1.4 Algebraic Combinations of Sequences

- Given that we have convergence of some sequences, we study here what we can prove about the convergence of various algebraic combinations of those sequences.

Theorem.

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences, L and M real numbers, for which $x_n \to L$ and $y_n \to M$. Then

(i) $c\, x_n \to c\, L$, where c is any constant;

(ii) $x_n + y_n \to L + M$;

(iii) $x_n\, y_n \to L\, M$;

(iv) If $L \neq 0$, then $x_n \neq 0$ for sufficiently large n, and $\dfrac{1}{x_n} \to \dfrac{1}{L}$;

(v) If $M \neq 0$, then $\dfrac{x_n}{y_n} \to \dfrac{L}{M}$.

- We will prove all of these as exercises.

Theorem.

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences, L and M real numbers, for which $x_n \to L$ and $y_n \to M$. Then

(i) $c\, x_n \to c\, L$, where c is any constant.

Comments on proof of (i)

- Given that we can make $|x_n - L|$ small for large n, we must show that we can make $|c\, x_n - c\, L|$ small.
- So for a given $\varepsilon > 0$, how small must we make $|x_n - L|$ in order that $|c\, x_n - c\, L| < \varepsilon$?

Exercise.

Write the proof of (i).

(*) • Given $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq N \Rightarrow |x_n - L| < \varepsilon]$

(**) • Want to prove $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq N \Rightarrow |c x_n - cL| < \varepsilon]$.

• So we must prove (**) by making use of (*).

• begin by trying to prove (**), making use of (*) when needed.

• (*) Enters at step 2 when you must figure out how to choose N. Hopefully the N you get from (*) will work in (**).

Theorem Let $\{x_n\}_n$, $\{y_n\}_n$ be sequences, L and M real numbers for which $x_n \to L$ and $y_n \to M$. Then $cx_n \to cL$, where c is any constant.

Proof The result is obvious in case $c = 0$, so we may assume $c \neq 0$.

Let $\varepsilon > 0$. Since $x_n \to L$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$ then $|x_n - L| < \frac{\varepsilon}{|c|}$.

Let $n \in \mathbb{N}$. Suppose $n \geq N$. Then

$$|cx_n - cL| = |c|\,|x_n - L| < |c| \frac{\varepsilon}{|c|} = \varepsilon.$$

$\square$

Theorem.

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences, L and M real numbers, for which $x_n \to L$ and $y_n \to M$. Then

(ii) $x_n + y_n \to L + M$;

Comments on proof of (ii)

- We must show that if we can make $|x_n - L|$ small for all sufficiently large n, and we can make $|y_n - M|$ small for sufficiently large n, then we can make $|(x_n + y_n) - (L + M)|$ suitably small for all sufficiently large n.
- Given that we can make $|x_n - L|$ smaller than a given positive real number for sufficiently large n,and we can make $|y_n - M|$ smaller than a given positive real number for sufficiently large n, how big should n be so that we are sure that both $|x_n - L|$ and $|y_n - M|$ are smaller than a given positive number ε?
- If we can force $|x_n - L|$ and $|y_n - M|$ both to be small for sufficiently large n, how can we be sure that $|(x_n + y_n) - (L + M)|$ is also small? What tool should we use?

Exercise.

Write the proof of (ii).

- Similar idea as in previous proof. Use the fact that you know $x_n \to L$ and $y_n \to M$ in order to deduce $x_n + y_n \to L+M$.

- Begin by trying to prove $x_n + y_n \to L+M$, making use of $x_n \to L$ and $y_n \to M$ when needed.

Theorem Let $\{x_n\}_n$ and $\{y_n\}_n$ be sequences, L and M real numbers for which $x_n \to L$ and $y_n \to M$. Then $x_n + y_n \to L + M$.

Proof Let $\varepsilon > 0$. Since $x_n \to L$, there exists $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_1$, then $|x_n - L| < \frac{\varepsilon}{2}$. Since $y_n \to M$, there exists $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_2$ then $|y_n - M| < \frac{\varepsilon}{2}$.

Choose $N \in \mathbb{N}$ such that $N \geq N_1$ and $N \geq N_2$. Let $n \in \mathbb{N}$. Suppose $n \geq N$. Then

$$|(x_n + y_n) - (L+M)| = |(x_n - L) + (y_n - M| \leq |x_n - L| + |y_n - M|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We've thus proven $x_n + y_n \to L + M$. $\square$

Theorem.

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences, L and M real numbers, for which $x_n \to L$ and $y_n \to M$. Then the following hold:

(iii) $x_n\, y_n \to L\, M$;

Comments on proof of (iii)

- We must show that if we can make $|x_n - L|$ small for all sufficiently large n, and we can make $|y_n - M|$ small for sufficiently large n, then we can make $|x_n\, y_n - L\, M|$ suitably small for all sufficiently large n.
- The trick is to add and subtract the right thing so that after simplifying we get a sum of terms each of which we can force to be suitably small for all large enough n.
- Try adding and subtracting $y_n L$ under the absolute value bars. This gives $|x_n\, y_n - L\, M| = |x_n y_n - y_n L + y_n L - LM| = |y_n(x_n - L) + L(y_n - M)|$.
- Now apply the triangle inequality to get $|x_n\, y_n - L\, M| \le |y_n|\, |x_n - L| + |L|\, |y_n - M|$.
- How do you know that you can force the term $|y_n|\, |x_n - L|$ to be suitably small? What property of the convergent sequence y_n should you make use of to do it?

Exercise.

Write the proof of (iii).

• Harder than the previous proofs.

• $|x_n y_n - LM| = |x_n y_n - y_n L + y_n L - LM|$

$\le |y_n|\, |x_n - L| + |L|\, |y_n - L|$

harder to control. use that $\{y_n\}_n$ is <u>bounded</u> and $y_n \to L$

easy to control using $y_n \to L$.

Theorem Let $\{x_n\}_n$ and $\{y_n\}_n$ be sequences, L and M real numbers for which $x_n \to L$ and $y_n \to M$. Then $x_n y_n \to LM$.

Proof We only consider the case with $L \neq 0$ and $M \neq 0$. Let $\varepsilon > 0$.

Since $\{y_n\}_n$ is convergent, it is bounded, so there exists $A \in \mathbb{R}$ such that $|y_n| < A$ for all $n \in \mathbb{N}$.

Since $y_n \to M$, there exists $N_1 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, if $n \geq N_1$ then

$$|y_n - M| < \frac{\varepsilon}{2A}.$$

Since $x_n \to L$, there exists $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N_2$ then

$$|x_n - L| < \frac{\varepsilon}{2|L|}.$$

Choose $N = \max\{N_1, N_2\}$. Let $n \in \mathbb{N}$ with $n \geq N$. Then

$$\begin{aligned} |x_n y_n - LM| &= |x_n y_n - y_n L + y_n L - LM| = |y_n(x_n - L) + L(y_n - M)| \\ &\leq |y_n|\,|x_n - L| + |L|\,|y_n - M| \\ &\leq A \cdot \frac{\varepsilon}{2A} + |L| \cdot \frac{\varepsilon}{2|L|} = \varepsilon, \end{aligned}$$

completing the proof. $\square$