ and let P' be a refinement of P. Then
$$L(f,P) \le L(f,P') \le U(f,P') \le U(f,P).$$
Consequently,
$$0 \le U(f,P') - L(f,P') \le U(f,P) - L(f,P).$$

Proof. That $L(f,P') \le U(f,P')$ is immediate from the fact that the supremum of a set of numbers is at least as big as the infimum of that set.

It follows from one of our homework assignments that
$$L(-f,P) = -U(f,P).$$
So if we could prove $U(f,P') \le U(f,P)$, then it would follow
$$L(f,P) = -U(-f,P) \le -U(-f,P') = L(f,P')$$

So we must prove $U(f,P') \le U(f,P)$. If we could do this in case P' has one more point than P, then it would follow by induction for any refinement of P.

So let's assume there exists $y \in [a,b]$ such that $P = P' \setminus \{y\}$. Then there exist consecutive points $x', x'' \in P$ such that $x' < y < x''$.

Define $M_{x',x''} := \sup\{f(x) : x' \leq x \leq x''\}$, with similar definitions for $M_{x',y}$ and $M_{y,x''}$. Clearly

$$M_{x',y} \leq M_{x',x''} \text{ and } M_{y,x''} \leq M_{x',x''}.$$

Thus

$$\begin{aligned} M_{x',y} \cdot (y - x') + M_{y,x''} \cdot (x'' - y) &\leq M_{x',x''} \cdot (y - x') + M_{x',x''} \cdot (x'' - x') \\ &= M_{x',x''} \cdot (x'' - x'). \end{aligned}$$

Adding to both sides the remaining terms of $\mathcal{U}(f,P)$, all of which are common to $\mathcal{U}(f,P')$, it follows that $\mathcal{U}(f,P') \leq \mathcal{U}(f,P)$.

As indicated at the beginning of the proof, we deduce

$$L(f,P) \leq L(f,P') \leq \mathcal{U}(f,P') \leq \mathcal{U}(f,P),$$

and so $\quad 0 \leq \mathcal{U}(f,P') - L(f,P') \leq \mathcal{U}(f,P) - L(f,P).$

$\square$

Definition: Upper and Lower Integrals

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Define the upper integral $\overline{\int_a^b} f(x)\,dx$ and the lower integral $\underline{\int_a^b} f(x)\,dx$ by

$$\overline{\int_a^b} f(x)\,dx := \inf\{U(f,P) : P \text{ a partition of } [a,b]\}$$

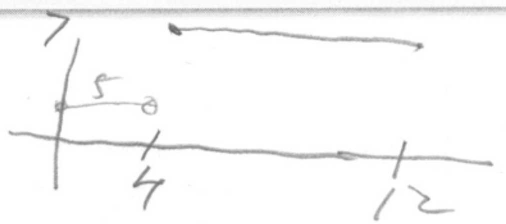
$$\underline{\int_a^b} f(x)\,dx := \sup\{L(f,P) : P \text{ a partition of } [a,b]\}$$

Exercise.

a) Let $f(x) = \begin{cases} 5 & \text{if } x < 4 \\ 7 & \text{if } x \geq 4 \end{cases}$ Calculate $\overline{\int_4^{12}} f(x)\,dx$ and $\underline{\int_4^{12}} f(x)\,dx$.

b) Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ Calculate $\overline{\int_4^{12}} f(x)\,dx$ and $\underline{\int_4^{12}} f(x)\,dx$.

(a)



For any P, $U(f,P) = L(f,P) = \sum_{i=1}^{n} 7\Delta_i = (7)(12-4) = 56$

(b) For any partition P, $M_i = 1$ and $m_i = 0$ since $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ dense. Then $U(f,P) = \sum_{i=1}^{n} 1\,\Delta_i = 1\cdot(12-4) = 8$ and $L(f,P) = \sum_{i=1}^{n} 0\cdot\Delta_i = 0$. Thus $\overline{\int_4^{12}} f(x)\,dx = 8$ and $\underline{\int_4^{12}} f(x)\,dx = 0$

Theorem (Basic properties of upper and lower integrals)

(i) Both the upper and lower integrals exist as real numbers.

