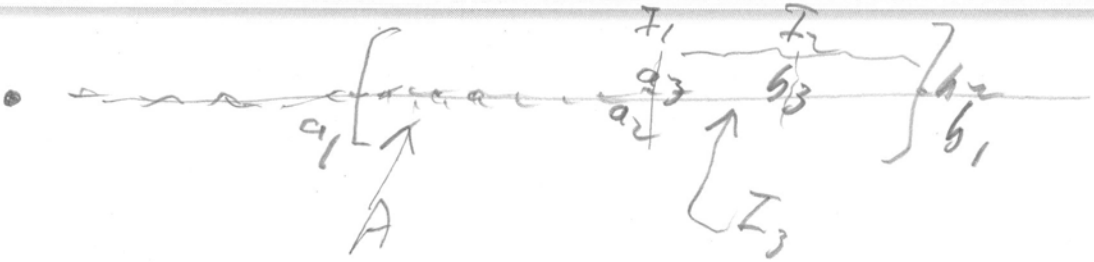
Consequences of the completeness axiom

Theorem

Let A be a nonempty set with an upper bound. Then the least upper bound of A exists.

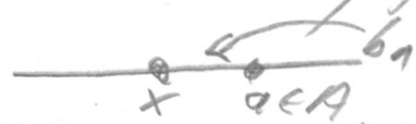
Exercise.

Use the method of interval halving and the completeness axiom to prove the above theorem.



- idea is to keep choosing intervals $[a_n, b_n]$ such that (i) a_n not an upper bound of A and (ii) b_n is an upper bound of A. and (iii) $b_n - a_n \to 0$.
- then a_n and b_n converge to the same number x and the hope is that x is the least upper bound of A.

- <u>x is an upper bound of A</u>: If not, $\exists a \in A$ such that $x < a$.



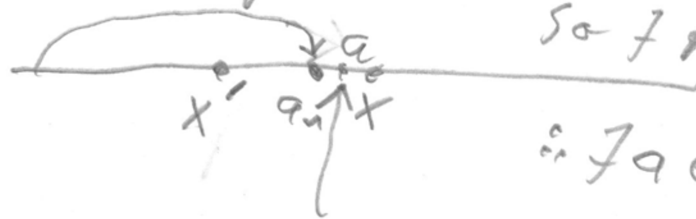
But $b_n \to x$ so $\exists n$ s.t. $b_n < a$ contradicting b_n an upper bound of A.

• x is the smallest of all upper bounds

If not $\exists x'$ s.t. $x' < x$ and x' an upper bound of A. But $a_n \to x$

So $\exists n$ s.t. $a_n > x'$

$\therefore \exists a \in A$ s.t.
$a > a_n > x'$



$\exists a \in A$ s.t.
$a > a_n$

But $x' < a \in A$ contradicts x' an upper bound of A.

Theorem Every nonempty, upper bounded subset of $\mathbb{R}$ has a least upper bound.

Proof Let A be a nonempty, upper bounded subset of $\mathbb{R}$.

Since A is upper bounded, there exists $M \in \mathbb{Z}$ such that

$$x \le M \text{ for all } x \in A.$$

We now inductively define a sequence of intervals $\{I_n\}_n = \{[a_n, b_n]\}_n$ such that for all n,

(i) I_{n+1} is either the left half or the right half of I_n;

and (ii) a_n is not an upper bound of A and b_n is an upper bound of A.

For the basis step, let x_0 be any element of A. Choose $a_1 = x_0 - 1$ and $b_1 = M$. Then a_1 and b_1 satisfy (ii) above.

For the inductive step, let $n \ge 1$ and suppose we have selected a_n and b_n. Let Y be the midpoint $\frac{a_n + b_n}{2}$ of $[a_n, b_n]$. Then either Y is an upper bound of A or it is not an upper bound of A. If Y is an upper bound of A, choose $a_{n+1} = a_n$ and $b_{n+1} = Y$; if Y is not an upper bound of A, choose $a_{n+1} = Y$ and $b_{n+1} = b_n$. This completes the proof of the induction.

By construction, for each n,

$$(*) \quad b_n - a_n = |I_n| = \frac{1}{2^{n-1}} \cdot |I_1|.$$

Also, by construction, $\{a_n\}_n$ is an increasing sequence and $\{b_n\}_n$ is a decreasing sequence. We have that M is an upper bound of $\{a_n\}_n$ and any element of A is a lower bound of $\{b_n\}_n$. Thus both $\{a_n\}_n$ and $\{b_n\}_n$ are convergent sequences.

By $\circledast$, $b_n - a_n \to 0$, so from this we can deduce that $\{a_n\}_n$ and $\{b_n\}_n$ both converge to the same number, L. We prove next that L is the least upper bound of A.

