1.8 Countability of the Rational Numbers; Countable and Uncountable Sets

- As you likely studied this topic quite a bit in Math 290, I won't develop the topic systematically. We'll state some relevant definitions and theorems, and also do some exercises.

Definitions

- Sets S and T are said to have the **same cardinality** provided there exists a function $f : S \to T$ which is one-to-one and onto.
- A set S is called **finite** if either it is empty or if there exists a natural number n such that S has the same cardinality as $\{1, 2, 3, \ldots, n\}$. If S is not finite, we say it is **infinite**.
- A set S is called **countable** if it has the same cardinality as $\mathbb{N}$. If S is infinite but not countable, it is called **uncountable**.

- For finite sets, the "n" in the definition can be proven to be unique to that set, and is called the **cardinality** of that set. Think of the n as being the number of elements of that set.
- There are infinite sets of different cardinalities. We single out the countable ones as being particularly important in the subject.
- Some books use the term "denumerable" to denote a set having the same cardinality as $\mathbb{N}$, and use the term countable to mean a set that is either finite or denumerable. So different authors use the term in different ways. There is nothing wrong with this, as long as they specify what they mean by the term. Our text doesn't use the term denumerable at all.

Enumeration of a countable set

- A set S is countable if and only if there exists f from $\mathbb{N}$ to S which is one-to-one and onto.
- If for each $n \in \mathbb{N}$ we let s_n denote $f(n)$, then this defines a sequence whose nth term is s_n. This gives the elements of S as an ordered list, each element of S appearing exactly once on the list.
- We refer to the sequence s_n as an **enumeration of the countable set** S.

Exercise.

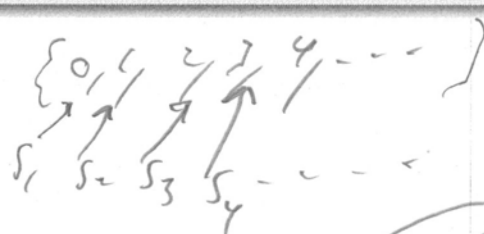
Show by means of a sketch how you would enumerate each of the following sets (and thus deduce they are countable).

a) $\mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \dots\}$

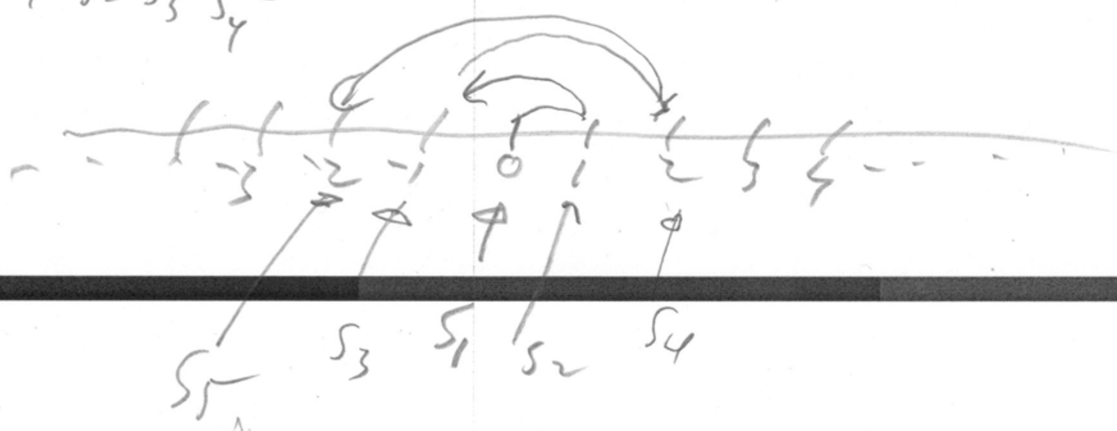
b) The set of integers, $\mathbb{Z}$

c) The set of ordered pairs of nonnegative integers.

