2.5 The Banach Space $C[a, b]$

- In this section we pursue a few of the ideas stated on the last slide of section 2.4.
- This means to take some of the ideas we've considered so far for real numbers and try to develop similar ideas in other settings, namely in **"function spaces"**.
- The first thing we developed for real numbers is the idea of a **sequence**, so we consider that first.

Sequences of functions

- Let D be any subset of $\mathbb{R}$.
- Suppose for each $n \in \mathbb{N}$ we have a real-valued function f_n with $f_n : D \to \mathbb{R}$. *(Note: The term "function" will always mean real-valued function.)*
- We refer to $\{f_n\}_{n=1}^{\infty}$ as a sequence of functions on D.

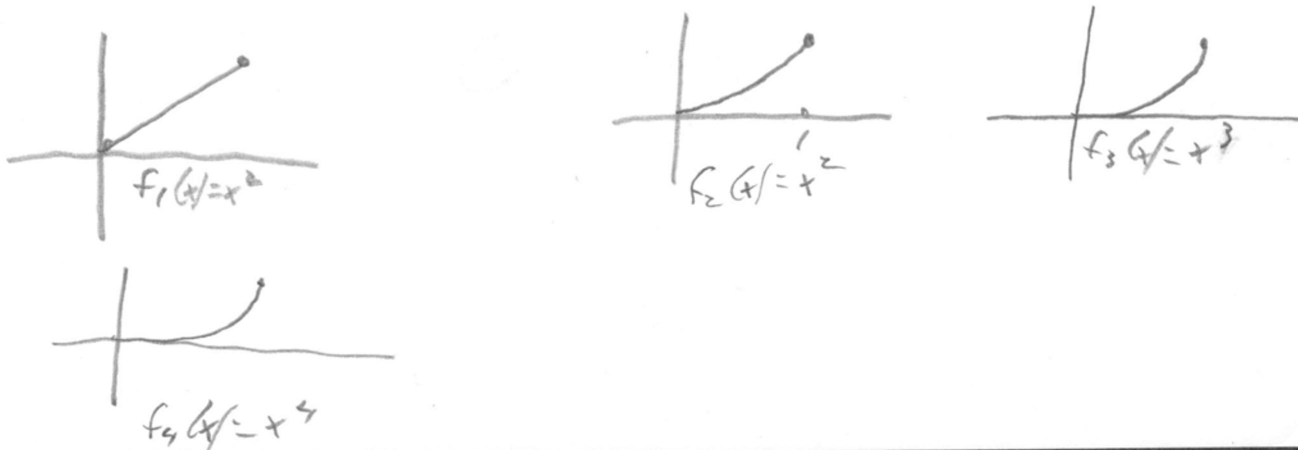
Exercise.

Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$.

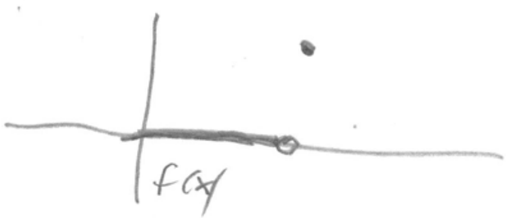
(i) Sketch the graph of a few terms of the sequence.

(ii) What is the apparent behavior of the sequence as you can see from the graph? Does it appear to go to some specific function?

(i)



(ii) Graphs are approaching the graph of $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$



Pointwise and uniform convergence of a sequence of functions

Pointwise Convergence of a sequence of functions

With D, f_n and f as on previous slide, we say that the sequence f_n converges pointwise to f if for each $x \in D$, we have $f_n(x) \to f(x)$. In symbols:

$$(\forall x \in D)[f_n(x) \to f(x)].$$

In more detail, this means the following holds:

$$(\forall x \in D)(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq N \Longrightarrow |f_n(x) - f(x)| < \varepsilon].$$

We also refer to f as the pointwise limit of the sequence f_n.

Exercise.

Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$.

(i) Does the sequence converge pointwise? If so, to what function $f(x)$?

(ii) What can you say about the continuity of the members of the sequence f_n and the continuity of f?

(i) $f_n \to f(x)$ pointwise, where

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

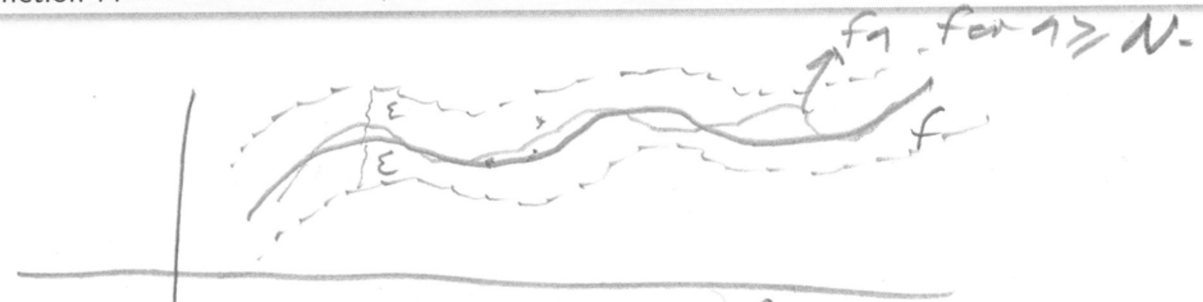
(ii) Note f_n is continuous, but $f(x)$ is not

continuous.