To do this we must show the following two things:

(i) L is an upper bound of A,

(ii) L is less than any other upper upper bound of A.

For (i), if L were not an upper bound of A, then there exists $x \in A$ such that $L < x$. Since $b_n \to L$, it follows there exists $n_x \in \mathbb{N}$ such that $b_{n_x} < x$. Since $x \in A$, this contradicts that b_{n_x} is an upper bound of A. Thus (i) is true.

For (ii), if there exists an upper bound L' of A such that $L' < L$, then since $a_n \to L$, we can deduce there exists $n_{L'} \in \mathbb{N}$ such that $a_{n_{L'}} > L'$. Since $a_{n_{L'}}$ is not an upper bound of A, there exists $x \in A$ with $a_{n_{L'}} < x$, so $L' < x$. This contradicts that L' is an upper bound of A. Thus (ii) is true, and so we are done. $\square$

Consequences of the completeness axiom

Theorem

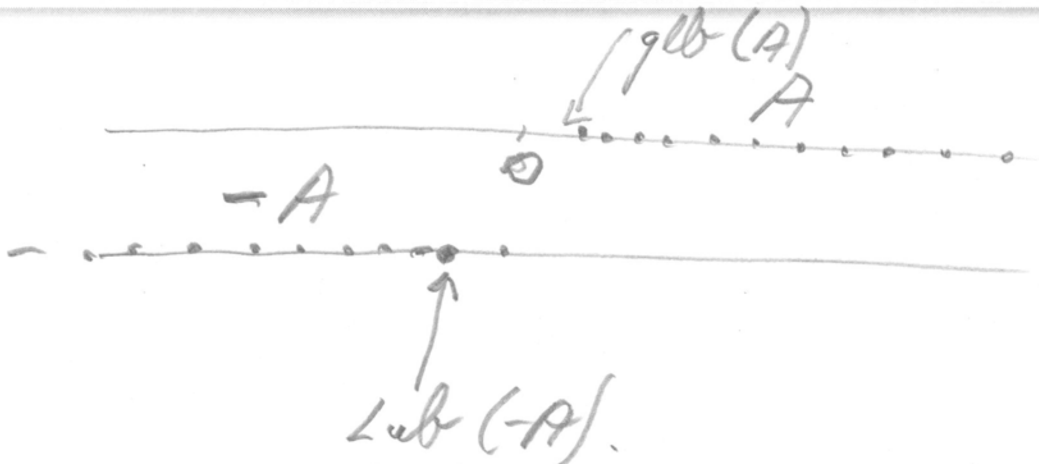
For any nonempty lower bounded set A, the greatest lower bound of A exists.

Exercise.

Prove the above theorem. But:

- Don't do it by going back to first principles, but rather by making use of the previous theorem.
- The idea is that if we define $-A$ to be $\{x \in \mathbb{R} : -x \in A\}$, then show that

$$glb(A) = -lub(-A).$$



The picture suggests $-lub(-A) = glb(A)$.

Exercise

(i) $\sup\{-1, -13, 5, 7, 9, 2\} = ?$

(ii) $\inf\{-1, -13, 5, 7, 9, 2\} = ?$

(iii) $\sup[6, 19.5] = ?$

(iv) $\inf(\{x \in \mathbb{R} : x < 10\}) = ?$

(v) $\sup(\{1 - 1/n : n \in \mathbb{N}\})$

(vi) Let A be a nonempty subset of $\mathbb{R}$. Is it necessarily true that $\sup(A) \in A$? Give examples to illustrate.

(i) 9 (ii) -13 (iii) 19.5 (iv) $-\infty$

(v) $\sup\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\} = 1$

(vi) No, not in general. For example problem (v).

Exercise.

(a) $\lambda < \sup(A)$: means λ not an upper bound,
so $(\exists a \in A)(\lambda < a)$.

(b) $\lambda > \sup(A)$: λ is an upper bound so $(\forall a \in A)(\lambda > a)$

(c) $(\forall a \in A)(a < \lambda)$

(d) $(\exists a \in A)(a < \lambda)$

Exercise.

Let A be a nonempty set. Answer the following questions using nothing more than the definitions of supremum and infimum.

a) Suppose λ is a number such that $\lambda < \sup(A)$. What can you deduce about the set A?

b) Suppose λ is a number such that $\lambda > \sup(A)$. What can you deduce about the set A?

c) Suppose λ is a number such that $\lambda < \inf(A)$. What can you deduce about the set A?

d) Suppose λ is a number such that $\lambda > \inf(A)$. What can you deduce about the set A?