- If $\{x_n\}_{n=1}^{\infty}$ is a sequence, say we choose a large $N \in \mathbb{N}$ and look at the members of the sequence x_n for any $n \geq N$. Let's informally call this "looking far out in the sequence".
- Then informally, the sequence is Cauchy provided given any $\varepsilon > 0$, if we look sufficiently far out in the sequence any pair of terms are within ε of each other.

Exercise.

To see if you understand what is a Cauchy sequence, consider the following sequence:

$$0, 1/2, 1, 2/3, 1/3, 0, 1/4, 2/4, 3/4, 1, 4/5, 3/5, 2/5, 1/5, 0,$$
$$1/6, 2/6, 3/6, 4/6, 5/6, 1, 6/7, 5/7, 4/7, 3/7, \ldots$$

a) Is it a Cauchy sequence? Why or why not?

b) What "closeness" property does this sequence satisfy?

(a) No: sequence has infinitely many 0's and 1's, so taking $\varepsilon = \frac{1}{2}$, for all N, there exist $m, n \geq N$ such that $|x_m - x_n| > \frac{1}{2}$.

(b) Consecutive terms get arbitrarily close; but that is not a strong enough condition to make the sequence Cauchy.

Theorem (Interval Halving Method Theorem)

Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of closed bounded intervals such that for each n, I_{n+1} is either the left half or the right half of I_n. For each n, let x_n be any point in I_n. Then the resulting sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

motivation

- $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, so for each N, if $m, n \geq N$ then $I_m \subseteq I_N$ and $I_n \subseteq I_N$.
- $|I_N| = \frac{1}{2^{N-1}} |I_1| \to$ very small if N big.
- So for $m, n \geq N$, $|x_m - x_n| \leq |I_N| = \frac{1}{2^{N-1}} |I_1| < \varepsilon$ if N big enough.

Theorem (Interval Halving Method)

Let $\{I_n\}_n$ be a sequence of closed, bounded intervals such that for each n, I_{n+1} is either the left half or the right half of I_n. For each n, let $x_n \in I_n$. Then $\{x_n\}_{n=1}^{\infty}$ is Cauchy.

Proof Let $|I_j|$ denote the length of interval I_j. Then $|I_2| = \frac{1}{2}|I_1|$, $|I_3| = \frac{1}{2^2}|I_1|$, $|I_4| = \frac{1}{2^3}|I_1|$, and in general

$$|I_j| = \frac{1}{2^{j-1}}|I_1|.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$\frac{1}{2^{N-1}}|I_1| < \varepsilon.$$

Let $m, n \in \mathbb{N}$ with $m, n \geq N$. Then $x_m \in I_m \subseteq I_N$ and $x_n \in I_n \subseteq I_N$, so $|x_m - x_n| \leq |I_N| = \frac{1}{2^{N-1}}|I_1| < \varepsilon$. Thus $\{x_j\}_{j=1}^{\infty}$ is Cauchy. $\square$

Application of the interval halving method

- The Interval Halving Method Theorem gives us a nice way to produce Cauchy sequences.
- The idea of using a decreasing sequence of closed bounded intervals, each one either the left half or the right half of the previous one, is an idea we'll use several times.
- The next theorem gives another application of this method. See if you can prove it yourself by using this idea of producing such a sequence of intervals. You will have to figure out at each step whether you should pick the left or the right half of the previous interval.

Theorem

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence with the following two properties:

- The sequence is decreasing: $(\forall n)[x_n \geq x_{n+1}]$
- The sequence is lower bounded: $(\exists M \in \mathbb{Z})(\forall n \in \mathbb{N})[x_n \geq M]$

Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

entire sequence inside

motivation

I_1

midpt

- If the x_n's don't cross to the left of the midpoint, then they are all in the right half.
- If one x_n crosses to left of the midpoint, then all successive terms are in the left half of I_1.

- The idea is to construct a decreasing sequence of intervals I_n such that $|I_n| \to 0$ and for each n <u>all the terms after a while</u> of $\{x_j\}_j$ lie in I_n.
- We can do this because $\{x_j\}_j$ is decreasing and lower bounded.
- The fact that $|I_n| \to 0$ allows us to deduce $\{x_j\}_j$ is Cauchy.

Theorem Every decreasing, lower bounded sequence is a Cauchy sequence.

Proof Let $\{x_n\}_{n=1}^{\infty}$ be a decreasing, lower bounded sequence. Since the sequence is lower bounded, there exists $m \in \mathbb{Z}$ such that for all n, $x_n \geq m$.

We inductively define a sequence of intervals $\{I_n\}_{n=1}^{\infty} = \{[a_n, b_n]\}_n$ with the following properties:

① For all n, I_{n+1} is either the left half or the right half of I_n;

② For all n, there exists $K_n \in \mathbb{N}$ such that for all $j \in \mathbb{N}$, if $j \geq K_n$ then $x_j \in I_n$.