So the pointwise limit of a sequence of continuous functions need not be continuous!

Exercise

Draw graphs which illustrate the idea of the definition of uniform convergence of a sequence f_n to a function f.



Given $\varepsilon > 0$, uniform cnvge means there exists N s.t. for all x, if $n \geq N$ then the graph of f_n is inside the ε-tube.

Pointwise and uniform convergence of a sequence of functions

Exercise

Consider the sequence $f_n : [0, 1/2] \to \mathbb{R}$, $f_n(x) = x^n$.

(i) What is the pointwise limit of the sequence?

(ii) For each n, calculate $\|f_n\|_\infty$.

(iii) Explain how you know that the sequence converges uniformly.

(iv) If you change the domain from $[0, 1/2]$ to $[0, 1)$, prove that the convergence of the sequence is not uniform.

(i) $f_n \to 0$ pointwise.

(ii) $\|f_n\|_\infty = \sup\{x^n : 0 \le x \le \frac{1}{2}\} = \frac{1}{2^n}$

(iii) so $\|f_n - 0\|_\infty = \|f_n\|_\infty = \frac{1}{2^n} \to 0$ as $n \to \infty$.

$\therefore f_n \to 0$ uniformly.

(iv) $\|f_n\|_\infty = \sup\{x^n : 0 \le x < 1\} = 1$

$\therefore \|f_n - 0\|_\infty = 1 \not\to 0$

$\therefore f_n \not\to 0$ uniformly.

Logical connection between pointwise and uniform convergence

Theorem If a sequence f_n converges uniformly on D to a function f, then it also converges pointwise. However, the converse is false in general, that is, there exists a domain D and a sequence of functions which is pointwise convergent but not uniformly convergent on D.

Exercise.

Prove the above theorem.

Suppose $f_n \to f$ uniformly. To see $f_n \to f$ pointwise, let $x \in D$. Then

$$|f_n(x) - f(x)| \leq \sup\{|f_n(y) - f(y)| : y \in D\}$$
$$= \|f_n - f\|_\infty$$

and by assumption, $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Thus $|f_n(x) - f(x)| \to 0$, i.e. $f_n(x) \to f(x)$. Thus $f_n \to f$ pointwise.

On an earlier slide we showed $f_n(x) = x^n$ converges to 0 pointwise on $[0,1)$ but not uniformly. $\square$

- Say $\{f_n\}$ is a sequence converging in some way to a function f.
- If we know that all of the f_n's are continuous, pointwise converge doesn't tell us that f is necessarily continuous. The next theorem tells us that if the convergence is uniform, then f must be continuous.

Theorem Let f_n a sequence of functions with domain D, and suppose that the sequence converges uniformly to a function f. If each f_n is continuous, then f is continuous.

- Simply put, it says that the uniform limit of continuous functions is continuous.
- The proof is a nice application of the triangle inequality.
- Start by giving yourself $\varepsilon > 0$. Use the uniform convergence convergence to produce f_N which is uniformly within $\varepsilon/3$ of f.
- Use the continuity of f_N and the triangle inequality to prove that f is continuous.

Exercise.

Write a proof of the theorem.

- Should begin the proof by trying to show f is continuous on D. So we should give ourself $x \in D$ and prove using an ε-δ argument that f is continuous at x.

- So we must show $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in D)\left[|y-x| < \delta \Rightarrow |f(y)-f(x)| < \varepsilon\right]$.
- That means you should next give yourself $\varepsilon > 0$ and your task is to choose δ with the above properties.

- The assumption that $f_n \to f$ uniformly allows you to find N s.t. $\|f_N - f\|_\infty < \frac{\varepsilon}{3}$.
- Then use the fact that f_N is continuous at x to produce δ that works for f_N.
- Then show using triangle inequality that the same δ works for f.

Theorem (Uniform Convergence Theorem)

Let $\{f_n\}_n$ be a sequence of functions with domain D. If each f_n is continuous on D and the convergence is uniform, then the limit function is continuous.

Proof Suppose $\{f_n\}_n$ converges uniformly on D to f. Let $x \in D$. We must prove that f is continuous at x.

Let $\varepsilon > 0$. By the uniform convergence, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$ then $\|f_n - f\|_\infty < \varepsilon/3$. In particular,

$$\|f_N - f\|_\infty < \frac{\varepsilon}{3}.$$

Since f_N is continuous, it is continuous at x, so there exists $\delta > 0$ such that for all $y \in D$, if $|y-x| < \delta$ then $|f_N(y) - f_N(x)| < \frac{\varepsilon}{3}$. So for any such y,

$$|f(y)-f(x)| = |f(y)-f_N(y)+f_N(y)-f_N(x)+f_N(x)-f(x)|$$

$$\leq |f(y)-f_N(y)| + |f_N(y)-f_N(x)| + |f_N(x)-f(x)|$$

$$\leq \|f-f_N\|_\infty + |f_N(y)-f_N(x)| + \|f_N - f\|_\infty$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

$\square$

Exercise

Give a simple proof that the sequence $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$ does not converge uniformly.

Pf The pointwise limit is $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1. \end{cases}$

Since each f_n is continuous, if f_n converged uniformly the limit function would be continuous. Since f is not continuous, it follows the convergence is not uniform. $\square$