1.5 The Bolzano-Weierstrass Theorem

Example 1

Let $\{x_n\}_{n=1}^{\infty}$ be the sequence $1, -2, 3, -4, 5, -6, 7, -8, 9, -10, \ldots$ so it is given by $x_n = (-1)^{n+1} n$.

Example 2

Let $\{x_n\}_{n=1}^{\infty}$ be the sequence $1, 2, 3, 1, 5, 6, 1, 8, 9, 1, 11, 12, 1, 14, 15, 1, 17, 18, 1, \ldots$

- Roughly speaking, a **subsequence** of a sequence $\{x_n\}_{n=1}^{\infty}$ is obtained by choosing infinitely many values from the given sequence, where each successive choice is made by taking a larger index (i.e. looking farther out in the sequence).

Exercise.

a) Give a few reasons you know that the sequences in Example 1 and Example 2 are divergent.

b) Write down a few examples of subsequences of the sequence in Example 1.

c) Write down a few examples of subsequence of the sequence in Example 2.

d) Do either of the sequences in Example 1 or Example 2 have the property that it has a convergent subsequence?

(a) They are both not Cauchy and both unbounded

(b) $\{1, 3, 5, 7, 9, \ldots\}$, $\{-2, -4, -6, -8, -10, \ldots\}$

$\{5, 9, 13, 17, \ldots\}$

(c) $\{1, 1, 1, 1, \ldots\}$, $\{2, 5, 8, 11, 14, \ldots\}$

(d) Not ex. 1, but ex 2 has $\{1, 1, 1, 1, \ldots\}$.

Example 3.

Let $\{x_n\}_{n=1}^{\infty}$ be the sequence $x_n = \cos n$.

Exercise.

a) Write out the first several terms of the sequence in Example 3.

b) Can you tell whether or not the sequence converges?

c) Can you tell whether or not the sequence has a convergent subsequence?

(a) $\cos 1 \approx 0.54,$

$\cos 2 \approx -0.42$

$\cos 3 \approx -0.99$

$\cos 4 \approx -0.65$

$\cos 5 \approx 0.281$

$\cos 6 \approx 0.96$

$\vdots$

(b) Seems to diverge, but unclear how to prove it.

(c) Not obvious.

Example 4.

Let's define a sequence $\{x_n\}_{n=1}^{\infty}$ as follows: For each $n \in \mathbb{N}$, let x_n be any number at all between 0 and 100.

Exercise.

a) Is the sequence in Example 4 convergent?

b) If we try to answer the question

"does the sequence in Example 4 have a convergent subsequence?",

why does this question appear to be harder to answer than the same question with Example 1 or with Example 2?

(a) You cannot decide.

(b) Not clear how to answer.

Before we state the theorem, let's first give a formal definition of subsequence of a sequence.

Definition

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. Let $n_1 < n_2 < n_3 < n_4 < \dots$ be a strictly increasing sequence of natural numbers. Let $\{y_k\}_{n=1}^{\infty}$ be the sequence defined by $y_k := x_{n_k}$. Then the sequence y_k is called a **subsequence** of the sequence x_n.

Alternate Definition

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. We say that sequence $\{y_n\}_{n=1}^{\infty}$ is a **subsequence** of $\{x_n\}_{n=1}^{\infty}$ if there exists a function $\phi : \mathbb{N} \to \mathbb{N}$ which is strictly increasing such that $y_n = x_{\phi(n)}$ for every n.

Exercise.

Consider the sequence of Example 2: $1, 2, 3, 1, 5, 6, 1, 8, 9, 1, 11, 12, 1, 14, 15, 1, 17, 18, 1, \dots$.

a) Using the alternate definition, what subsequence do we get if we take $\phi(n) = 2n$?

b) What choice of $\phi(n)$ gives the subsequence 1, 1, 1, 1, ...?

(a) $\varphi(n) = 2n$.

$\{x_{\varphi(n)}\}_{n=1}^{\infty} = \{x_2, x_4, x_6, x_8, \dots\}$

$= \{2, 1, 6, 8, 1, 12, 14, 1, \dots\}$.

(b) $\varphi(n) = 3n-2$

Alternate Definition

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. We say that sequence $\{y_n\}_{n=1}^{\infty}$ is a **subsequence** of $\{x_n\}_{n=1}^{\infty}$ if there exists a function $\phi : \mathbb{N} \to \mathbb{N}$ which is strictly increasing such that $y_n = x_{\phi(n)}$ for every n.

Exercise.

Suppose $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ are three sequences for which $\{y_n\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ is a subsequence of $\{y_n\}_{n=1}^{\infty}$. Prove that $\{z_n\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$

We're given there exist $\varphi_1 : \mathbb{N} \to \mathbb{N}$ and $\varphi_2 : \mathbb{N} \to \mathbb{N}$ both strictly increasing such that for each n

$$y_n = x_{\varphi_1(n)} \text{ and } z_n = y_{\varphi_2(n)}$$

Thus

$$z_n = x_{\varphi_2 \circ \varphi_1(n)}.$$

Result follows from fact that the

composition of strictly increasing functions is strictly increasing.

