Exercise.

Consider the statement that $(-1)^n$ diverges.

a) Intuitively why do you believe it is true?

b) Write a proof that it is true.

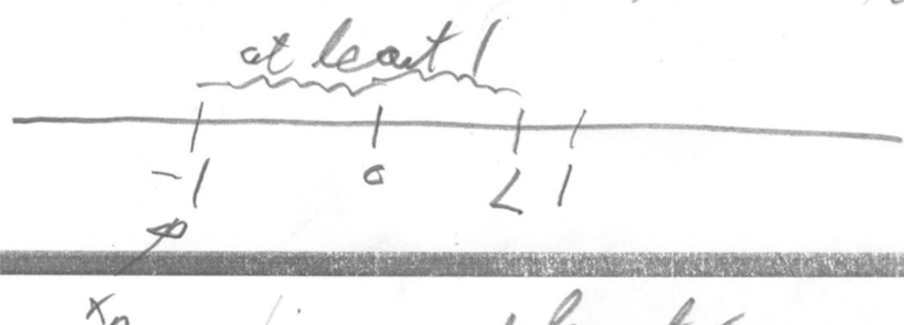
(a) The sequence oscillates between -1 and 1, so it doesn't get close to any particular number.

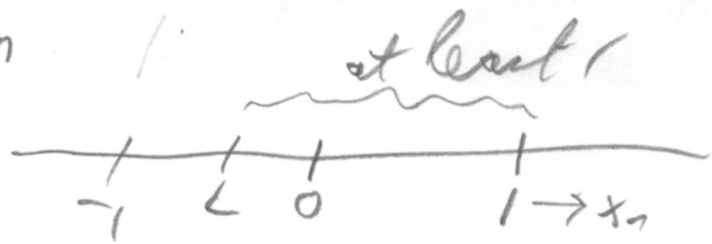
(b) cvge:

$$(\exists L\in\mathbb{R})(\forall \varepsilon>0)(\exists N\in\mathbb{N})(\forall n\in\mathbb{N})\left[n\geq N \Rightarrow |x_n-L|<\varepsilon\right].$$

dvge

$$(\forall L\in\mathbb{R})(\exists \varepsilon>0)(\forall N\in\mathbb{N})(\exists n\in\mathbb{N})\left[n\geq N \text{ and } |x_n-L|\geq\varepsilon\right].$$





Proof Let $L\in\mathbb{R}$. Choose $\varepsilon=1$. Let N

be a natural number.

Case 1 $L \geq 0$. Choose n to be odd and greater than N. Then

$$|x_n - L| = |-1 - L| \geq 1 = \varepsilon$$

(since $L \geq 0$).

Case 2 $L < 0$. Choose n to be even and greater than N. Then

$$|x_n - L| = |1 - L| \geq 1 = \varepsilon.$$

Then $\{(-1)^n\}_n$ is divergent. $\square$

Definition: Cauchy sequence

A sequence x_n is called a ***Cauchy sequence*** if the following is true:

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \in \mathbb{N})[m, n \geq N \Longrightarrow |x_m - x_n| < \varepsilon]$$

Exercise.

a) Intuitively what does it mean to say that a sequence is a Cauchy sequence?

b) Write in symbols and in words what it means to say that a sequence is not Cauchy.

c) Intuitively what does it mean to say that a sequence is not Cauchy?

(a) Far out in the sequence the terms are <u>all</u> close to each other.

(b) $(\exists \varepsilon > 0)(\forall N \in \mathbb{N})(\exists m, n \in \mathbb{N})(m, n \geq N \text{ and } |x_m - x_n| \geq \varepsilon)$

(c) No matter how far out in the sequence you look, you can find at least two

terms not that close together.

Theorem.

If a sequence is convergent, then it is a Cauchy sequence.

Proof. $\boxed{\text{Discussion}}$

Given $\boxed{\{x_n\}_n \to L}$

Given: $(\exists L \in \mathbb{R})(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq N \Rightarrow |x_n - L| < \varepsilon]$.

want: $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \in \mathbb{N})[m, n \geq N \Rightarrow |x_m - x_n| < \varepsilon]$.

$\forall n \geq N$: $|x_n - L| < \varepsilon$.

$\Rightarrow m, n \geq N$: $|x_m - x_n| = |x_m - L + L - x_n|$

$\leq |x_m - L| + |L - x_n|$

Theorem If a sequence is convergent, then it is Cauchy.

Proof Let $\{x_n\}_n$ be a sequence convergent to some number x.
Let $\varepsilon > 0$. Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$ then $|x_n - x| < \frac{\varepsilon}{2}$.

