1.6 The Nested Intervals Theorem

- We've seen several applications of the interval halving method.
- Starting with a closed interval $I_1 = [a_1, b_1]$, the method produces a sequence of closed intervals $I_n = [a_n, b_n]$, such that for each $n \geq 1$, I_n is either the left half or the right half of I_{n-1}. If we randomly pick a point $x_n \in I_n$ for each n, then the resulting sequence is necessarily a Cauchy sequence.
- Thus by the completeness axiom of $\mathbb{R}$, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to some real number x.

Exercise.

Prove that $x \in \bigcap_{n=1}^{\infty} I_n$.

<u>Proof</u> We must show that for all $n \in \mathbb{N}$, $x \in I_n$.
So let $n \in I_1$. Then for all $m \geq n$,

$$x_m \in I_m \subseteq I_n,$$

So $a_n \leq x_m \leq b_n$ for all $m \geq n$. Letting $m \to \infty$ we deduce $a_n \leq x \leq b_n$. This means

$x \in I_n$. Since this true for all n, $x \in \bigcap_{n=1}^{\infty} I_n$.

$\square$

Theorem (Nested Intervals Theorem)

Let $I_n = [a_n, b_n]$ be a sequence of closed intervals satisfying each of the following conditions:

(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$,

(ii) $b_n - a_n \to 0$ as $n \to \infty$.

Then $\bigcap_{n=1}^{\infty} I_n$ consists of exactly one real number x. Moreover both sequences a_n and b_n converge to x.

Exercise.

Use the comments to write the proof of the Nested Intervals Theorem.

<u>Next Time</u>

Exercise

a) The theorem is false if we replace the condition that the intervals I_n be closed by the condition that they be open. Show a counterexample which shows that the intersection can be chosen to be empty.

b) The theorem is false if we omit the condition that $b_n - a_n \to 0$. Give a counterexample.

(a) $I_n = (0, \frac{1}{n})$. Then $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$

and $|I_n| \to 0$, yet $\bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \phi$

(since there is no x such that $0 < x < \frac{1}{n}$ for all n.

(b) $I_n = [-2 - \frac{1}{n}, 2 + \frac{1}{n}]$.

Then $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, but $\bigcap_{n=1}^{\infty} I_n = [-2, 2]$

$\bigcap_{n=1}^{\infty} I_n =$

lots of pts.