The sequential characterization of continuity and theorems proved in the previous section make the following theorem immediate:

Theorem (Arithmetic properties of continuity)

Let f and g be continuous at $a \in D_f \cap D_g$. Then we have each of the following:

1. $f \pm g$ is continuous at a.
2. $f \cdot g$ is continuous at a.
3. f/g is continuous at a provided $g(a) \neq 0$.

Proof of 1 for $f+g$

Let $\{x_n\}_1$ be a sequence in $D_f \cap D_g$ converging to a. By continuity of f and g at a, $f(x_n) \to f(a)$ and $g(x_n) \to g(a)$. Thus by a result in section 1.4,

$f(x_n) + g(x_n) \to f(a) + g(a)$. Thus $f+g$ is continuous at a. $\square$

Exercises.

Define $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$.

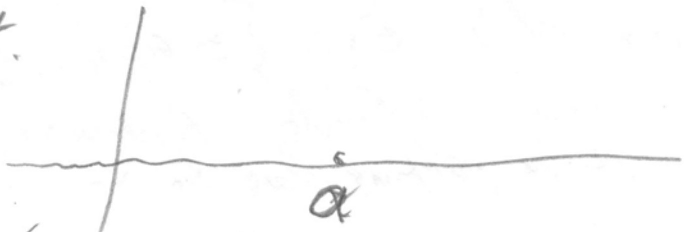
a) Draw the graph of f (if it's not too much trouble).

b) For which x values do you believe f is continuous?

c) Prove your claim in (b) above.

(a) cannot.

(b)



Nowhere continuous

(c) If "a" rational, $f(a)=1$, can choose $x_n \to a$ s.t. x_n irrational.

(Why are irrationals dense?

Let (c,d) be an open interval. How to show (c,d) contains an irrational?

Pf choose e, f rational so that

$$c < e < f < d.$$

Then $e + \frac{f-e}{\sqrt{2}}$ is in (e,f).

Thus the irrationals are dense in $\mathbb{R}$.

Proof that f not continuous anywhere. Let $a \in \mathbb{R}$. We show f not cont. at a. If $a \in \mathbb{Q}$, then choose $x_n \to a$ s.t. x_n irrational. Then $f(x_n) = 0 \to 0 \neq 1 = f(a)$. Thus f not continuous at a. If $a \notin \mathbb{Q}$, choose $x_n \to a$, x_n rational. Then $f(x_n) = 1 \longrightarrow 1 \neq 0 = f(a)$. Thus f not continuous anywhere.

$\square$

Exercise

Each positive rational number can be written uniquely in the form m/n, where m and n have no factors in common. For the following function, we agree to write each positive rational in this way. Define

$$f : (0,1) \to \mathbb{R} \text{ by } f(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ \dfrac{1}{n} & \text{if } x \text{ rational of the form } \dfrac{m}{n} \end{cases}$$

a) Prove that f is discontinuous at every rational in $(0,1)$.

b) Prove that f is continuous at every irrational in $(0,1)$.
Hints for question 2:
   (i) Start with $x_0 \in (0,1) \setminus \mathbb{Q}$ and $\varepsilon > 0$. You must produce $\delta > 0$ such that for all x, if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| = f(x) < \varepsilon$.
   (ii) Let $E_\varepsilon := \left\{ \dfrac{m}{n} \in (0,1) : \dfrac{1}{n} \geq \varepsilon \right\}$. How big a set is E_ε? Can you count how many elements there are in it?
   (iii) Is there a number $\min \left\{ \left| \frac{m}{n} - x_0 \right| : \frac{m}{n} \in E_\varepsilon \right\}$? How do you know?
   (iv) If you took δ to be 1/2 of the minimum on the previous line, does it do what we'd like it to do?

(b) Given any irrational a, must show

$$\lim_{x \to a} f(x) = f(a) = 0$$

• idea is you must show that if $\frac{m}{n}$ close

to irrational a, then $\frac{1}{n}$ must get near 0, i.e. the n must be big.

• Look at $\left\{ \frac{m}{n} \in (0,1) : \frac{1}{n} \geq \varepsilon \right\}$.

$$\frac{1}{n} \geq \varepsilon \Rightarrow n \leq \frac{1}{\varepsilon} \Rightarrow m < n \leq \frac{1}{\varepsilon}$$

so E_ε is finite. This is the key to the proof.

Theorem Define $f:(0,1)\to\mathbb{R}$ by $f(x)=\begin{cases}0 & \text{if } x \text{ irrational}\\ \frac{1}{n} & \text{if } x=\frac{m}{n}\in\mathbb{Q}\end{cases}$

where in the above, it is assumed $m,n\in\mathbb{N}$ with no prime factors in common. Then f is continuous at each irrational and discontinuous at each rational.

Proof Let $x_0\in(0,1)$. Suppose first $x_0\in\mathbb{Q}$. Since the set of irrationals is dense in $\mathbb{R}$, there exists a sequence $\{x_n\}_n$ of irrationals such that $x_n\to x_0$. Then $f(x_n)\not\to f(x_0)$ since $f(x_n)=0$ for all n but $f(x_0)\neq 0$. Thus f is not continuous at x_0.

Lastly, suppose x_0 is irrational. We will prove f is continuous at x_0 by showing $\lim\limits_{x\to x_0} f(x)=f(x_0)$. For this purpose, let $\varepsilon>0$ and define

$$E_\varepsilon=\left\{\frac{m}{n}\in(0,1):\frac{1}{n}\geq\varepsilon\right\},$$

where in writing $\frac{m}{n}$ we always assume any common factors have been cancelled.

For any such $\frac{m}{n}\in E_\varepsilon$, $n\leq\frac{1}{\varepsilon}$ so $m<n\leq\frac{1}{\varepsilon}$. Thus if K is any integer greater than $\frac{1}{\varepsilon}$, then E_ε has less than K^2 elements. In particular, E_ε is a finite set.

Consider $\left\{\left|\frac{m}{n}-x_0\right| : \frac{m}{n} \in E_\varepsilon\right\}$. Since E_ε is a finite set and x_0 is irrational, this is a finite set of strictly positive numbers. Thus the number δ defined by

$$\delta := \frac{1}{2}\min\left\{\left|\frac{m}{n}-x_0\right| : \frac{m}{n} \in E_\varepsilon\right\}$$

is positive.

Let $x \in (0,1)$. Suppose $|x-x_0|<\delta$. If x is irrational, then

$$|f(x)-f(x_0)| = |0-0| < \varepsilon.$$

If $x=\frac{m}{n}$ is rational, by the choice of δ, $\frac{m}{n} \notin E_\varepsilon$, so

$$|f(x)-f(x_0)| = \left|\frac{1}{n}-0\right| = \frac{1}{n} < \varepsilon.$$

We've shown that $\lim_{x\to x_0} f(x) = f(x_0)$, concluding the proof that f is continuous at x_0.

$\square$