$\{f_n\}_1^\infty$ a sequence of functions,

$$f_n : D \to \mathbb{R}, \text{ where } D \subseteq \mathbb{R}.$$

① We say $\{f_n\}_1^\infty$ converges pointwise if for each $x \in D$, $\{f_n(x)\}$ converges to some real number.

Let call $\lim_{n \to \infty} f_n(x)$ the number $f(x)$. Then $f : D \to \mathbb{R}$. We say $\{f_n\}_1^\infty$ converges pointwise to f.

② Infinity norm

If $g : D \to \mathbb{R}$, if g is a bounded function, we associate the number $\|g\|_\infty$, defined by

$$\|g\|_\infty = \sup\{|g(x)| : x \in D\}.$$

③ We say $\{f_n\}_1^\infty$ converges uniformly to f if

$$\|f_n - f\|_\infty \to 0 \text{ as } n \to \infty.$$

Idea of a Norm

2 examples of norms

1. $x \in \mathbb{R}$ $\qquad |x|$

2. $\|f\|_\infty$

We next formulate convergence and Cauchy sequences in any normed space:

Definition

Let $(V, \|\cdot\|)$ be a real normed space. Let v_n be a sequence in V, and let $v \in V$.

(i) We say that the sequence v_n converges in norm to v provided $\|v_n - v\| \to 0$, i.e.

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq N \Longrightarrow \|v_n - v\| < \varepsilon].$$

(ii) We say that the sequence v_n is a Cauchy sequence provided the following is true:

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \in \mathbb{N})[m, n \geq N \Longrightarrow \|v_m - v_n\| < \varepsilon].$$

Theorem In any normed space, if v_n converges to v in norm, then v_n is a Cauchy sequence.

Exercise.

Write the proof of the above theorem.

Pf Say $v_n \to v$. To show $\{v_n\}$ Cauchy, let $\varepsilon > 0$. Since $v_n \to v$ we can choose $N \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N$ then $\|v_n - v\| < \frac{\varepsilon}{2}$. Let

$m, n \in \mathbb{N}$. Suppose $m, n \geq N$. Then

$$\begin{aligned} \|v_m - v_n\| &= \|v_m - v + v - v_n\| \\ &\leq \|v_m - v\| + \|v_n - v\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

$\square$

In section 2.4 we saw that $C[a,b]$ with the sup norm is an example of a normed space. We now show that it has the stronger property of being a Banach space.

Theorem Let $[a,b]$ be a closed bounded interval. Let $C[a,b]$ be the normed space of continuous real-valued function on $[a,b]$, equipped with the sup norm. Then $C[a,b]$ is a Banach space.

Some hints on the proof:

- We must give ourselves a sequence f_n which is Cauchy relative to the sup norm.
- Do you see why it's true that for each $x \in [a,b]$, we have $f_n(x)$ is a Cauchy sequence of real numbers?
- Why does this allow us to associate a new real number which we will call $f(x)$?
- Now try to prove that the sequence f_n converges in norm to f.
- How do you know that $f \in C[a,b]$?

Exercise.

Write the proof of the above theorem.

completeness

- Show if $\{f_n\}$ Cauchy in the sup norm, then converges in sup norm (i.e. uniformly).
- ① Find if it cvges pointwise:

Fix $x \in [a,b]$. Look at $\{f_n(x)\}_n$

$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \to 0$ as $m,n \to \infty$.

$\therefore$ $\{f_n(x)\}_n$ is Cauchy seq of reals.

$\therefore$ cvges. Call the limit $f(x)$.

Try to show $f_n \to f$ uniformly.

Theorem $C[a,b]$ is complete with respect to the sup norm.

Proof Let $\{f_n\}_n$ be a sequence in $C[a,b]$ which is Cauchy with respect to the sup norm. We must show there exists $f \in C[a,b]$ such that $f_n \to f$ in the sup norm.

Let $\varepsilon > 0$. Since $\{f_n\}_n$ is Cauchy in the sup norm, there exists $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, if $m, n \geq N$ then $\|f_m - f_n\|_\infty < \varepsilon$.

Let $x \in [a,b]$. Then for each $m, n \geq N$,

$$(*) \quad |f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty < \varepsilon.$$

Thus $\{f_n(x)\}_n$ is a Cauchy sequence of numbers. By the Completeness Axiom of $\mathbb{R}$, the sequence converges to some real number. Let's call that real number $f(x)$.

Since we can do this for each $x \in [a,b]$, we have defined f as a function from $[a,b]$ to $\mathbb{R}$. We will be done if we can prove $f \in C[a,b]$ and $f_n \to f$ in the sup norm.

Let ε and N be as above, and let $m \geq n \geq N$. Then by $(*)$,

$$f_n(x) - \varepsilon \leq f_m(x) \leq f_n(x) + \varepsilon.$$

Letting $m \to \infty$ and rewriting, we get $|f(x) - f_n(x)| \leq \varepsilon$.

Taking the supremum of this over all $x \in [a,b]$, we get

$$\|f - f_n\|_\infty \leq \varepsilon.$$

This proves $f_n \to f$ uniformly.

Since the uniform limit of continuous functions is continuous, it follows $f \in C[a,b]$.

$\square$