(ii) We always have $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$

(iii) For any partition P of $[a,b]$, we have $L(f,P) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(f,P)$ and so consequently $\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le U(f,P) - L(f,P)$.

Exericse.

Write the proof of the theorem.

• Any P is a refinement of the partition $\{a,b\}$.

$\therefore L(f,\{a,b\}) \le L(f,P) \le U(f,P) \le U(f,\{a,b\})$

• $L(f,\{a,b\}) = (\inf\{f(x): a \le x \le b\})(b-a)$

$\le U(f,P)$ any P.

$\therefore \{U(f,P): P$ any partition$\}$

bounded below $L(f,\{a,b\})$.

$\therefore \overline{\int_a^b} f(x)\,dx$ exists.

If P_1 and P_2 any partitions of $[a,b]$, then

$$L(f,P_1) \le U(f,P_2).$$

__Pf__ we have

$$L(f,P_1) \le L(f,P_1 \cup P_2) \quad \text{(Refinement thm)}$$

$$\le U(f,P_1 \cup P_2)$$

$$\le U(f,P_2) \quad \text{(Refinement thm)}$$

Theorem (Basic properties of upper and lower integrals)

Let $f:[a,b]\to\mathbb{R}$ be bounded.

(i) Both $\overline{\int_a^b} f(x)\,dx$ and $\underline{\int_a^b} f(x)\,dx$ exist as real numbers.

(ii) $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$

(iii) For any partition P of $[a,b]$ we have

$$L(f,P) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(f,P)$$

and so consequently

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le U(f,P) - L(f,P).$$

Proof First note that any partition of $[a,b]$ is a refinement of the partition $\{a,b\}$, so by the Refinement Theorem, $L(f,\{a,b\})$ is a lower bound of $\{L(f,P) : P$ a partition of $[a,b]\}$ and $U(f,\{a,b\})$ is an upper bound of $\{U(f,P) :$ P a partition of $[a,b]\}$. Thus by the Least Upper Bound and Greatest Lower Bound properties of $\mathbb{R}$, (i) follows.

For (ii) observe that if P_1 and P_2 are any partitions of $[a,b]$, then $P_1 \cup P_2$ is a refinement of both P_1 and P_2, so by the Refinement Theorem,

$$L(f,P_1) \le L(f,P_1 \cup P_2) \le U(f,P_1 \cup P_2) \le U(f,P_2).$$

Thus $L(f,P_1)$ is a lower bound of $\{U(f,P) : P$ a partition of $[a,b]\}$

and so by definition,

$$L(f, P_1) \leq \overline{\int_a^b} f(x)\,dx.$$

Since this is true for all P_1, then $\overline{\int_a^b} f(x)\,dx$ is an upper bound of $\{L(f,P) : P \text{ a partition of } [a,b]\}$. Thus, by definition, $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$, completing the proof of (ii).

Since it is obvious that for any partition P of $[a,b]$, $L(f,P) \leq \underline{\int_a^b} f(x)\,dx$ and $U(f,P) \geq \overline{\int_a^b} f(x)\,dx$, (iii) follows from these and (ii).

$\square$

- A useful criterion that $f \in \mathscr{R}[a,b]$ is given by the following theorem:

Theorem (Partition Characterization of Integrability)

Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable on $[a,b]$ if and only if for all $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

Idea of the proof:

- $\Rightarrow$ follows from the Refinement Theorem and the definitions of sup and inf.
- $\Leftarrow$ follows from part (iii) of the theorem on the previous slide.

Exercise.

Write the proof of the theorem.

