A few comments on homework 9

We say $A \subseteq \mathbb{R}$ is <u>closed</u> if $\mathbb{R} - A$ is open.

- A set can be both open and closed (unlike a door).

<u>ex</u> $\emptyset$ is open (vacuously true).

$\therefore \mathbb{R} - \emptyset = \mathbb{R}$ is closed.

<u>ex</u> $\emptyset$ is closed (since $\underset{\text{open}}{\mathbb{R} - \emptyset = \mathbb{R}}$ is open).

So for #2,

$$\mathbb{R} = \underbrace{(-\infty, 1] \cup [0, \infty)}_{\text{closed}}$$

In fact we can show the finite union of closed sets is closed.

[$F_1, \dots, F_n$ closed sets.

$$\mathbb{R} \setminus (F_1 \cup F_2 \cup \dots \cup F_n) = (\mathbb{R} - F_1) \cap (\mathbb{R} - F_2) \cap \dots \cap (\mathbb{R} - F_n)$$

finite intersection of open is open (exercise)

② In ①, showed $[a,\infty)$ is closed and (b,∞) not closed. So why not use that to do ②:

$$\bigcup_{n=1}^{\infty} \left[\tfrac{1}{n}, \infty\right) = (0,\infty).$$

closed — not closed

① ⓐ Showing (b,∞) not closed doesn't follow from showing (b,∞) is open.

You must show $\mathbb{R} \setminus (b,\infty)$ is not open.

$\mathbb{R} \setminus (b,\infty) = (-\infty, \boxed{b}]$ → focus on b to show not open.

<u>Nicer solution of ②</u>

$$\mathbb{Q} = \bigcup_{x \in \mathbb{Q}} \{x\}$$

$\mathbb{R} \setminus \{x\} = (-\infty, x) \cup (x, \infty)$ open, so $\{x\}$ is closed.

$\mathbb{Q}$ is not closed because the set of irrationals is not open.

<u>Exercise</u>: Show set of irrationals is not open.

2.3 Some Properties of Continuous Functions

In this section we look at some properties, some quite deep, shared by all continuous functions. They are known as the following:

1. Preservation of sign property
2. Intermediate Value Property
3. Uniform continuity property

Before we comment on the Intermediate Value Property, do the following exercise.

Exercise.

The sets $(-\infty, 10)$, $(\infty, 10]$, $(1, 4)$, $[1, 7]$, $[1, \infty)$ are all intervals (and we could have included other kinds of intervals on this list). What property characterizes all of them? In other words, define the term "interval".

An interval is a set I with the property that for all x and y in I, all the numbers between x and y lie in I.

Sign property of continuous functions

Theorem 1 (Preservation of sign)

Let f be a function and let $p \in D_f$. Suppose f is continuous at p. If $f(p) > 0$, then there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, if $x \in D_f \cap I_\delta(p)$, then $f(x) > 0$.

Exercise : Comments on the theorem and its proof

a) Intuitively, why do we expect it to be true?

b) If we weaken the continuity hypothesis, give a counterexample to show the resulting statement is false.

c) The continuity of f at p says what precisely? Use the $\varepsilon - \delta$ definition of continuity.

d) Are we making use of the continuity definition or are we proving that it holds here? Why?

e) If we could guarantee that $f(x) > f(p)/3$ for all x in some open interval centered at p, we would be done. So what ε should we work with in the $\varepsilon - \delta$ definition of continuity?

f) Write the proof of the theorem.

$f(p) > 0.$

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D_f)(0 < |x-p| < \delta$
$\Rightarrow |f(x) - f(p)| < \varepsilon)$

$\varepsilon < f(x) - f(p) < \varepsilon$

$f(p) - \varepsilon < f(x) < f(p) + \varepsilon.$

want $f(p) - \varepsilon > 0$ / so use $\varepsilon = \frac{f(p)}{2}$!

Theorem (Preservation of Sign) Let $f: D_f \to \mathbb{R}$ and let $p \in D_f$. Suppose f is continuous at p. If $f(p) > 0$, then there exists $\delta > 0$ such for all $x \in \mathbb{R}$, if $x \in D_f \cap I_\delta(p)$, then $f(x) > 0$.

Proof Let f and p be as in the statement. Since $f(p) > 0$, we can choose ε such that

$$0 < \varepsilon < \frac{2}{3} f(p).$$

Corresponding to this $\varepsilon > 0$, since f is continuous at p, there exists $\delta > 0$ such that for any $x \in D_f$, if $|x - p| < \delta$ then $|f(x) - f(p)| < \varepsilon$. For any such x, we have $f(x) - f(p) > -\varepsilon$, so

$$f(x) > f(p) - \varepsilon > f(p) - \frac{2}{3} f(p) = \frac{1}{3} f(p) > 0.$$

$\square$

Theorem 2 (Intermediate Value Theorem)

Let I be an interval and f a function whose domain contains I. If f is continuous, then for all $a, b \in I$ with $a < b$ and all real numbers k, if k is strictly between $f(a)$ and $f(b)$, then there exists c such that $a < c < b$ and $f(c) = k$.