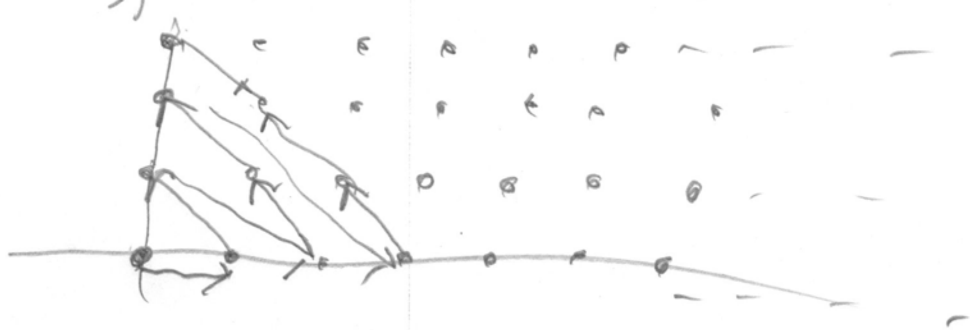
(a)



(b)



(c)



A few theorems on countable sets

Theorem 1. Any infinite subset of a countable set is countable.

Theorem 2. The finite union of countable sets is countable. That is, if $A_1, \ldots, A_n$ is a finite collection of sets each of which is countable, then $A_1 \cup A_2 \cup \cdots \cup A_n$ is countable.

Theorem 3. The countable union of countable sets is countable. That is, if $\{A_n\}_{n=1}^{\infty}$ is a countable collection of sets each of which is countable, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

Exercise

Decide if each of the following sets is countable or uncountable. For each one, justify by making use of a result on an earlier slide or one of the above theorems.

(i) The set of odd integers.

(ii) The set of rational numbers.

(i) countable. Follows from Theorem 1 and the fact that $\mathbb{Z}$ is countable (shown on slide 2)

(ii) • For each $n \in \mathbb{N}$ let $R_n = \{\frac{m}{n} : m \in \mathbb{Z}\}$. Then R_n is countable (since in 1-1 correspondence with $\mathbb{Z}$).
• $\mathbb{Q} = \bigcup_{n=1}^{\infty} R_n$ so $\mathbb{Q}$ is countable by Theorem 3

Exercise.

a) What can you say about the countable union $\bigcup_{n=1}^{\infty} A_n$ of sets such that each A_n is finite?

b) Let n be a fixed natural number. Let S_n be the set of subsets of the set $\{1, 2, 3, \ldots, n\}$. Is S_n finite or infinite? If finite, what is its cardinality?

c) Let S be the set of all finite subsets of the natural numbers $\mathbb{N}$. Is S countable or uncountable? Why?

(a) Finite or infinite.

(b) Finite. Each S_n has 2^n elements.

(c) Note $S = \bigcup_{n=0}^{\infty} S_n$, with S_n as in (b). Then

Obviously S is infinite, so by (a), S is countable.

Theorem. The Cartesian product $A \times B$ of two sets A and B each of which is countable is a countable set.

Exercise.

a) This is a ~~special~~ general case of a result we proved in an earlier slide. Which result is that?

b) What general result could we use in order to deduce the above theorem using that special case?

c) Using the result of the above theorem, how could we deduce that the Cartesian product of 50 countable sets is countable?

d) What is the strongest theorem which can be proven which generalizes the result of part (c)?

(a) on slide 2 we showed the set of ordered pairs of nonnegative integers is countable.

(b) If A has same cardinality as B, and C has same cardinality as D, then $A \times C$ has same cardinality as $B \times D$.

(c) Induction

(d) The Cartesian product of any <u>finite</u> number

of countable sets is countable.

