- Intuitively a function is continuous provided you can draw its graph without lifting your pencil.
- But what to do if we have a function whose graph cannot be drawn? How to decide if it is continuous?

For example, how to decide continuity of each the following functions?

1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$.
2. Each positive rational number can be written uniquely in the form m/n, where m and n have no factors in common. For the following function, we agree to write each positive rational in this way.
   Define $g : (0,1) \to \mathbb{R}$ by $g(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ \dfrac{1}{n} & \text{if } x \text{ rational of the form } \dfrac{m}{n} \end{cases}$
3. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\cos(12^n x)}{2^n}$

- These functions all exist, yet there is no way we can draw any of their graphs.
- So how can we decide whether or not they are continuous?

We need a very precise definition of continuity, one which is sufficient to deal with the above questions.

Definition

Let D be a nonempty subset of $\mathbb{R}$ and let $a \in \mathbb{R}$. We say that "a" is a **cluster point** of D provided the following is true:

$$(\forall \delta > 0)(\exists x \in D \setminus \{a\})[0 < |x - a| < \delta].$$

Equivalently, "a" is a cluster point of D provided there exists a sequence x_n in $D \setminus \{a\}$ such that $x_n \to a$.

Exercise

Find all the cluster points of the following sets. Prove you are correct.

1. $A = \{0, 1, 2, 3, 4, 5\}$
2. $B = (0, 10)$
3. $C = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$.
4. Can you generalize the result in the previous part?
5. $\mathbb{Z}$
6. $\mathbb{Q}$.

(1) $\emptyset$ (no cluster points)

(2) $[0, 10]$

(3) $\{1\}$

(4) Let D be a countable whose elements we view as the members of a certain infinite sequence. Then the cluster points of D are the numbers which are the limits of convergent subsequences of that sequence.

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Definition

Let f be a real-valued function with domain denoted by D_f. Let $a \in \mathbb{R}$ be a cluster point of D_f. Let L be a real number. Then we define $\lim_{x \to a} f(x) = L$ by

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D_f)[0 < |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon].$$

- Since we know quite a bit about sequences, it would be convenient to characterize the definition of limit in terms of convergence of certain sequences.

Theorem. Let f be a real-valued function on some domain $D_f \subseteq \mathbb{R}$, let $a \in \mathbb{R}$ be a cluster point of D_f, and let $L \in \mathbb{R}$. Then the following are equivalent:

(i) $\lim_{x \to a} f(x) = L$;

(ii) For every sequence x_k in $D_f \setminus \{a\}$, if $x_k \to a$, then $f(x_k) \to L$.

Comments on the proof

$\Longrightarrow$: This is done with a direct proof using the working definitions of $\lim_{x \to a} f(x) = L$ and convergence of sequences. But be sure you are clear on what you are assuming to be true and what you are trying to prove.

$\Longleftarrow$: Our textbook proves this direction by the method of contradiction. But I think it's just as easy to do it using contraposition, i.e. assuming that (i) is false, see if you can prove that (ii) is false.

Notes: $\Longrightarrow$: Assume $\lim_{x \to a} f(x) = L$. You must prove that for any sequence $\{x_k\}_k$ in D_f, if $x_k \to a$ then $f(x_k) \to L$. The key is begin with assumption a given sequence $\{x_k\}$ converges to a. Then try to prove that $f(x_k) \to L$. In order to do that, you will need to make use of the assumption that $\lim_{x\to a} f(x) = L$.

The point is that you shouldn't begin by writing down what you get from $\lim_{x\to a} f(x) = L$. Only make use of it when it is time.

$\Longleftarrow$: Proving by contraposition means assuming $\lim_{x\to a} f(x) \neq L$ and using this prove the existence of a sequence $\{x_k\}_k$ in D_f such that $x_k \to a$ but $f(x_k) \not\to L$.

This is not hard to do, because the statement $\lim_{x\to a} f(x) \neq L$ allows us to make a pretty strong statement.

Now for the detailed proofs.

Theorem (Sequence characterization and limits)

Let f be a real-valued function on some domain $D_f \subseteq \mathbb{R}$, let a be a cluster point of D_f, and let $L \in \mathbb{R}$. Then the following are equivalent:

(i) $\lim\limits_{x \to a} f(x) = L$;

(ii) For every sequence $\{x_k\}_k$ in $D_f \setminus \{a\}$, if $x_k \to a$ then $f(x_k) \to L$.

Proof $\Longrightarrow$: Suppose $\lim\limits_{x\to a} f(x) = L$. Let $\{x_k\}_k$ be a sequence in $D_f \setminus \{a\}$ converging to a. We must prove $f(x_k) \to L$.

Let $\varepsilon > 0$. Since $\lim\limits_{x \to a} f(x) = L$, there exists $\delta > 0$ such that for all $x \in D_f$,

$$\circledast \quad 0 < |x-a| < \delta \Longrightarrow |f(x) - L| < \varepsilon.$$

Since $x_k \to a$, there exists $K \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, if $k \geq K$ then $|x_k - a| < \delta$. For $k \geq K$, by $\circledast$ we have $|f(x_k) - L| < \varepsilon$. This completes the proof that $f(x_k) - L$.

$\Longleftarrow$: We argue by contraposition. Suppose $\lim\limits_{x \to a} f(x) \neq L$. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists $x \in D_f \setminus \{a\}$ with $0 < |x - a| < \delta$ and $|f(x) - L| \geq \varepsilon$.

For each $n \in \mathbb{N}$, taking $\delta = \frac{1}{n}$, there exists $x_n \in D_f \setminus \{a\}$ with $|x_n - a| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon$. Thus $x_n \to a$ but $f(x_n) \not\to L$. $\square$

The sequential characterization of limits makes it easy to prove the following result.

Theorem. Let f, g be two real-valued functions with domains D_f, D_g. Let $a \in \mathbb{R}$ be a cluster point of $D_f \cap D_g$. Suppose that L and M are numbers such that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then

(i) $\lim_{x \to a} f(x) + g(x) = L + M$,

(ii) $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M$

(iii) $\lim_{x \to a} f(x)/g(x) = L/M$ provided $M \neq 0$ and a is a cluster point of $D_{f/g}$.

Exercise.

Write the proof of the above theorem.

Let's just do the first one.

(i) Proof. Let $\{x_k\}_k$ be a sequence in $D_f \cap D_g \setminus \{a\}$ such that $x_k \to a$. Since $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then by the theorem on slide 6, $f(x_k) \to L$ and $g(x_k) \to M$. Thus by the theorem in section 1.4, $f(x_k) + g(x_k) \to L + M$. Then again by the theorem on slide 6, $\lim_{x \to a} f(x) + g(x) = L + M$. $\square$