Exercise: Comments on the IVT and its proof

a) Draw some graphs of functions to illustrate what the theorem is saying.

b) Illustrate by sketch some counterexamples which show why we have the hypothesis that f is continuous.

c) Why is it sufficient to prove the result in case $f(a) < f(b)$?

d) Explain how to find a sequence of closed intervals $I_n = [a_n, b_n]$ such that

   (i) $I_1 = [a, b]$

   (ii) for all n, I_{n+1} is either the left half or right half of I_n

   (iii) For all n, $f(a_n) \leq k \leq f(b_n)$

e) What does the Nested Intervals Theorem tell you about the sequence a_n and b_n?

f) What can you say about the sequences $f(a_n)$ and $f(b_n)$? Make the most of the continuity assumption of f.

g) Write the proof of the Intermediate Value Theorem.

Theorem (Intermediate Value Theorem)

Let I be an interval and f a real-valued function whose domain contains I. If f is continuous, then for all $a, b \in I$ with $a < b$ and all real numbers k, if k is strictly between $f(a)$ and $f(b)$, then there exists c such that $a < c < b$ and $f(c) = k$.

Proof Let f, I, a, b, k be as in the statement. Without loss of generality we may assume $f(a) < f(b)$, for otherwise we would replace f by $-f$.

We show by induction the existence of a sequence $\{I_n\}_n$ of closed bounded intervals $I_n := [a_n, b_n]$ with the following properties:

1. $I_1 = [a, b]$;
2. For each n, I_{n+1} is the left or right half of I_n;
3. For all n, $f(a_n) \le k \le f(b_n)$.

For the basis step, define $I_1 = [a, b]$ and note that 3 holds by assumption.

For the inductive step, let $n \geq 1$ and assume we have selected $I_1, I_2, \ldots, I_n$. Choose $I_{n+1} = [a_{n+1}, b_{n+1}]$ as follows:

$$I_{n+1} = \begin{cases} \text{left half of } I_n & \text{if } f\left(\frac{a_n+b_n}{2}\right) \geq K \\ \\ \text{right half of } I_n & \text{if } f\left(\frac{a_n+b_n}{2}\right) < K \end{cases}$$

This completes the proof of the induction.

By the Nested Intervals Theorem, there exists a unique $x \in \bigcap_n I_n$ such that $a_n \to x$ and $b_n \to x$.

By continuity of f, $f(a_n) \to f(x)$ and $f(b_n) \to f(x)$. We also have $f(a_n) \leq K \leq f(b_n)$ for all n, so taking the limit we get

$$f(x) = \lim_{n\to\infty} f(a_n) \leq K \leq \lim_{n\to\infty} f(b_n) = f(x),$$

from which we see $f(x) = K$.

$\square$

Application of the Intermediate Value Theorem (IVT)

Exercise: Existence of nth roots

Let k be a positive real number, and let $n \in \mathbb{N}$.

a) What is the formal definition of $k^{1/n}$, i.e. what does it mean to say a real number x satisfies

$$x = k^{1/n}?$$

b) Prove that the real number $k^{1/n}$ exists by making appropriate use of the Intermediate Value Theorem.

Hint: The IVT allows you to use two inequalities to deduce that a desirable equality can be achieved.

(a) $k^{\frac{1}{n}}$ is defined to be the unique real number x such $x > 0$ and $x^n = k$

How do we know it exists?

$x^n = k$

Apply IVT to $f(x) = x^n$.

Need a and b s.t. $f(a) < k < f(b)$.

$a^n < k < b^n$

Take $a = 0$. Which b works?

need $k < b^n$

$k < (k+1)^n$

Proof Let $f:\mathbb{R}\to\mathbb{R}$ be $f(x)=x^n$.
Then f is continuous. Note

$$f(0)=0<K$$

and

$$f(K+1)=(K+1)^n>K.$$

Thus by IVT there exists x between 0 and $K+1$ s.t. $f(x)=K$, i.e. $x^n=K$. By def^n, $x=K^{\frac{1}{n}}$. $\square$

Definition

Let f be a real-valued function on a set D_f. We say that f is **uniformly continuous** if the following holds:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x_1 \in D_f)\forall x_2 \in D_f)[|x_2 - x_1| < \delta \Longrightarrow |f(x_2) - f(x_1)| < \varepsilon].$$

- Note the difference between mere continuity and uniform continuity is just the order in which "$\forall x_1 \in D_f$" appears in the definition. But it makes a great difference.