Theorem (Partition Characterization of Integrability)

Let $f:[a,b]\to\mathbb{R}$ be bounded. Then f is Riemann integrable if and only if for each $\varepsilon>0$ there exists a partition P of $[a,b]$ such that

$$U(f,P)-L(f,P)<\varepsilon.$$

Proof $\Rightarrow$: Suppose f is Riemann integrable. Let $\varepsilon>0$. Then by definition of sup and inf, there exist partitions P_1 and P_2 of $[a,b]$ such that

$$\overline{\int_a^b} f(x)dx+\frac{\varepsilon}{2}>U(f,P_1) \text{ and } \underline{\int_a^b} f(x)-\frac{\varepsilon}{2}<L(f,P_2).$$

Then by the Refinement Theorem we get

$$\underline{\int_a^b} f(x)dx-\frac{\varepsilon}{2}<L(f,P_2)\le L(f,P_1\cup P_2)\le U(f,P_1\cup P_2)\le U(f,P_1)<\overline{\int_a^b} f(x)dx+\frac{\varepsilon}{2}.$$

It follows from this and the integrability of f that

$$U(f,P_1\cup P_2)-L(f,P_1\cup P_2)<\overline{\int_a^b} f(x)dx+\frac{\varepsilon}{2}-\left(\underline{\int_a^b} f(x)\,dx-\frac{\varepsilon}{2}\right)$$

$$=\varepsilon.$$

$\Leftarrow$: Let $\varepsilon>0$. Then there exists a partition P of $[a,b]$ such that $U(f,P)-L(f,P)<\varepsilon$. Thus by our theorem on basic properties of upper and lower integrals,

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \le U(f,P)-L(f,P) < \varepsilon.$$

Since $\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx$ is independent of P and the above holds for all $\varepsilon>0$, it follows

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx = 0.$$

$\square$

Theorem (Linearity of the integral)

Let $f, g \in \mathscr{R}[a,b]$. Then we have each of the following.

(i) $f+g \in \mathscr{R}[a,b]$ and $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(ii) $-f \in \mathscr{R}[a,b]$ and $\displaystyle\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$.

(iii) For any real α, we have $\alpha f \in \mathscr{R}[a,b]$ and $\displaystyle\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$.

Hints:

(i) Show first that for any partition P of $[a,b]$ we have $U(f+g,P) \leq U(f,P)+U(g,P)$ and $L(f+g,P) \geq L(f,P)+L(g,P)$.

(ii) Recall $U(-f,P) = -L(f,P)$ and $L(-f,P) = -U(f,P)$.

(iii) Easy in case $\alpha > 0$. The case of $\alpha < 0$ follows from this and (ii).

Exericise.

Prove the theorem.

Theorem (Linearity of the integral)

Let $f, g \in \mathcal{R}[a,b]$. Then the following hold.

(i) $f+g \in \mathcal{R}[a,b]$ and $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(ii) For any $\alpha \in \mathbb{R}$, $\alpha f \in \mathcal{R}[a,b]$ and $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$.

Proof of (i): Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. For each $i \geq 1$, let $M_{i,f} := \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$, with similar definitions for $M_{i,g}$ and $M_{i,f+g}$.

Let $\tilde{x} \in [x_{i-1}, x_i]$. Then $f(\tilde{x}) + g(\tilde{x}) \leq M_{i,f} + M_{i,g}$, so by definition,

$$M_{i,f+g} \leq M_{i,f} + M_{i,g}.$$

Multiplying by $x_i - x_{i-1}$ and summing on i gives

$$(1)\quad \mathcal{U}(f+g,P) \leq \mathcal{U}(f,P) + \mathcal{U}(g,P).$$

A similar argument shows

$$(2)\quad L(f+g,P) \geq L(f,P) + L(g,P).$$

Let $\varepsilon > 0$. There exist partitions P_1, P_2 of $[a,b]$ such that

$$\overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2} > \mathcal{U}(f,P_1) \text{ and } \overline{\int_a^b} g(x)\,dx + \frac{\varepsilon}{2} > \mathcal{U}(g,P_2).$$

