1.7 The Heine-Borel Covering Theorem; open sets, compact sets

- This section gives another application of the interval halving method, this time to a particularly famous theorem of analysis, the $\boxed{\textit{Heine – Borel Covering Theorem}}$.
- It also introduces two very important kinds of sets, namely $\boxed{\textit{open sets}}$ and $\boxed{\textit{compact sets}}$.
- The Heine-Borel theorem says that closed bounded intervals $[a, b]$ are examples of compact sets.
- The concept of open set is what is needed in order to define convergence and to formulate the idea of continuity.
- One can formulate the definition of open set in other settings where various notions of convergence are needed:
  - For example, it is formulated in $\mathbb{R}^n$ in order to study multivariable calculus.
  - In a branch of mathematics known as "functional analysis" where we study sets of functions, we're interested in convergence of sequences of functions, so one requires a notion of open set in that setting.
- The compact sets are typically infinite, but they have a property in common with finite sets with very far-reaching applications.

Notation

Let $x \in \mathbb{R}$ and let $r > 0$. The notation $I_r(x)$ refers to the open interval centered at x of radius r, that is

$$I_r(x) := \{y \in \mathbb{R} : |y - x| < r\} = \{y \in \mathbb{R} : -r < y - x < r\} = (x - r, x + r).$$

Exercise.

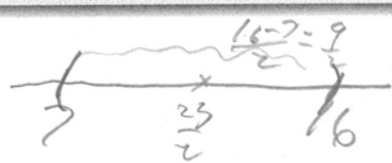
Consider the set $U = \{x \in \mathbb{R} : 7 < x < 16\}$.

a) Sketch this set on a number line.

b) Identify this set using the "I" notation of the above definition.

c) Now identify $I_7(1.3)$ using set-builder notation and then using interval notation. Sketch it on a number line.

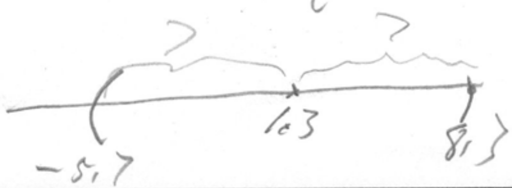
(a)



(b) $U = I_{\frac{9}{2}}(23/2)$

(c) $I_7(1.3) = \{x \in \mathbb{R} : |x - 1.3| < 7\} = \{x : -5.7 < x < 8.3\}$

$= (-5.7, 8.3).$



Definition

Let O be a subset of $\mathbb{R}$. We call O an **open set** if for each x in O there exists an open interval centered at x which is contained in O. Thus O is open provided

$$(\forall x \in O)(\exists r > 0)[I_r(x) \subseteq O].$$

- Note that the r in the above definition will usually depend on the given x.

Exercise.

Let $a, b \in \mathbb{R}$ such that $a < b$.

a) Prove that $\{4\}$ is not an open set.

b) Prove that the closed interval $[a, b]$ is not open.

c) Prove that the open interval (a, b) is an open set.

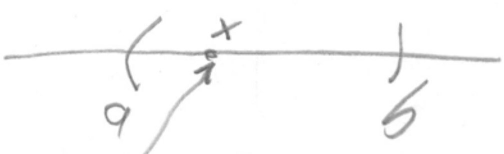
Not open $(\exists x \in O)(\forall r > 0)[I_r(x) \not\subseteq O]$

- A similar argument as the one used in the exercise shows that half open intervals like $(-\infty, b)$ and (a, ∞) are also open sets.

(a) choose $x = 4$. Let $r > 0$. Then $I_r(4) = (4-r, 4+r)$ contains points other than 4 (for example $4 + \frac{r}{2}$) so $I_r(4) \not\subseteq \{4\}$. Thus $\{4\}$ is not open.

(b) we'll do it as a homework.

(c) idea

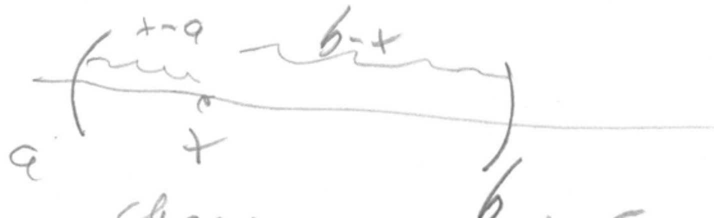


$x \in (a, b)$. Must find $r > 0$ so that $(x-r, x+r) \subseteq (a, b)$.

we need $a < x - r$ and $x + r < b$.