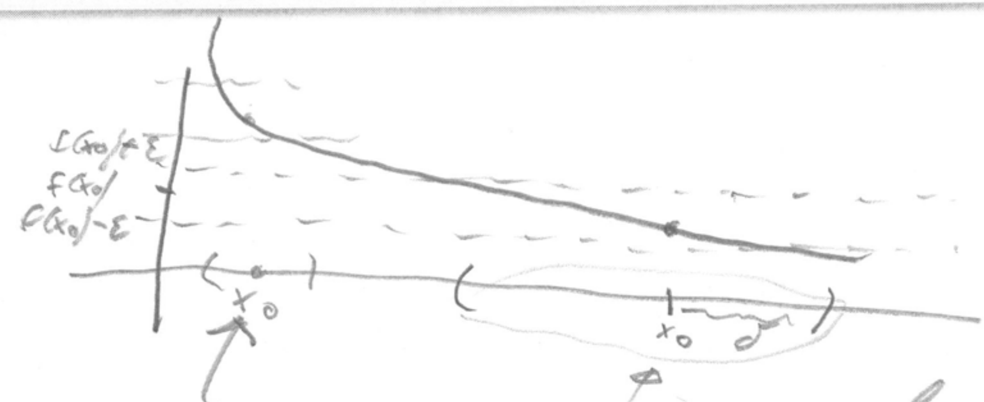
Exercise

Consider the function $f(x) = 1/x$, $0 < x < 1$.

a) Illustrate by sketch why you believe f is not uniformly continuous.

b) Prove f is not uniformly continuous.

9



much smaller δ needed here for the same ε.

this δ works for this x_0 and the given $\varepsilon > 0$

but you would need a much smaller δ for a a smaller x_0 and the same ε.

Fundamental Theorem on Uniform Continuity

Theorem Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous.

- This is an important result and often applied in real analysis books.
- The text has a nice proof using the Bolzano-Weierstrass theorem, so you should be sure to work through that proof.
- I'm going to show another proof which seems to me is more natural, and is very typical of so-called "compactness arguments".
- The idea is that given $\varepsilon > 0$, the statement of continuity at a single point $p \in D_f$ produces an open interval centered at p. As p varies over D_f, the set of all such open intervals gives an open cover of $[a, b]$.
- We can then pass to a finite subcover (why?), and this finite subcover allows us to produce a single δ that does what we'd like for the given ε (i.e. it allows us to confirm uniform continuity).

Exercise.

Write a proof of the theorem.

- This is the key idea.
- For a given $\varepsilon > 0$, the continuity of f allows us to produce a special open

interval about each $x \in [a,b]$ on which we have control over f. We can refer to this as making many "local statements" about f.

- The compactness of $[a,b]$ allows us to turn those local statements into a "global statement"

Theorem (Fundamental Theorem on Uniform Continuity) Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous.

Proof Let $\varepsilon > 0$. For each $x \in [a,b]$, since f is continuous at x, there exists $\delta_x > 0$ such that for all $z \in [a,b]$,

$$(*) \qquad \text{if } |z-x| < 2\delta_x \text{ then } |f(z)-f(x)| < \frac{\varepsilon}{2}.$$

The set $\{I_{\delta_x}(x) : x \in [a,b]\}$ is an open cover of $[a,b]$, so by the Heine-Borel Theorem, finitely many of these open intervals cover $[a,b]$. Thus, there exists $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in [a,b]$ such that

$$[a,b] \subseteq I_{\delta_{x_1}}(x_1) \cup \ldots \cup I_{\delta_{x_n}}(x_n).$$

Choose $\delta = \min\{\delta_{x_1}, \ldots, \delta_{x_n}\}$.

Let $x, y \in [a,b]$ with $|x-y| < \delta$. There exists $i \in \mathbb{N}$ such that $x \in I_{\delta_{x_i}}(x_i)$, i.e. $|x - x_i| < \delta_{x_i}$. Also, since $|x-y| < \delta$,

$$|y-x_i| = |y-x+x-x_i| \leq |y-x| + |x-x_i| < \delta + \delta_{x_i} \leq \delta_{x_i} + \delta_{x_i} = 2\delta_{x_i},$$

so by $(*)$, $|f(x) - f(x_i)| < \frac{\varepsilon}{2}$ and $|f(y) - f(x_i)| < \frac{\varepsilon}{2}$. Thus

$$|f(x)-f(y)| = |f(x)-f(x_i)+f(x_i)-f(y)| \leq |f(x)-f(x_i)| + |f(x_i)-f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$