$\square$

Is it possible that boundedness or unboundedness has something to do with whether or not a sequence has a convergent subsequence?

Exercise.

1 Write down two sequences with the following properties:

- Both sequences consist entirely of positive numbers.
- Both sequences are unbounded.
- The first sequence does not have a convergent subsequence, but the second sequence does have a convergent subsequence.

Exercise.

2 So what does the above exercise tell you about what the Bolzano-Weierstrass theorem can definitely not be?

① $\{10, 1, 10, 2, 10, 3, 10, 4, 10, 5, \ldots\}$

$\{10, 11, 12, 13, 14, \ldots\}$

② Probably require boundedness of the sequence as a hypothesis.

Exercise.

a) Recall that Example 4 had a sequence $\{x_n\}_{n=1}^{\infty}$ consisting of randomly chosen numbers between 0 and 100. Why is it plausible that this sequence has a convergent subsequence?

b) This example suggests a conjecture as to what might be a sufficient condition to guarantee that a sequence must have a convergent subsequence. What is that condition?

<u>Thm</u> Let $\{x_n\}$ be a sequence. If it has a bounded subsequence, then it has a convergent subsequence.

Theorem (Bolzano-Weierstrass)

Let $\{x_n\}_{n=1}^{\infty}$ be any bounded sequence. Then $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Comments on the proof

- It is sufficient to show that the sequence has a Cauchy subsequence. The result will then follow from the completeness axiom of $\mathbb{R}$.
- This is done using the method of interval halving. We show how to construct a nested decreasing sequence of intervals $I_n = [a_n, b_n]$ such that $(b_n - a_n) \to 0$ and a subsequence y_n of the original sequence such that $y_n \in I_n$ for each n. Then y_n is automatically Cauchy.
- Let $I_1 = [a_1, b_1]$ be any closed interval which contains the entire sequence. This is possible since the sequence is bounded. Let $n_1 := 1$ and $y_1 := x_{n_1} = x_1$.
- The key idea which makes the proof work is the fact that

  either the left half or the right half of I_1 must contain infinitely many terms of the sequence,

  so we let I_2 be the half which does contain infinitely terms of the sequence, and we let y_2 be a specific term x_{n_2} of the sequence which is in I_2.
- After that, we apply the same idea on I_2 to produce an interval I_3 and a number y_3. The rest of the intervals and points are obtained in a similar manner.

Theorem (Bolzano-Weierstrass)

Let x_n be any bounded sequence. Then x_n has a convergent subsequence.

Exercise.

Use the comments on the previous slide to write the proof of the Bolzano-Weierstrass Theorem

Theorem (Bolzano–Weierstrass) Let $\{x_n\}_n$ be any bounded sequence. Then $\{x_n\}_n$ has a convergent subsequence.

Proof We will inductively define a sequence $\{I_n\}_n$ of closed, bounded intervals and a sequence $\{N_n\}_n$ of natural numbers with the following properties:

1. For each n, I_{n+1} is either the left half or the right half of I_n;
2. For each n there are infinitely many k such that $x_k \in I_n$;
3. $\{N_n\}_n$ is a strictly increasing sequence of natural numbers;
4. For every n, $x_{N_n} \in I_n$.

Basis step: Since $\{x_n\}_n$ is a bounded sequence, there exists $M \in \mathbb{N}$ such that for all n, $|x_n| \leq M$. Thus the sequence $\{x_n\}_n$ lies in $[-M, M]$, so we define $I_1 = [-M, M]$ and $N_1 = 1$.

Inductive step: Let $n \geq 1$, and suppose we have selected $I_1, I_2, \ldots, I_n$ and $N_1, N_2, \ldots, N_n$ with the above properties.

Since I_n contains x_k for infinitely many indices k, either the left half or the right half of I_n must also contain x_k for infinitely many indices k (since the union of two finite sets is finite). Then define

$$I_{n+1} = \begin{cases} \text{left half of } I_n & \text{if } x_k \text{ is in the left half of } I_n \text{ for infinitely many } k \\ \text{right half of } I_n & \text{otherwise} \end{cases}$$

Since $x_k \in I_{n+1}$ for infinitely many indices k, there must be such an index k which is strictly greater than N_n. Let N_{n+1} be any such index.

This completes the proof of the induction.

Since $x_{N_n} \in I_n$ for each n, it follows immediately from the Interval Halving Theorem that $\{x_{N_n}\}_n$ is a Cauchy sequence, and so by the Completeness Axiom it converges. $\square$

Example

Revisit Examples 3 and 4. Explain how we know that those sequences must have convergent subsequences.

Each is bounded, so by the Weierstrass-Bolzano Theorem, each has a convergent subsequence.

Exercise.

Formulate a theorem like the Bolzano-Weierstrass Theorem which applies to all sequences, including sequences which are unbounded. Thus it should read

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that *(you fill in the blank)*.

Then $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Thm Any sequence with a bounded subsequence must have a convergent subsequence.

Pf By Weierstrass-Bolzano, the bounded subsequence must have a convergent subsequence. The result then follows from the result that for a sequence $\{x_n\}_n$ a subsequence of a subsequence of $\{x_n\}_n$ is itself a subsequence of $\{x_n\}_n$. $\square$