$r < x - a$ and $r < b - x$

Since $x - a > 0$ and $b - x$ are positive we should be able to choose such an r.



choose $r = \min\{x-a, b-x\}$. This should work.

$\mathcal{O} \subseteq \mathbb{R}$ is open if

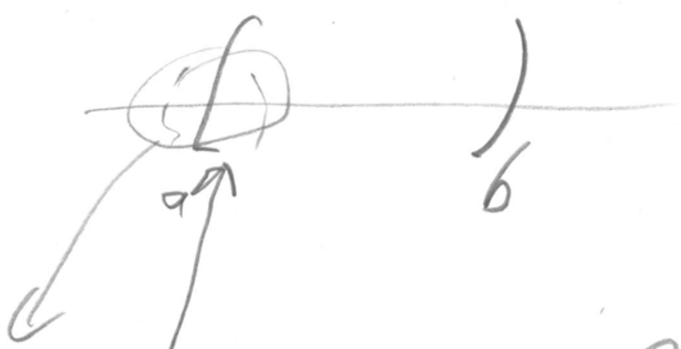
$$(\forall x \in \mathcal{O})(\exists r > 0)(I_r(x) \subseteq \mathcal{O})$$

($\iff$ all points of $\mathcal{O}$ are completely surrounded by pts. of $\mathcal{O}$).

$\mathcal{O}$ is not open if:

$$(\exists x \in \mathcal{O})(\forall r > 0)(I_r(x) \not\subseteq \mathcal{O}).$$

ex



never contained in $[a, b)$.

Exercise.

Define the following collection of sets:

$$U_1 = \{1,3\},\ U_2 = \{1/2, 7, 0\},\ U_3 = \{9, 10, 11\},\ U_4 = \{1, 7, 10, 15\}.$$

Let's view this as an indexed collection of sets $\mathscr{U}$. In the following, make sure to use correct set notation (in this case, listing notation).

a) Identify the indexing set A.

b) Identify the family of sets which we've called $\mathscr{U}$.

c) Identify $\bigcup_{i \in A} U_i$.

d) Identify an element in $\mathscr{U}$, any element will do.

e) Identify any element in $\bigcup_{i \in A} U_i$.

(a) $A = \{1, 2, 3, 4\}$.

(b) $\mathscr{U} = \{\{1,3\}, \{\frac{1}{2}, 7, 0\}, \{9, 10, 11\}, \{1, 7, 10, 15\}\}$

(c) $\bigcup_{i \in A} U_i = \{1, 3, \frac{1}{2}, 7, 0, 9, 10, 11, 15\}$.

(d) $\{1,3\} \in \mathscr{U}$.

(e) $1 \in \bigcup_{i \in A} U_i$

Definition of cover

Let S be a subset of $\mathbb{R}$ and let $\mathscr{U} = \{U_\alpha : \alpha \in A\}$ be an indexed family of sets. We say that **the family of sets is a cover of** S **(or that the family covers** S**)** provided

$$S \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

Exercise.

Look again at the family $U_1 = \{1, 3\}$, $U_2 = \{1/2, 7, 0\}$, $U_3 = \{9, 10, 11\}$, $U_4 = \{1, 7, 10, 15\}$ of the previous exercise.

a) Describe what are all the sets S which are covered by this family $\{U_1, U_2, U_3, U_4\}$.

b) Write down a few different sets S which are covered by this family of sets.

c) Write down a subset of $\mathbb{R}$ which is not covered by this family of sets.

d) Write down a subset S of $\mathbb{R}$ which is covered by $\{U_1, U_2, U_3, U_4\}$, but which is also covered by $\{U_1, U_2, U_3\}$.

(a) Any subset of the union $\{1, 3, \frac{1}{2}, 7, 0, 9, 10, 11, 15\}$ is covered by this family.

(b) $\{1, 3, \frac{1}{2}\}$, $\{7, 10, 15, \frac{1}{2}\}$ are covered

by the family.

(c) $\{1, 3, 2, 16\}$ is not covered by this family.

(d) $\{1, 3, \frac{1}{2}, 10, 11\}$ is covered by $\{U_1, U_2, U_3, U_4\}$ but also by $\{U_1, U_2, U_3\}$.

