Consequences of the completeness axiom

Theorem (bounded monotone sequences)

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Then x_n converges to a real number if any of the following is true:

(i) x_n is upper bounded and either increasing or strictly increasing;

(ii) x_n is lower bounded and either decreasing or strictly decreasing;

(iii) x_n is bounded and monotone or strictly monotone.

Exercise.

Write the proof of the above theorem by making appropriate use of some of the exercises we've already done above.

<u>Proof</u>. We showed earlier in this section that any such sequence is Cauchy. It thus follows from the completeness of R that any such sequence is convergent. $\square$

Exercise.

Prove that the sequence $x_n = \left(1 + \frac{1}{n}\right)^n$ converges to a real number between 2 and 3. Do it by making use of the result of the previous exercise. Before doing it, recall the following two things:

- The binomial theorem
- The formula for an infinite geometric series

- $x_1 = 2$, $x_2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2.25$. So if we're hoping to use the previous theorem to prove convergence, we would have to prove the sequence is increasing.
- So we try to show $\{x_n\}_n$ is increasing and upper bounded.
- The fact that $\left(1+\frac{1}{n}\right)^n$ is raised to a positive integer power suggests the binomial theorem might be helpful, since it allows us to expand $\left(1+\frac{1}{n}\right)^n$ to a sum of terms which we can study.

- We can then compare the terms we get in the expansion of $\left(1+\frac{1}{n}\right)^n$ with the terms we get in the expansion of $\left(1+\frac{1}{n+1}\right)^{n+1}$.

Exercise $\left\{\left(1+\frac{1}{n}\right)^n\right\}_n$ converges to a real number between 2 and 3.

Proof To prove convergence it is sufficient to show the sequence is increasing and upper bounded.

To show increasing we use the binomial theorem:

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\frac{1}{n^k} = 1 + \sum_{k=1}^{n} \frac{n!}{k!\,(n-k)!}\cdot\frac{1}{n^k}$$

$$= 1 + \sum_{k=1}^{n} \frac{1\cdot 2\cdot 3\cdot 4\cdots(n-k)\,\overbrace{(n-k+1)(n-k+2)\cdots n}^{k \text{ factors}}}{k!\;1\cdot 2\cdot 3\cdots(n-k)}\cdot\frac{1}{n^k}$$

$$= 1 + \sum_{k=1}^{n} \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k!\,n^k}$$

$$= 1 + \sum_{k=1}^{n} \frac{1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\cdots\left(1-\frac{(k-1)}{n}\right)}{k!}$$

$$< 1 + \sum_{k=1}^{n+1} \frac{\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{(k-1)}{n+1}\right)}{k!}$$

$$= \left(1+\frac{1}{n+1}\right)^{n+1}.$$

Thus the sequence is increasing.

Also, using something shown above,

$$\left(1+\frac{1}{n}\right)^n = 1 + \sum_{k=1}^{n} \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(k-1)}{n}\right)}{k!}$$

Each factor less than 1

$$\leq \sum_{k=0}^{n} \frac{1}{k!}$$

$$= 1 + \frac{1}{1!} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdots n}$$

$$\leq 1 + \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{2\cdot 2} + \frac{1}{2\cdot 2\cdot 2} + \cdots + \frac{1}{\underbrace{2\cdot 2\cdot 2\cdots 2}_{n-1 \text{ factors}}}\right)$$

$$= 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^{n-1}}\right)$$

$$= 1 + \frac{1 - 1/2^n}{1-\frac{1}{2}} < 1 + \frac{1}{1-\frac{1}{2}} = 1 + 2 = 3.$$

Thus the sequence is increasing and bounded above by 3, and so converges to a number, e such that $e \leq 3$.

When $n=1$, $\left(1+\frac{1}{n}\right)^n = 2$, so $2 \leq e \leq 3$. $\square$

Consequences of the completeness axiom

- We define $\sqrt{2}$ to be a positive real number x with the property that $x^2 = 2$.
- But defining it in this way, doesn't prove that there is such a real number $\sqrt{2}$.

Theorem There exists a positive real number x such that $x^2 = 2$.

Lemma 1. Suppose y_n is a sequence which converges to a number y. Then y_n^2 converges to y^2.

Lemma 2. Suppose y_n is a sequence which converges to a number y. Suppose that x is a number such that $y_n \leq x$ for all n. Then $y \leq x$.

Lemma 3. Suppose that x_n is a sequence that converges to a number x, and suppose y_n is a sequence such that the sequence $x_n - y_n$ converges to 0. Then y_n converges to x.

Exercise.

Prove the above theorem by first proving the three lemmas and then using the results of those lemmas. In your proof, make use of the interval halving method and clearly show how the completeness axiom of $\mathbb{R}$ is being used.

- Lemma 1 we'll do later. Lemmas 2 and 3 we'll

do in a homework assignment.

- We prove the theorem using the Interval Halving Theorem.
- The idea is to produce a sequence of intervals $[a_n, b_n]$ s.t. $a_n^2 < 2 < b_n^2$, so $\underset{a_n}{[} \;\; \underset{\sqrt{2}}{} \;\; \underset{b_n}{]}$.

and then use the Interval Halving Thm. to deduce $a_n \to x$ such that $x^2 = 2$.

Theorem There exists a positive real number x such that $x^2 = 2$.

For the proof we make use of the following lemma.

Lemma (i) If $y_n \to y$, then $y_n^2 \to y^2$.

(ii) If $y_n \to y$ and $y_n \leq x$ for all x, then $y \leq x$.

(iii) If $x_n \to x$ and $y_n - x_n \to 0$, then $y_n \to x$.

Proof of the theorem We inductively define a sequence of intervals $\{I_n\}_n = \{[a_n, b_n]\}$ with the following properties:

① For all n, I_{n+1} is either the left or the right half of I_n;

② $a_n^2 < 2$ and $2 < b_n^2$.

For the basis step, choose $I_1 = [1, 2]$.

For the inductive step, let $n \geq 1$ and suppose we have selected $I_1, \ldots, I_n$ with the properties described above in ① and ②.

Let $y = \frac{a_n + b_n}{2}$ be the midpoint of I_n. Then define

$$I_{n+1} = \begin{cases} \text{left half of } I_n & \text{if } y^2 > 2 \\ \text{right half of } I_n & \text{if } y^2 < 2 \end{cases}$$

Since $a_n^2 < 2 < b_n^2$, this completes the induction.

By the Interval Halving Theorem, $\{a_n\}_n$ and $\{b_n\}_n$ are both Cauchy sequences, so by the Completeness of $\mathbb{R}$, both sequences converge. But $b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} \to 0$ as $n \to \infty$, so by the lemma, both sequences converge to the same number x.

By the lemma, $a_n^2 \to x^2$ and $b_n^2 \to x^2$.

Also, for all n,

$$a_n^2 < 2 < b_n^2,$$

so again by the lemma,

$$x^2 \leq 2 \text{ and } 2 \leq x^2.$$

Thus $x^2 = 2$, and we are done. $\square$