Let $m, n \in \mathbb{N}$. Suppose $m, n \geq N$. Then

$$|x_m - x_n| = |x_m - x + x - x_n| \leq |x_m - x| + |x - x_n|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

proving that $\{x_n\}_n$ is a Cauchy sequence. $\square$

"Theorem: If a sequence is convergent, then it is a Cauchy sequence."

Exercise.

a) What does the contrapositive of the above theorem say? Is it true?

b) Earlier in this section we proved that the sequence $(-1)^n$ is divergent. Give a simpler alternate proof.

(a) $\boxed{\text{Not Cauchy} \Longrightarrow \text{not cvgt}}$ is true.

(b) $\left[(\exists \varepsilon > 0)(\forall N \in \mathbb{N})(\exists m, n \in \mathbb{N})\left(m, n \geq N \text{ and } |x_m - x_n| \geq \varepsilon \right) \right]$.

Pf Choose $\varepsilon = 1$. Let $N \in \mathbb{N}$. Choose $m = 2N$ and $n = 2N+1$. Then $m, n \geq N$ and

$|x_m - x_n| = |1 - (-1)| = 2 \geq 1$.

Definition: Bounded sequence

A sequence x_n is called ***bounded*** if the following is true:

$$(\exists M \in \mathbb{R})(\forall n \in \mathbb{N})[|x_n| < M]$$

If a sequence is not bounded, we say it is ***unbounded***.

Exercise.

a) Informally what does boundedness of a sequence say about the sequence?

b) Give an example of a bounded sequence which is not convergent.

c) Write down in symbols and in words what it means to say that a sequence is unbounded.

d) Give an example of an unbounded sequence.

(a) Terms don't get too large or too negative.

(b) $\{(-1)^n\}_n$

(c) $(\forall M \in \mathbb{R})(\exists n \in \mathbb{N})(|x_n| \geq M)$.

(d) $\{1, 2, 3, 4, 5, 6, 7, \ldots\}$

$\{0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots\}$

$\{0, 1, 0, 2, 0, 3, 0, 4, 0, 5, \ldots\}$

Theorem.

If a sequence is a Cauchy sequence, then it is bounded.

Proof. $\boxed{\text{Discussion}}$

Given: $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall m, n \in \mathbb{N})[m, n \geq N \Rightarrow |x_m - x_n| < \varepsilon]$.

Want: $(\exists M \in \mathbb{R})(\forall n \in \mathbb{N})[|x_n| \leq M]$.

If $\varepsilon = 1$, $\exists N$ s.t. $|x_m - x_n| < 1$ for all $m, n \geq N$.

$\therefore |x_m - x_N| < 1$

$\therefore |x_m| = |x_m - x_N + x_N|$
$\leq |x_m - x_N| + |x_N|$
$\leq 1 + |x_N|$

Theorem. If a sequence is Cauchy then it is bounded.

Proof Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Then the following statement is true:

$$(\forall \varepsilon>0)(\exists N\in \mathbb{N})(\forall m,n\in \mathbb{N})[m,n\geq N \Longrightarrow |x_m-x_n|<\varepsilon].$$

Then taking $\varepsilon=1$, there exists $N\in\mathbb{N}$ such that for all $m,n\in\mathbb{N}$, if $m,n\geq N$ then $|x_m-x_n|<1$. In particular, if $m\geq N$ then $|x_m-x_N|<1$.
Thus if $m\geq N$,

$$|x_m|=|x_m-x_N+x_N|\leq|x_m-x_N|+|x_N|\leq 1+|x_N|.$$

Since the maximum of any finite list of numbers exists, we can define

$$M:=\max\{|x_1|,|x_2|,\cdots,|x_{N-1}|,1+|x_N|\}.$$

Then for any $n\in\mathbb{N}$ we have $|x_n|\leq M$. We've shown $\{x_n\}_n$ is a bounded sequence. $\square$

Exercise.

a) Informally, what does it mean to say that $x_n \to \infty$?

b) True or false: If $x_n \to \infty$, then x_n is unbounded.

c) Can you give an example of an unbounded sequence x_n such that x_n doesn't diverge to ∞ or $-\infty$? If not explain why not, and if true give such an example.

d) Is it true or false that every bounded sequence is a Cauchy sequence? If it is true prove it, and if false then give a counterexample.

e) Can you give an example of a Cauchy sequence which is not bounded?

(a) The terms ultimately all get large.

(b) True: "$x_n \to \infty$" means <u>all</u> terms with a big enough index get large; and "x_n unbounded" means infinitely many terms get large.

(c) $\{0, 1, 0, 2, 0, 3, 0, 4, 0, 5, \cdots\}$

(d) False $\{-1, 1, -1, 1, -1, 1, -1, \cdots\}$

(e) No, every Cauchy sequence is bounded.