Theorem (Nested Intervals Theorem)

Let $I_n = [a_n, b_n]$ be a sequence of closed intervals satisfying each of the following conditions:

(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots,$

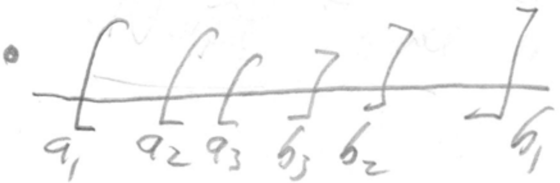
(ii) $b_n - a_n \to 0$ as $n \to \infty$.

Then $\bigcap_{n=1}^{\infty} I_n$ consists of exactly one real number x. Moreover both sequences a_n and b_n converge to x.

Exercise.

Use the comments to write the proof of the Nested Intervals Theorem.

<u>Motivation</u>



The picture suggests the sequence $\{a_n\}$ of left endpoints is increasing and upper bounded (b_1 is an upper bound).

$\therefore$ a_n converges to some $x \in \mathbb{R}$.

- $b_n - a_n \to 0$ tells us also $b_n \to x$
- Since $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, we know $I_m \subseteq I_n$ if $m \geq n$

$\therefore$ $a_n \leq a_m \leq b_n$ for $m \geq n$.

So letting $m \to \infty$ and using homework 2

$$a_n \leq \lim_{m \to \infty} a_m \leq b_n$$

i.e. $a_n \le x \le b_n$, so $x \in I_n$.

This is true for all n, so $x \in \bigcap_{n=1}^{\infty} I_n$.

- If x and $y \in I_n$ for all n,

then $|x-y| \le b_n - a_n$ for all n.

But $b_n - a_n \to 0$, so $\therefore x = y$.

Theorem (Nested Intervals Theorem)

Let $\{I_n\}_n = \{[a_n, b_n]\}_n$ be a sequence of closed, bounded intervals satisfying each of the following conditions:

(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$

(ii) $b_n - a_n \to 0$.

Then $\bigcap_{n=1}^{\infty} I_n$ is nonempty and consists of exactly one real number x. Moreover $a_n \to x$ and $b_n \to x$.

Proof Since for each n we have $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$, it follows $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$. Thus $\{a_n\}_n$ is increasing and $\{b_n\}_n$ is decreasing. In addition it follows from this that $a_1 \leq b_n$ and $a_n \leq b_1$ for all n. Thus $\{a_n\}_n$ is increasing and upper bounded, and $\{b_n\}_n$ is decreasing and lower bounded. Thus, by the Completeness Axiom of $\mathbb{R}$, both $\{a_n\}_n$ and $\{b_n\}_n$ converge. Since $b_n - a_n \to 0$, they both converge to the same number, x.

We show next that $x \in \bigcap_{n=1}^{\infty} I_n$. Let $n \in \mathbb{N}$. For all $m \geq n$ we have $a_m \in I_m \subseteq I_n$. So

$$a_n \leq a_m \leq b_n.$$

Letting $m \to \infty$ in this inequality, we deduce

$$a_n \leq x \leq b_n.$$

Thus $x \in I_n$. Since this holds for all $n \in \mathbb{N}$, by definition $x \in \bigcap_{n=1}^{\infty} I_n$.

Finally, we show there cannot be more than one point in $\bigcap_{n=1}^{\infty} I_n$. Say $x, y \in \bigcap_{n=1}^{\infty} I_n$. Then for all n, x and y are in I_n, so for every n,

$$|x-y| \leq b_n - a_n.$$

Letting $n \to \infty$, we get $|x-y| \leq 0$, so $x = y$. $\square$

Before we give a second proof of the exercise, we first prove a theorem which establishes an equivalent formulation that a set S is dense in $\mathbb{R}$.

Theorem

Let S be a subset of $\mathbb{R}$. Then S is dense in $\mathbb{R}$ if and only if every open interval contains a point of S.

Comments on the proof of the theorem

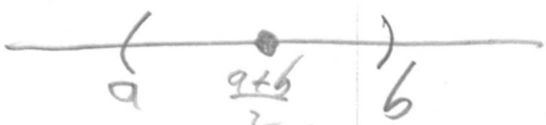
- You have to prove two things here:
  - (i) Assuming that S is dense in $\mathbb{R}$, you have to prove that any open interval contains at least one point of S.
  - (ii) Assuming that S has the property that every open interval contains at least one point of S, you have to prove that S is dense in $\mathbb{R}$.
- (i) Give yourself an open interval I. Let x be any point of I. Assuming that $x \notin S$, how can you use the density of S to deduce there are lots of points of S in I?
- (ii) Let x be any point of $\mathbb{R}$. Assuming that $x \notin S$, how can you use the fact that every open interval contains a point of S to deduce that there is a sequence of points of S converging to S?

Exercise.