- In part (d) of the previous exercise, we refer to $\{U_1, U_2, U_3\}$ as being a **subcover** of the cover $\{U_1, U_2, U_3, U_4\}$ because it is a subset of the original cover $\{U_1, U_2, U_3, U_4\}$ and it is also a cover of the set S written down in that exercise.

Definition

Let S be a subset of $\mathbb{R}$ and let $\mathscr{U} = \{U_\alpha : \alpha \in A\}$ be an indexed family of sets which is a cover of S. Let $B \subseteq A$, i.e. B is a subset of the indexing set A. If it is the case that $\mathscr{U} = \{U_\alpha : \alpha \in B\}$ is also a cover of S, then we say that $\mathscr{U} = \{U_\alpha : \alpha \in B\}$ is a **subcover** of the cover $\mathscr{U} = \{U_\alpha : \alpha \in A\}$.

Exercise.

a) Write down a specific cover of $S = \mathbb{R}$ consisting of finitely many open intervals.

b) Write down a specific cover of $\mathbb{R}$ consisting of countably many open intervals such that this cover does not have a subcover consisting of finitely many intervals.

(a) $\{(-\infty, 1), (0, \infty)\}$.

(b) $\{(-1,1), (-2,2), (-3,3), \ldots\} = \{(-n,n) : n \in \mathbb{N}\}$

or

$\{(0,2), (1,3), (2,4), (3,5), \ldots\} \cup \{(-1,1), (-2,0), (-3,-1), \ldots\}$.

Let $S \subseteq \mathbb{R}$. Three obvious examples of covers of S.

① $\{\mathbb{R}\}$: a cover of S consisting of one set.

② $\{S\}$: a cover of S consisting of one set.

$\{\{x\} : x \in S\}$.

If S is an infinite set, this is a covering of S with infinitely many sets. But this particular cover has no finite subcover, i.e. no finite number of these sets cover S.

- In this section, we are particularly interested in
  - open covers of sets
  - the number of elements in the indexing set of that open cover

Definition

For an open cover $\{U_\alpha : \alpha \in A\}$ of a set $S \subseteq \mathbb{R}$, let's call it

- **a finite open cover** of S if A is a finite set;
- **a countable open cover** of S if A is a countable set;
- **an uncountable open cover** of S if A is an uncountable set.

Exercise.

Write down specific open covers of $\mathbb{R}$ of the following types:

a) A finite open cover

b) A countable open cover that does not have a finite subcover

c) A countable open cover that does have a finite subcover

d) An uncountable open cover

(a) $\{(-\infty, 1), (0, \infty)\}$.

(b) The examples in (b) on slide 7.

(c) $\{(-\infty, 1), (0, \infty)\} \cup \{(-n, n) : n \in \mathbb{N}\}$.

This is the finite subcover.

(d) $\{I_1(t) : t \in \mathbb{R}\}$.

Theorem. Let $\{U_\alpha : \alpha \in A\}$ be a nonempty family of open sets (where the indexing set A can have any cardinality whatsoever) . Then $\bigcup_{\alpha \in A} U_\alpha$ is an open set.

Exercise.

Prove the above theorem.

- $x \in \bigcup_{\alpha \in A} U_\alpha \Rightarrow x \in U_{\alpha_0}$ some $\alpha_0 \in A$
- $\exists r$ s.t. $I_r(x) \subseteq U_{\alpha_0}$ (since U_{α_0} open)
- So the same r should work for $\bigcup_{\alpha \in A} U_\alpha$.

Theorem Let $\{U_\alpha : \alpha \in A\}$ be a nonempty family of open sets, where the indexing set A can have any cardinality whatsoever. Then $\bigcup_{\alpha \in A} U_\alpha$ is an open set.

Proof. Let $x \in \bigcup_{\alpha \in A} U_\alpha$. Then there exists $\alpha_0 \in A$ such that $x \in U_{\alpha_0}$. Since U_{α_0} is open, there exists $r > 0$ such that $I_r(x) \subseteq U_{\alpha_0}$. Since $U_{\alpha_0} \subseteq \bigcup_{\alpha \in A} U_\alpha$, it follows $I_r(x) \subseteq \bigcup_{\alpha \in A} U_\alpha$. Thus $\bigcup_{\alpha \in A} U_\alpha$ is open. $\square$

Theorem. A nonempty subset U of $\mathbb{R}$ is open if and only if it is a union of a family of nonempty open intervals.

