2.4 The Extreme Value Theorem and Some of its Consequences

- The Extreme Value Theorem deals with the question of when we can be sure that for a given function f,

  (1) the values $f(x)$ don't get too big or too small,

  (2) and f takes on both its absolute maximum value and absolute minimum value.

- We'll see that it gives another important application of the idea of compactness.

Exercise

Phrase boundedness using the terms supremum and infimum, that is, try to complete the sentences

"f is upper bounded if and only if ......"

"f is lower bounded if and only if ......"

"f is bounded if and only if ......"

using the words supremum and infimum somehow.

f upper bounded $\Longleftrightarrow$ $\sup\{f(x) : x \in D_f\} < \infty$

f lower bdd $\Longleftrightarrow$ $\inf\{f(x) : x \in D_f\} > -\infty$

f bounded $\Longleftrightarrow$ $\sup\{|f(x)| : x \in D_f\} < \infty$.

Some examples

Exercise

Give some examples (in pictures) of functions which illustrates various things:

a) A function can be continuous but not bounded.

b) A function can be continuous, but might not take on its supremum value, or not take on its infimum value.

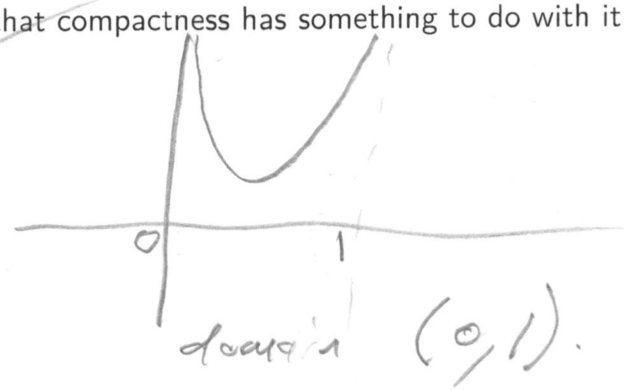
c) A function can be continuous, and does take on both its supremum value and its infimum value.

d) A function can be discontinuous, but bounded.

e) A function can be discontinuous on a closed bounded interval, and not take on its supremum or its infimum value.

- You might notice that the above negative examples involving continuous functions all have domains which are not closed bounded intervals. This suggests that compactness has something to do with it.

a



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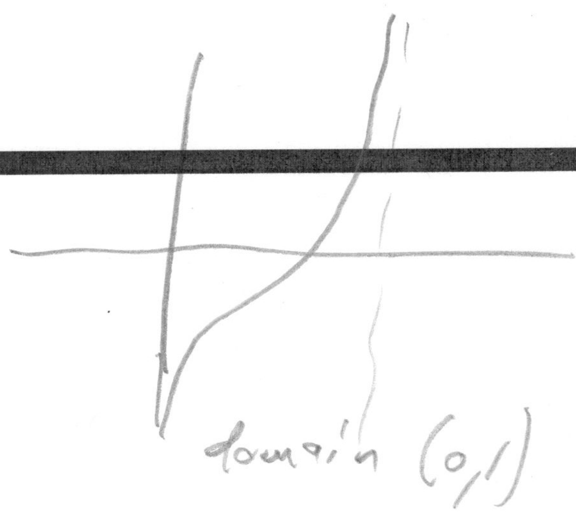

