Theorem.

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences, L and M real numbers, for which $x_n \to L$ and $y_n \to M$. Then the following hold:

(iii) $x_n\, y_n \to L\, M$;

Comments on proof of (iii)

- We must show that if we can make $|x_n - L|$ small for all sufficiently large n, and we can make $|y_n - M|$ small for sufficiently large n, then we can make $|x_n\, y_n - L\, M|$ suitably small for all sufficiently large n.
- The trick is to add and subtract the right thing so that after simplifying we get a sum of terms each of which we can force to be suitably small for all large enough n.
- Try adding and subtracting $y_n L$ under the absolute value bars. This gives $|x_n\, y_n - L\, M| = |x_n y_n - y_n L + y_n L - LM| = |y_n(x_n - L) + L(y_n - M)|$.
- Now apply the triangle inequality to get $|x_n\, y_n - L\, M| \le |y_n|\, |x_n - L| + |L|\, |y_n - M|$.
- How do you know that you can force the term $|y_n|\, |x_n - L|$ to be suitably small? What property of the convergent sequence y_n should you make use of to do it?

Exercise.

Write the proof of (iii).

• Harder than the previous proofs.

• $|x_n y_n - LM| = |x_n y_n - y_n L + y_n L - LM|$

$\le |y_n|\,|x_n - L| + |L|\,|y_n - L|$

Harder to control. use that $\{y_n\}_n$ is <u>bounded</u> and $x_n \to L$

easy to control using $y_n \to L$.

Theorem Let $\{x_n\}_n$ and $\{y_n\}_n$ be sequences, L and M real numbers for which $x_n \to L$ and $y_n \to M$. Then $x_n y_n \to LM$.

Proof We only consider the case with $L \neq 0$ and $M \neq 0$. Let $\varepsilon > 0$.

Since $\{y_n\}_n$ is convergent, it is bounded, so there exists $A \in \mathbb{R}$ such that $|y_n| < A$ for all $n \in \mathbb{N}$.

Since $y_n \to M$, there exists $N_1 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n \geq N_1$ then

$$|y_n - M| < \frac{\varepsilon}{2A}.$$

Since $x_n \to L$, there exists $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_2$ then

$$|x_n - L| < \frac{\varepsilon}{2|L|}.$$

Choose $N = \max\{N_1, N_2\}$. Let $n \in \mathbb{N}$ with $n \geq N$. Then

$$\begin{aligned} |x_n y_n - LM| &= |x_n y_n - y_n L + y_n L - LM| = |y_n(x_n - L) + L(y_n - M)| \\ &\leq |y_n|\,|x_n - L| + |L|\,|y_n - M| \\ &\leq A \cdot \frac{\varepsilon}{2A} + |L| \cdot \frac{\varepsilon}{2|L|} = \varepsilon, \end{aligned}$$

completing the proof. $\square$

Theorem.

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences, L and M real numbers, for which $x_n \to L$ and $y_n \to M$. Then

(iv) If $L \neq 0$, then $x_n \neq 0$ for sufficiently large n, and $\dfrac{1}{x_n} \to \dfrac{1}{L}$;

Comments on proof of (iv)

- We first need to ensure that knowing $L \neq 0$ is enough to deduce $x_n \neq 0$ for all n after a while.
- So this means proving that $|x_n|$ is bounded away from 0 for all n after a while.
- Since $L \neq 0$ and the terms of x_n get close to L for all n sufficiently large, it must be possible to prove $x_n \neq 0$ for n large. But how to prove it?
- Try to do it using the reverse triangle inequality.
- Next must show that if we can make $|x_n - L|$ small for all sufficiently large n, then we can make $|1/x_n - 1/L|$ suitably small for all sufficiently large n.
- Rewriting we get $|1/x_n - 1/L| = \left|\dfrac{x_n - L}{x_n L}\right|$.
- Use the facts that $|x_n|$ is bounded away from 0 and that we can make $|x_n - L|$ as small as we wish to show that for all sufficiently large n we can make $\left|\dfrac{x_n - L}{x_n L}\right|$ suitably small for all n after a while.

Exercise.

Write the proof of (iv).

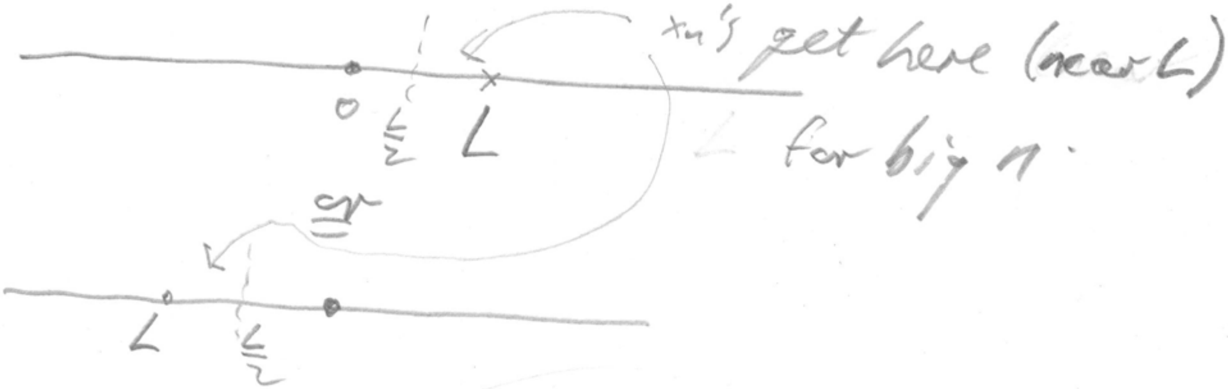
Notes: $\left|\dfrac{1}{x_n} - \dfrac{1}{L}\right| = \left|\dfrac{x_n - L}{x_n L}\right|$ → Need to show this is small for all n big enough.