Use the above comments to write a proof of this theorem.

<u>Motivation for the proof of this theorem</u>

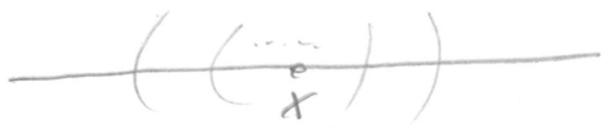
$\Longrightarrow$: Assume S dense. Let (a,b) be any interval



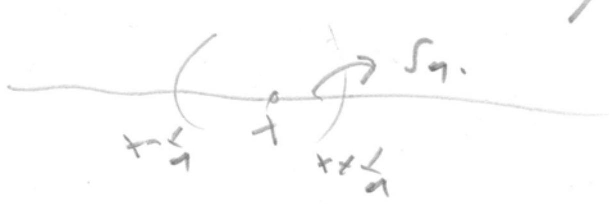
- we must find a point of S in this interval.
- By assumption there is a sequence in S converging to $\frac{a+b}{2}$, so we should be able

to find points so close to $\frac{a+b}{2}$ that they find themselves in the interval (a, b).

$\Longleftarrow$: Assume all open intervals contain points of S. Let $x \in \mathbb{R}$.



- Need to look at a sequence of open intervals shrinking down to x. We can pick a point of S from each one in order to construct a sequence in S converging to x.



Definition. A subset S of $\mathbb{R}$ is said to be <u>dense</u> if for every real number x, there exists a sequence $\{s_n\}_n$ in S such that $s_n \to x$.

Theorem (Characterization of denseness) A subset S of $\mathbb{R}$ is dense if and only if every open interval contains a point of S.

Proof $\Rightarrow$ Let S be a dense subset of $\mathbb{R}$. Let (a,b) be an open interval. Let $y = \frac{a+b}{2}$ be the midpoint of (a,b). Since S is dense, there exists a sequence $\{s_n\}_n$ in S such that $s_n \to y$. Choose $\varepsilon = \frac{b-a}{2}$. Then there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$ then $|s_n - y| < \varepsilon$. In particular,

$$-\frac{(b-a)}{2} < s_N - \left(\frac{a+b}{2}\right) < \frac{b-a}{2}.$$

This says $a < s_N < b$, i.e. $s_N \in (a,b)$.

Conversely, suppose that S has the property that every open interval contains a point of S. Let $x \in \mathbb{R}$. Consider the sequence of open intervals $\{I_n\}_n$, $I_n = (x - \frac{1}{n}, x + \frac{1}{n})$. By assumption, for each n, there exists $s_n \in S \cap I_n$. Since $|s_n - x| < \frac{1}{n}$, it follows $s_n \to x$. $\square$

Now let's return to the exercise and give a second proof using this alternate formulation of density.

Theorem. $\mathbb{Q}$ is dense in $\mathbb{R}$.

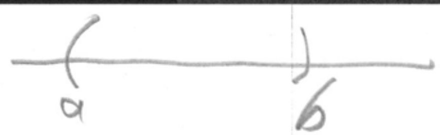
Comments on the second proof of the exercise.

- For this proof we have to show that every open interval (a, b) contains a rational number m/n.
- The m and the n have a different role to play. Think of n as determining a "grid size" $1/n$, so that the various multiples of m/n allow us to partition $\mathbb{R}$ into closely spaced rationals (the spacing determined by how large is n).
- The closer is a and b, the more challenging it is to find a rational in between, and so the smaller the required grid size.
- A measure of the closeness of a and b is the quantity $\varepsilon := b - a$.
- This suggests that the grid size should be less than ε, so we'd like to choose $n \in \mathbb{N}$ such that $0 < 1/n < \varepsilon$. How do we know we can do that?
- Notice that since $0 < 1/n < b - a$ then $nb - na > 1$.
- Since the spacing between nb and na is more than 1, what does that allow us to do?

Exercise.

Use the above comments to write a second proof that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Motivation



- Show any interval (a,b) contains a rational $\frac{m}{n}$.
- The n determines the grid size. Make sure the grid is fine enough to get inside (a,b).

- So choose n big enough that $\frac{1}{n} < b-a$.

- Then we expect one of these grid points to get inside (a,b).

- To show it, note $nb - na > 1$ (from $\frac{1}{n} < b-a$)

So there must be an integer m between na and nb:

So $na < m < nb$

$$\Rightarrow a < \frac{m}{n} < b.$$

Theorem $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let (a,b) be an open interval. We will be done if we can prove there exists a rational number in (a,b).

Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < b-a$. Then $nb - na > 1$. Thus nb and na are real numbers which differ by more than 1, so there must exist an integer m such

$$na < m < nb.$$

Dividing through by n, we get

$$a < \frac{m}{n} < b,$$

i.e. $\frac{m}{n}$ is a rational number in (a,b). $\square$