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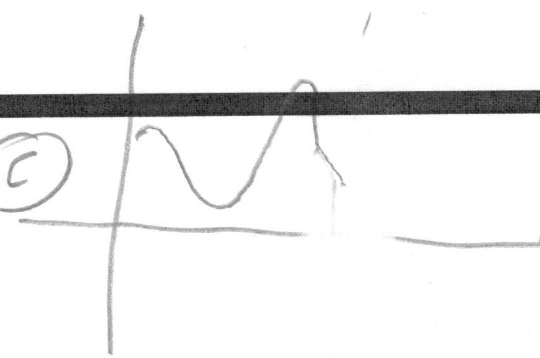
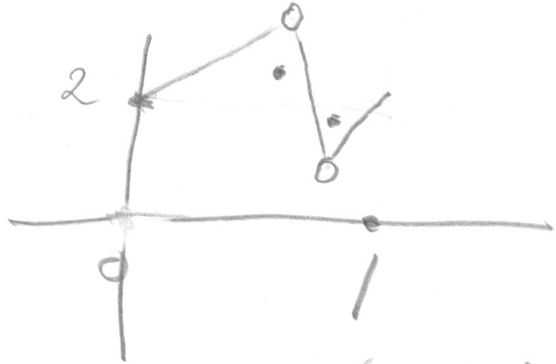
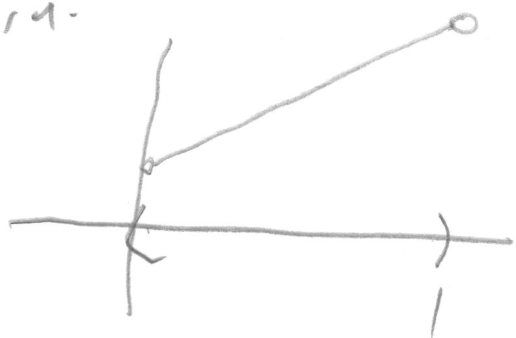


(e)



(f) f continuous and bounded on a bounded interval but does not achieve its absolute max or min.



Theorem (Extreme Value Theorem)

Let f be a real-valued continuous function with domain a closed bounded interval $[a, b]$. Then f is bounded, and f has both a maximum and minimum value on $[a, b]$.

- This theorem is one of the most important of the subject.
- The proof will make use of the Heine-Borel theorem, the Bolzano-Weierstrass theorem, and the sequential characterization of continuity.

- We'll only do proof for max.
- Proof has 2 steps.
  1. Prove f is upper bounded.
  2. f achieves its supremum.

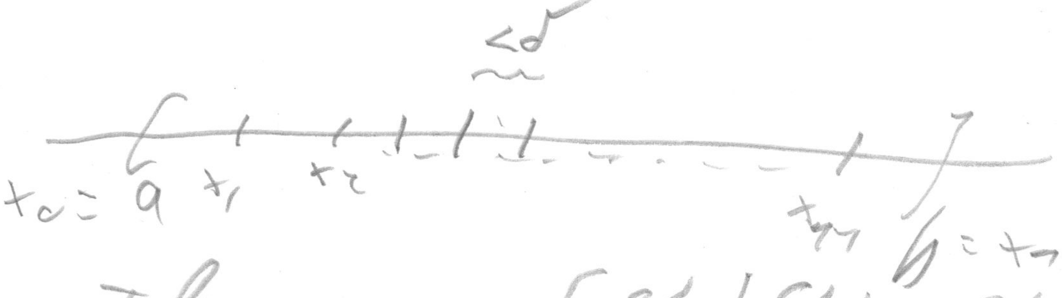
(1) • automatically f uniformly continuous.

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall z, w \in [a,b])[|z-w| < \delta \Rightarrow |f(z) - f(w)| < \varepsilon]$

Take $\varepsilon = 1$

$|f(z) - f(w)| < 1$.

use δ to partition $[a,b]$:



Take $\max\{f(x_0), f(x_1), \dots, f(x_n)\}$

So every $f(x)$ is within 1 of that max.

② <u>Show f achieves its max.</u>

- Let $M = \sup\{f(x) : a \le x \le b\} \in \mathbb{R}$.
- $\exists \{x_n\}_n$ in $[a,b]$ s.t. $f(x_n) \to M$.
- By Bolzano-Weierstrass $\exists$ x_{n_j} converging to some $x \in [a,b]$.
- Does $f(x_{n_j}) \longrightarrow f(x)$?

  Yes by continuity of f!

Theorem (Extreme Value Theorem)

Let f be a real-valued continuous function with domain a closed bounded interval $[a, b]$. Then f is bounded, and f has both a maximum and minimum value on $[a, b]$.

Exercise.

Write the proof of the Extreme Value Theorem.