For the basis step of the induction, choose $I_1 = [m, x_1]$. Since $\{x_j\}_j$ is decreasing, $x_j \in I_1$ for all j, so we can take $K_1 = 1$.

For the inductive step, let $n \geq 1$, and suppose we have selected $I_j = [a_j, b_j]$ and K_j for $1 \leq j \leq n$, all having the desired properties. We must explain how to obtain I_{n+1} and K_{n+1}.

In particular we know that if $j \geq K_n$ then $x_j \in I_n$. Let Y be the midpoint of I_n. There are two possibilities:

<u>case 1</u>: $x_j \geq Y$ for all $j \geq K_n$; <u>case 2</u>: there exists $j_0 > K_n$ such $x_{j_0} < Y$.

If case 1 occurs, let I_{n+1} be the right half of I_n and let $K_{n+1} = K_n$. If case 2 occurs, let I_{n+1} be the left half of I_n and let $K_{n+1} = j_0$. Since $\{x_j\}_j$ is decreasing, it follows that $x_j \in N_{n+1}$ for all $j \geq K_{n+1}$. This completes the induction.

We now complete the proof that $\{x_j\}_j$ is Cauchy. Let $\varepsilon > 0$. By construction, for all j we have

$$|I_n| = \frac{1}{2^{n-1}} |I_1|.$$

Choose n such that $|I_n| = \frac{1}{2^{n-1}} |I_1| < \varepsilon$. Let $j, k \in \mathbb{N}$. Suppose $j, k \geq K_n$. Then x_j and $x_k \in I_n$. Since $|I_n| < \varepsilon$, it follows $|x_j - x_k| < \varepsilon$. This completes the proof that $\{x_j\}_j$ is a Cauchy sequence.

$\square$

Application of the interval halving method

- The next theorem is another important result.
- It could be proved using a similar technique as the previous theorem, but a better approach is to try to deduce it from the theorem you just proved on the previous page,
- i.e. if you know that any decreasing lower bounded sequence is Cauchy, deduce from this that any increasing upper bounded sequence is Cauchy.

Theorem

Any increasing upper bounded sequence is a Cauchy sequence.

- idea is that if $\{x_n\}_n$ increases and is upper bounded, then $\{-x_n\}_n$ is decreasing and lower bounded.
- We can then apply the previous theorem.

Theorem Every increasing upper bounded sequence is Cauchy.

Proof Let $\{x_n\}_n$ be an increasing, upper bounded sequence. Then the sequence $\{-x_n\}_n$ is decreasing and lower bounded. Thus by a previously proven theorem, $\{-x_n\}_n$ is Cauchy.

Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, if $m, n \geq N$, then $|(-x_m) - (-x_n)| < \varepsilon$. But this just says $|x_n - x_m| < \varepsilon$. Thus $\{x_n\}_n$ is a Cauchy sequence. $\square$

A Cauchy sequence of rationals doesn't necessarily converge to a rational

We make use of the following ideas concerning decimal expansions:

- A rational number is defined to be the ratio of two integers.
- Each rational number can be represented by a decimal expansion which is either terminating or repeating.
- Conversely, each decimal expansion which is either terminating or repeating represents a unique rational number.
- A decimal expansion which is neither terminating nor repeating does not represent a rational number.

Exercise.

a) Write down a few examples of terminating decimal expansions.

b) Write down a few examples of repeating, nonterminating decimal expansions.

c) If $a = 0.123572223796$ and $b = 0.12357721900457$, what is an upper bound for $|a - b|$? For this upper bound, look for the smallest reciprocal power of 10 you can get away with.

d) If $a = 0.342715$ and $b = 0.3422222\ldots$, what is an upper bound for $|a - b|$? Again, look for the smallest reciprocal power of 10 you can get away with.

e) Write down an example of a nonrepeating, nonterminating decimal expansion that uses only 0's and 1's.

f) For the example you just wrote down, write down a rational number that is within 0.001 of that number and which has a terminating decimal.

g) For the same example, write down a rational number that is within 0.000001 of that number and which has a nonterminating, repeating decimal.

(a) 0.3975

(b) 0.33333...., 0.1237942942942...

(c) $|a-b| = 0.0000055\ldots < 10^{-5}$

(d) $|a-b| < 10^{-3}$

(e) 0.101001000100001....

(f) 0.101, 0.101001

(g) 0.

A Cauchy sequence of rationals doesn't necessarily converge to a rational

Exercise.

Use the above ideas to construct a sequence of rational numbers which is a Cauchy sequence, yet which doesn't converge to a rational number.

$x_1 = 0.1$

$x_2 = 0.10$

$x_3 = 0.101$

$x_4 = 0.101001$

$x_5 = 0.1010010001$

$x_6 = 0.101001000100001$

$\vdots$

Is Cauchy but not convergent to a rational number, since if it converged to anything, it would have to converge to $0.101001000100001 0\ldots$ and this does not represent a rational.