Then

$$\begin{aligned}
\overline{\int_a^b} f(x)\,dx + \overline{\int_a^b} g(x)\,dx + \varepsilon &> \mathcal{U}(f,P_1) + \mathcal{U}(g,P_2) \\
&\geq \mathcal{U}(f,P_1 \cup P_2) + \mathcal{U}(g,P_1 \cup P_2) && \text{(Refinement Thm)} \\
&\geq \mathcal{U}(f+g,P_1 \cup P_2) && \text{(by (1))} \\
&\geq \overline{\int_a^b} (f(x)+g(x))\,dx.
\end{aligned}$$

Since these integrals are independent of ε, it follows

$$\overline{\int_a^b}(f(x)+g(x))\,dx \le \overline{\int_a^b} f(x)\,dx + \overline{\int_a^b} g(x)\,dx.$$

A similar argument using ② instead of ① shows

$$\underline{\int_a^b}(f(x)+g(x))\,dx \ge \underline{\int_a^b} f(x)\,dx + \underline{\int_a^b} g(x)\,dx.$$

Thus

$$\underline{\int_a^b} f(x)\,dx + \underline{\int_a^b} g(x)\,dx \le \underline{\int_a^b}(f(x)+g(x))\,dx \le \overline{\int_a^b}(f(x)+g(x))\,dx \le \overline{\int_a^b} f(x)\,dx + \overline{\int_a^b} g(x)\,dx.$$

Since $f, g \in \mathcal{R}[a,b]$, the smallest and largest in this string of inequalities are equal, so all four terms are equal, and the common value is $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx$. This completes the proof of (i).

<u>Proof of (ii)</u> Suppose first $\alpha \in \mathbb{R}$ with $\alpha > 0$. It is obvious that for any partition P of $[a,b]$, $U(\alpha f, P) = \alpha U(f,P)$ and $L(\alpha f, P) = \alpha L(f,P)$. It follows that $\overline{\int_a^b} \alpha f(x)\,dx = \alpha \overline{\int_a^b} f(x)\,dx$ and $\underline{\int_a^b} \alpha f(x)\,dx = \alpha \underline{\int_a^b} f(x)\,dx$. Since $f \in \mathcal{R}[a,b]$ we get

$$\underline{\int_a^b} \alpha f(x)\,dx = \alpha \underline{\int_a^b} f(x)\,dx = \alpha \overline{\int_a^b} f(x)\,dx = \overline{\int_a^b} \alpha f(x)\,dx,$$

proving $\alpha f \in \mathcal{R}[a,b]$ and $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$.

To prove (ii) in case $\alpha < 0$, we first prove that $-f \in \mathcal{R}[a,b]$ and $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$. As we observed in our proof of the Refinement Theorem, for any partition P of $[a,b]$, $U(-f,P) = -L(f,P)$, so

$$
\begin{aligned}
(3)\qquad \overline{\int_a^b} -f(x)\,dx &= \inf\{U(-f,P) : P \text{ a partition of } [a,b]\} \\
&= \inf\{-L(f,P) : P \text{ a partition of } [a,b]\} \\
&= -\sup\{L(f,P) : P \text{ a partition of } [a,b]\} \\
&= -\underline{\int_a^b} f(x)\,dx
\end{aligned}
$$

and since $f \in \mathcal{R}[a,b]$, we get

$$(4)\qquad \overline{\int_a^b} -f(x)\,dx = -\int_a^b f(x)\,dx.$$

Repeating the equalities in (3) with f replaced by $-f$, we get

$$\overline{\int_a^b} -(-f(x))\,dx = -\underline{\int_a^b} -f(x)\,dx,$$

i.e. $-\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} -f(x)\,dx$, which says

$$(5)\qquad \underline{\int_a^b} -f(x)\,dx = -\int_a^b f(x)\,dx.$$

It follows from (4) and (5) that $-f \in \mathcal{R}[a,b]$ and

$$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx.$$

Finally, if $\alpha < 0$, then (ii) holds by viewing α as $-(-\alpha)$ and applying previously proven cases. $\square$