Exercise.

a) Let $\mathscr{P}_1(\mathbb{Q})$ denote the set of polynomials of degree 1 having rational coefficients. What results considered on an earlier slide allows us to deduce that $\mathscr{P}_1(\mathbb{Q})$ is a countable set?

b) Fix any $n \in \mathbb{N}$. Let $\mathscr{P}_n(\mathbb{Q})$ denote the set of polynomials of degree n having rational coefficients. What results considered on an earlier slide allows us to deduce that $\mathscr{P}_n(\mathbb{Q})$ is a countable set?

c) Let $\mathscr{P}(\mathbb{Q})$ denote the set of all polynomials having rational coefficients. What results considered allows us to deduce that $\mathscr{P}(\mathbb{Q})$ is a countable set?

(a) Each $p(x) \in \mathscr{P}_1(\mathbb{Q})$ is of the form $p(x) = a_0 + a_1 x$ for $a_0, a_1 \in \mathbb{Q}$. So $\mathscr{P}_1(\mathbb{Q})$ can be identified with $\mathbb{Q} \times \mathbb{Q}$, so it is countable by previous slide.

(b) Result (d) on previous slide.

(c) $\mathscr{P}(\mathbb{Q}) = \bigcup_{n=0}^{\infty} \mathscr{P}_n(\mathbb{Q})$, so follows from Theorem 3 on slide 3.

Theorem. The set of functions $f : \mathbb{N} \longrightarrow \{0,1\}$ is an uncountable set.

Exercise.

a) Say we wish to prove the above theorem using the method of contradiction. Then we would assume that there exists an enumeration $\{f_1, f_2, f_3, \ldots\}$ of all of the functions from $\mathbb{N}$ into the two-point set $\{0,1\}$. See if you can get a contradiction from this by producing a function $f : \mathbb{N} \to \{0,1\}$ which is not any of the functions in the enumeration.

b) The above argument is very famous. What is the name of that argument and why is it given that name?

(a) Lets create an infinite matrix in which row n lists all the values of f_n:

$$\begin{array}{llllll}
f_1 : & f_1(1) & f_1(2) & f_1(3) & f_1(4) & \cdots \\
f_2 : & f_2(1) & f_2(2) & f_2(3) & f_2(4) & \cdots \\
f_3 : & f_3(1) & f_3(2) & f_3(3) & f_3(4) & \cdots \\
 & & & \vdots & &
\end{array}$$

Look at the diagonal. We create a function $f : \mathbb{N} \to \{0,1\}$ by $f(n) = \begin{cases} 0 & \text{if } f_n(n) = 1 \\ 1 & \text{if } f_n(n) = 0 \end{cases}$

Then $f \neq f_n$ for any n. This contradicts our claim that we had enumerated all functions from $\mathbb{N} \to \{0,1\}$.

(b) It is known as Cantor's <u>diagonalization</u> argument.

Exercise.

For each of the following, give the simplest possible explanation.

a) Is the set of all infinite sequences consisting of $0's$ and $1's$ countable or uncountable? Why?

b) We saw earlier that the set of all finite subsets of $\mathbb{N}$ is a countable set. What about the set of all subsets of $\mathbb{N}$? Is it countable or uncountable? Why?

c) Is the set of all real numbers from 0 to 1, i.e. $(0,1)$, a countable or uncountable set? Why?

d) Is $\mathbb{R}$ countable or uncountable? Why?

(a) Uncountable. Each such sequence is a function $f: \mathbb{N} \to \{0,1\}$, and we showed on slide 7 this is an uncountable set.

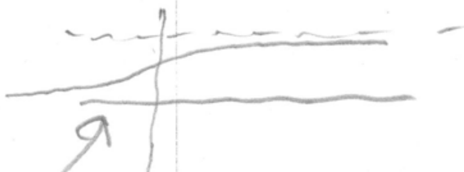
(b) Each subset A of $\mathbb{N}$ can be identified with a unique $f: \mathbb{N} \to \{0,1\}$, namely $f(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \notin A \end{cases}$

So follows from (a) that this set is uncountable.

(c) Each real number can be identified with its

binary expansion, which is a sequence of 0's and 1's. So it is uncountable by (a).

(d) Uncountable.



$\mathbb{R}$ has same cardinality as $(0,1)$.

bijection of $(0,1)$ and $\mathbb{R}$.