To make this small we need to (i) show we can bound $|x_n L|$ away from 0 (since $|x_n L|$ near 0 makes $\dfrac{1}{|x_n L|}$ very big.

and (ii) make $|x_n - L|$ small for n big.

- To do (i) and (ii) we can use the facts that (a) $x_n \to L$ and (b) $L \neq 0$.

- To do (i) above, the idea is that since $x_n \to L$ and $L \neq 0$, we should be able to prove $|x_n| \geq \frac{|L|}{2}$ if n big enough:



- In order to achieve this we reverse triangle inequality:

$$|x_n| = |x_n - L + L| \geq |L| - |x_n - L| \geq \frac{|L|}{2}$$

provided $|x_n - L| < \frac{|L|}{2}$ for n big

we can achieve this if we take $\varepsilon = \frac{|L|}{2}$ in definition of $x_n \to L$.

Theorem Let $\{x_n\}_n$ be a sequence, $L \in \mathbb{R}$ such $x_n \to L$ and $L \neq 0$. Then $\frac{1}{x_n} \to \frac{1}{L}$.

Proof Let $\varepsilon > 0$. Since $x_n \to L$ and $L \neq 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_1$ then

$$|x_n - L| < \frac{|L|}{2}.$$

For any $n \geq N_1$,

$$\frac{|L|}{2} > |x_n - L| \geq |L| - |x_n|$$

and so

$$|x_n| > |L| - \frac{|L|}{2} = \frac{|L|}{2}.$$

Since $x_n \to L$, we can choose $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_2$ then

$$|x_n - L| < \frac{L^2 \varepsilon}{2}.$$

Choose $N = \max\{N_1, N_2\}$. Let $n \in \mathbb{N}$. Suppose $n \geq N$. Then

$$\left|\frac{1}{x_n} - \frac{1}{L}\right| = \frac{|x_n - L|}{|x_n L|} < \frac{\frac{L^2 \varepsilon}{2}}{\frac{|L|}{2} \cdot |L|} = \varepsilon. \quad \square$$

Theorem.

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences, L and M real numbers, for which $x_n \to L$ and $y_n \to M$. Then

(v) If $M \neq 0$, then $\dfrac{x_n}{y_n} \to \dfrac{L}{M}$.

Comments on proof of (v)

- This part of the theorem is the most complex one of the theorem.
- However, do you see that now that we have proved the other parts of the theorem we can prove part (v) very easily?

Exercise.

Write the proof of (v).

Proof By part (iv) $\frac{1}{y_n} \to \frac{1}{M}$, so by part (iii), $x_n \cdot \frac{1}{y_n} \to L \cdot \frac{1}{M}$, i.e. $\frac{x_n}{y_n} \to \frac{L}{M}$. $\square$

Exercise

Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ be two sequences such that $x_n \to 0$, but we specify nothing more about the sequence y_n.

a) It is not necessarily true that $x_n\, y_n \to 0$. Intuitively why don't you believe that in general $x_n\, y_n$ has to converge to 0?

b) Give a few specific counterexamples to illustrate what can go wrong.

c) What additional property could you assign to the sequence y_n so that one can prove that $x_n\, y_n \to 0$? Try to make your property as weak a condition on y_n as you can.

(a) The y_n can be so badly behaved that $x_n y_n$ does not converge.

(b) $\frac{1}{n} \to 0$, but $n^2 \cdot \frac{1}{n} = n$ does not cvge.

$\frac{1}{n} \to 0$, but $\frac{1}{n} \cdot 2n = 2 \to 2$

(c) We expect if $x_n \to 0$ and $\{y_n\}$ bounded, then $x_n y_n \to 0$

- A useful result is

Theorem. If $x_n \to L$ and $y_n \to M$ where L and M are real numbers, and if in addition $x_n < y_n$ for all n, then $L \leq M$.

- A special case of this is when $y_n = 0$ for all n:

  if $x_n \to L$ for some real number L and $x_n < 0$ for all n, then $L \leq 0$.

- We proved this last result earlier in the course.

Exercise

Write a proof of the theorem which makes use of the above result. Make your proof as simple as possible, but show all relevant details.

ex $1-\frac{1}{n} < 1$ all n, yet $\lim_{n\to\infty}(1-\frac{1}{n}) = 1$, not <1

Thm If $x_n \to L$, $y_n \to M$ and $x_n < y_n$ for all n, then $L \leq M$.

Lemma If $z_n < 0$ for all n and $z_n \to z$, then $z \leq 0$.

Assuming the lemma,

Pf of the Thm Let $z_n = x_n - y_n$. By parts (i) and (ii) (page 1), $z_n \to L - M$. By the lemma $L - M \leq 0$, i.e. $L \leq M$.