Exercise.

Prove the above theorem.

$\Leftarrow$: we've seen this.

$\Rightarrow$: Let U be open. Let $x \in U$.

$\subseteq U$.

$\forall x \in U \; \exists r_x > 0$ s.t. $I_{r_x}(x) \subseteq U$.

$$U = \bigcup_{x \in U} I_{r_x}(x)$$

Theorem (Open set Characterization)

A nonempty subset U of $\mathbb{R}$ is open if and only if it can written as a union of open intervals.

Proof

$\Leftarrow$ Suppose U can be written as a union of open intervals. We proved earlier that any open interval is an open set and any union of open sets is an open set. Thus U is open.

$\Rightarrow$ Let U be a nonempty open set. For each $x \in U$ there exists $r_x > 0$ such that $I_{r_x}(x) \subseteq U$. Then

$$① \quad \bigcup_{x \in U} I_{r_x}(x) \subseteq U.$$

To prove the reverse inclusion, let $y \in U$. Since $y \in I_{r_y}(y)$, it follows $y \in \bigcup_{x \in U} I_{r_x}(x)$. This proves

$$② \quad U \subseteq \bigcup_{x \in U} I_{r_x}(x).$$

From ① and ②, $U = \bigcup_{x \in U} I_{r_x}(x)$, and so we have shown U can be written as a union of open intervals.

$\square$

Let's review the definition of open cover of a set and finite subcover of an open cover of a set:

Open cover of a set

Let S be any subset of $\mathbb{R}$. An **open cover** of S is a family of sets U_α indexed by some set A such that the following hold:

(i) U_α is open for each $\alpha \in A$;

(ii) $S \subseteq \bigcup_{\alpha \in A} U_\alpha$.

Finite subcover of an open cover of a set

Let S be any subset of $\mathbb{R}$ and let $\{U_\alpha : \alpha \in A\}$ be an open cover of S. We say that this open cover **has a finite subcover** if there exists a set B such that the following two things hold:

- B is a finite subset of A;
- $\{U_\alpha : \alpha \in B\}$ is a cover of S.

Exercise.

From an "economics" point of view, explain in words what is the benefit of a given open cover $\{U_\alpha : \alpha \in A\}$ of a set S having a finite subcover?

For a given family of sets $\{U_\alpha : \alpha \in A\}$ which covers S, it makes sense to cover S with as few of the sets of the given cover as possible.

Exercise.

Let S be a nonempty set with the property that every cover of S has a finite subcover.

a) Does the set $\{1, 2, 3, \ldots, 1000\}$ have this property?

b) Does the set $[0, 1]$ have this property?

c) What kind of set must S be if it has the above property?

(a) Yes $\{1,2,3,\ldots,1000\}$ has only 1000 elements, any cover of it must have a subcover containing at most 1000 of the sets of that cover.

(b) No. For example $\{\{x\} : x \in [0,1]\}$ is a cover of $[0,1]$. But each element of the cover contains only one element, so any finite subcover could cover at most finitely many points of $[0,1]$, so not all of $[0,1]$.

(c) The only sets S with the property that every cover has a finite subcover are the finite sets.

Exercise.

a) Prove that $\{1, 2, 3\}$ is compact.

b) Prove that $\mathbb{R}$ is not compact.

c) Prove that $(0, 1)$ is not compact.

- A similar argument as in (a) above shows that any finite set is compact.
- After doing the above exercise, you might wonder if there exist any infinite subsets of $\mathbb{R}$ which are compact.
- The point of the Heine-Borel theorem is that it shows that there are lots of infinite sets which are compact.

~~S compact, Every open cover of S has a finite subcover.~~

Not compact There exists an cover of S with no finite subcover

(b) $\{(-n, n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$ since $\bigcup_{n \in \mathbb{N}} (-n, n) = \mathbb{R}$ and

each $(-n, n)$ is open. However any finite number of these sets $(-n_1, n_1)$, $(-n_2, n_2)$, ... $(-n_k, n_k)$ cannot cover $\mathbb{R}$ since the union is $(-n, n)$ where $n = \max\{n_1, \dots, n_k\}$.