Theorem (Extreme Value Theorem)

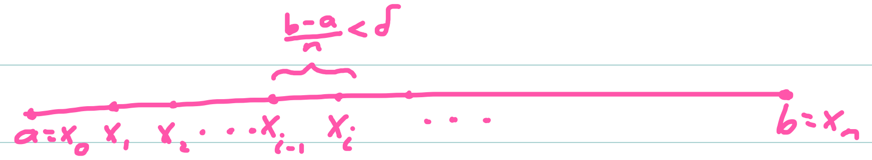
Let f be a real-valued continuous function with domain a closed bounded interval $[a,b]$. Then f is bounded, and f has both a maximum and minimum value on $[a,b]$.

Proof We will prove f is upper bounded and has a maximum value. The proof concerning the minimum value follows by applying these results to $-f$.

We first show f is bounded. Since f is continuous and the domain $[a,b]$ is compact, it follows f is uniformly continuous. Taking $\varepsilon=1$ in the definition of uniform continuity, there exists $\delta>0$ such that for any $z,w\in[a,b]$, if $|z-w|<\delta$ then $|f(z)-f(w)|<1$.

Choose $n\in\mathbb{N}$ with $n>\frac{b-a}{\delta}$. Let

$$x_0=a=a+\frac{0(b-a)}{n},\quad x_1=a+\frac{1\cdot(b-a)}{n},\quad x_2=a+\frac{2(b-a)}{n},$$
$$x_3=a+\frac{3(b-a)}{n},\ \ldots,\ x_n=a+n\frac{(b-a)}{n}=b$$



Let $K=\max\{|f(x_i)|: 0\le i\le n\}$.

Let $x \in [a,b]$. Then there exists i such that $|x - x_i| < \frac{b-a}{n} < \delta$, so $|f(x) - f(x_i)| < 1$. Thus

$$|f(x)| = |f(x) - f(x_i) + f(x_i)| \leq |f(x) - f(x_i)| + |f(x_i)| < 1 + K.$$

We have thus shown f is bounded.

To prove f has a maximum, let

$$M = \sup\{f(x) : x \in [a,b]\}.$$

We know that M exists as a real number by the Least Upper Bound property of $\mathbb{R}$. We must show there exists $x \in [a,b]$ such that $f(x) = M$.

By definition of supremum, for each $n \in \mathbb{N}$ there exists $x_n \in [a,b]$ such that $M - \frac{1}{n} < f(x_n) \leq M$. Thus $f(x_n) \to M$.

Since $\{x_n\}_n$ is a bounded sequence, by the Bolzano-Weierstrass Theorem, it has a convergent subsequence $\{x_{n_j}\}_j$. Thus there exists x such that $x_{n_j} \to x$. Since $a \leq x_{n_j} \leq b$ for all j, it follows $a \leq x \leq b$.

Since f is continuous, $f(x_{n_j}) \to f(x)$. But we showed above that $f(x_{n_j}) \to M$, so $f(x) = M$. $\square$

Exercise

Prove that $f(x) = x^2 - x + 1 + \cos x$ has a minimum value on $\mathbb{R}$.

Hints:

(i) *Begin by completing the square.*

(ii) *You can't immediately make use of the EVT because $\mathbb{R}$ isn't a closed bounded interval. However, you can reduce the problem to a problem on a closed bounded interval using the expression for $f(x)$ in (i).*

$x^2 - x + 1 = x^2 - x + \frac{1}{4} + \frac{3}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} + \cos x.$

Note. $f(0) = 2$.

$\therefore \inf\{f(x) : x \in \mathbb{R}\} \le 2.$

$x \geq 10 + \frac{1}{2}$: then $f(x) \geq 100 + \frac{3}{4} - 1 > 2.$

$x \le -(10 + \frac{1}{2})$: then $f(x) \geq 121 + \frac{3}{4} - 1 > 2.$

$\therefore \inf\{f(x) : x \in \mathbb{R}\}$

$= \inf\{f(x) : -(10 + \frac{1}{2}) \le x \le 10 + \frac{1}{2}\}.$

By Extreme Value Thm f has a minimum value for $x \in \left[-\frac{21}{2}, \frac{21}{2}\right]$.