2.2 Continuous Functions

- Shortly we will give the formal definitions concerning continuity of a functions. In the definitions, we distinguish between
  (i) what it means that a given function is continuous at a point of its domain
  (ii) and what it means that a given function is continuous on its domain.
- Points "a" of the domain are either cluster points or not cluster points.
- Cluster points "a" have the property that there are sequences in $D_f \setminus \{a\}$ converging to a;
- Points of D_f which are not cluster points cannot be approached in $D_f \setminus \{a\}$, so they are also referred to as **isolated points** of D_f, i.e. they have the property that

$$(\exists \delta > 0)[D_f \cap (a-\delta, a+\delta) = \{a\}].$$

Exercise.

If $D_f = ((0,4] \setminus \{2\}) \cup \{10\}$, then what are the cluster points and what are the isolated points of D_f?

cluster points $\quad [0,4] \setminus \{2\} = (0,2) \cup (2,4]$.

isolated points $\quad \{10\}$

Definition

Let f be a real-valued function with domain D_f. Let $a \in D_f$. We say that f **is continuous at** a **provided**

(i) a **is an isolated point of** D_f **or**

(ii) a **is a cluster point of** D_f **and** $\lim_{x \to a} f(x) = f(a)$.

Exercise.

Recall we have a formulation of the statement $\lim_{x \to a} f(x) = f(a)$ in terms of sequences provided a is a cluster point of D_f. We claim that f is continuous at $a \in D_f$ if and only if for every sequence x_n in D_f, if $x_n \to a$ then $f(x_n) \to f(a)$.
Prove that this sequential characterization of continuity works even if a is an isolated point of D_f.

It is because the statement

"for every sequence $\{x_n\}_n$ in D_f if $x_n \to a$ then $f(x_n) \to f(a)$"

is vacuously true in case a is an isolated point, for in that case there

is no sequence in $D_f \setminus \{a\}$ converging to a.

Definition

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Exercise.

Writing down the ε, δ definition of limit gives a third way to say that f is continuous at $a \in D_f$, namely f is continuous at $a \in D_f$ if and only if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R}[x \in D_f \cap (a-\delta, a+\delta) \implies |f(x) - f(a)| < \varepsilon].$$

Explain why this definition automatically predicts continuity at any isolated point a of D_f.

If a is isolated, then for δ sufficiently small, there is no x in $D_f \cap (a-\delta, a+\delta)$ other than a, and $|f(a)-f(a)| = 0 < \varepsilon$.

So for isolated a, the above statement is always true.

- We'll find the sequential characterization of continuity particularly useful since we've already seen some useful theorems concerning sequences.

Exercise.

a) The simplest example of a continuous function is $f(x) = x$. Prove the continuity of it in the most elementary proof you can think of.

b) What set of functions can we generate from $f(x) = x$ using the operations of addition, multiplication, and division? If you were to try to prove this rigorously, what theorem would you need to use?

c) In order to prove continuity of the functions in (b), what theorem should we be motivated to try to prove?

(a) To prove $f(x) = x$ is continuous it is easiest to use the sequence characterization of continuity (criterion (2) on slide 4) because it says

"if $\{x_n\}_n$ is any sequence in D_f such that $x_n \to a$, then $x_n \to a$,"

(because $f(x_n) = x_n$ and $f(a) = a$) and this is obviously a true statement.