The sup norm, and general definition of a norm

Theorem Let f, g be bounded functions on $[a, b]$, and let α be a real number. Then

1. $\|f\|_\infty = 0$ if and only if $f(x) = 0$ for all $x \in [a, b]$.
2. $\|\alpha f\|_\infty = |\alpha| \, \|f\|_\infty$.
3. (triangle inequality) $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

- (1) and (2) follow quite easily from the definition of supremum.
- For (3), try to show that the right side is an upper bound of the set of values $\{|f(x) + g(x)| : x \in [a, b]\}$.

Exercise.

Write the proof of the theorem.

(3) Let $x \in [a,b]$. Then

$$|f(x)| \leq \|f\|_\infty, \quad |g(x)| \leq \|g\|_\infty,$$

so $$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty$$

Thus $\|f\|_\infty + \|g\|_\infty$ is an upper bound for

$\{|f(x) + g(x)| : x \in [a,b]\}$. Thus by definition

$$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

$\square$

3 examples of normed spaces

① $V = \mathbb{R}$ $\quad \|x\| = |x|$.

② $V = \mathbb{Q}$ (where scalars are also $\mathbb{Q}$)

$\|x\| := |x|$.

③ $C[a,b]$ = set of continuous real-valued functions on $[a,b]$,

$\|f\| := \|f\|_\infty = \sup\{|f(t)| : a \le t \le b\}$.

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- The set of **bounded real-valued functions** on a fixed interval $[a, b]$ forms what is called a **vector space**. This means that if f, g are any two such functions, then $f + g$ is such a function, αf is such a function for any real number α, and furthermore basic rules of arithmetic hold (these rules are described on page 59 of the text, we won't list them here).
- Similarly if we look instead at the **continuous real-valued functions** on $[a, b]$, these also form a vector space. Do you see why? What property of continuity is needed here?
- The three properties described on the previous slide are worth abstracting and giving a name. We do that next.