1.2 Limits of Sequences & Cauchy Sequences

Definition: Sequence

A ***sequence*** is defined to be a function from $\mathbb{N}$ to $\mathbb{R}$. If a sequence is named x, we will refer to $x(n)$ as x_n. We will usually denote the entire sequence by $\{x_n\}_{n=1}^{\infty}$, but sometimes by abuse of notation we may denote the entire sequence by x_n.

- We also think of a sequence as being an endless list of real numbers.
- So the sequence $x_n = 1/n$ gives rise to the list

$$1, 1/2, 1/3, 1/4, 1/5, 1/6, \ldots.$$

- We're particularly interested in the behavior of the sequence x_n far out in the list, i.e. very large n.

Definition: Convergence and divergence of a sequence

(i) Let x_n be a sequence and L a real number. We say that x_n converges to L provided

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq N \Longrightarrow |x_n - L| < \varepsilon]$$

(ii) If x_n converges to L, we write $x_n \to L$, or $\lim_{n\to\infty} x_n = L$.

(iii) If x_n is a sequence and there does not exist a real number L such that $x_n \to L$, then we say that x_n diverges.

Exercise 1.2.1.

a) Informally, what does it mean to say that the sequence x_n converges to the number L?

b) Suppose the following statement is true:

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})[n \geq N \Longrightarrow |x_n - L| < 100\varepsilon]$$

Is it true that $x_n \to L$?

c) Write in symbols and in words what it means to say that a sequence does not converge to a number L.

d) Informally, what does it mean to say that a sequence does not converge to a given number L?

(a) All the terms sufficiently far out in the sequence (i.e. for n sufficiently large) get arbitrarily close to L.

(b) Yes. As ε varies over all positive real numbers, so does $100\,\varepsilon$.

(c) $(\exists \varepsilon > 0)(\forall N \in \mathbb{N})(\exists n \in \mathbb{N})[n \geq N$ and $|x_n - L| \geq \varepsilon]$

$\sim[P \Rightarrow Q] = P \wedge \sim Q$

Exercise 1.2.2.

Consider the statement $1/n \to 0$.

(i) Intuitively why do you believe it is true?

(ii) Write a proof that it is true.

(i) If n gets very large, $\frac{1}{n}$ gets very close to 0.

must prove this:

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})\left[n \geq N \implies \left|\frac{1}{n}\right| < \varepsilon\right]$$

(ii) Pf Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$.

~~Let $n \in \mathbb{N}$.~~ Suppose $n \geq N$. Then

$$\left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

We've thus shown $\frac{1}{n} \to 0$. $\square$

Exercise 1.2.3.

Consider the statement $1/\sqrt{n} \to 0$.

(i) Intuitively why do you believe it is true?

(ii) Write a proof that it is true.

(i) If n gets really large, so does $\sqrt{n}$, and thus we expect $\frac{1}{\sqrt{n}}$ to get arbitrarily close to 0.

(ii) [Show $(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \in \mathbb{N})\left(n \geq N \Rightarrow \left|\frac{1}{\sqrt{n}}\right| < \varepsilon\right)$

- work backwards from $\frac{1}{\sqrt{n}} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < \sqrt{n}$

$\Leftrightarrow \frac{1}{\varepsilon^2} < n.$

So we want $N > \frac{1}{\varepsilon^2}$]

motivation

Proof. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon^2}$.

Let $n \in \mathbb{N}$. Suppose $n \geq N$. Then

$$\left|\frac{1}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \varepsilon. \qquad \square$$

Exercise.

Consider the statement that a convergent sequence has a unique number to which it converges.

a) Intuitively why do you believe this is true?

b) Write a proof that it is true.

(a) If the terms of a sequence ultimately get close to a number, they can't get close to a different number.

(b) Say the sequence $\{x_n\}_{n=1}^{\infty}$ converges to L_1 and to L_2. We will show $L_1 = L_2$. We do it by showing $|L_1 - L_2| = 0$.

Let $\varepsilon > 0$. Since $x_n \to L_1$, we know there exists $N_1 \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N_1$ then $|x_n - L_1| < \frac{\varepsilon}{2}$. Since $x_n \to L_2$,

we know there exists $N_2 \in \mathbb{N}$ s.t. for all $n \in \mathbb{N}$, if $n \geq N_2$ then $|x_n - L_2| < \frac{\varepsilon}{2}$.

Let $n \in \mathbb{N}$ s.t. $n \geq N_1$ and $n \geq N_2$. Then

$$|L_1 - L_2| = |(L_1 - x_n) + (x_n - L_2)| \leq |L_1 - x_n| + |x_n - L_2|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $|L_1 - L_2| = 0$, i.e. $L_1 = L_2$. $\square$

Exercise.

Fix r such that $0 < r < 1$. Consider the statement that $r^n \to 0$.

a) Intuitively why do you believe the statement is true?

b) Write a proof that it is true. Make use of properties of the natural logarithm and exponential functions (even though you haven't yet been given rigorous definitions).

c) Since we haven't rigorously developed the definition and properties of the log and exponential functions, we should try to write a proof that doesn't make use of them. So write a proof that $r^n \to 0$ which does not make use of logs or exponentials.

(a) Since $0<r<1$ the powers of r get smaller and smaller.

(b) Motivation. $(\forall \varepsilon>0)(\exists N\in\mathbb{N})(\forall n\in\mathbb{N})(n\geq N \Rightarrow |r^n|<\varepsilon)$.

- work backwards from

negative

$r^n<\varepsilon$.

$n\log r = \log r^n < \log\varepsilon$

$n > \frac{\log\varepsilon}{\log r}$

So we want $N > \frac{\log\varepsilon}{\log r}$.

Proof Let $\varepsilon>0$. Choose $N\in\mathbb{N}$ s.t. $N>\frac{\log\varepsilon}{\log r}$. Let $n\in\mathbb{N}$. Suppose $n\geq N$. Then $\ln r<0$, we

$n\geq N>\frac{\log\varepsilon}{\log r}$.

Since $0 < r < 1$, it follows $\log r < 0$, so

$$\log r^n = n \log r < \log \varepsilon.$$

Since log is an increasing function, we deduce

$$|r^n| = r^n < \varepsilon.$$

$\square$

ⓒ **Motivation**

- $0 < r < 1$
  $\Rightarrow \frac{1}{r} > 1.$
- Put $y = \frac{1}{r} - 1.$
  Then $y > 0.$
- $\frac{1}{r} = 1 + y \Rightarrow r = \frac{1}{1+y}.$

$\therefore$ $r^n = \frac{1}{(1+y)^n}$

So $r^n \to 0 \Longleftrightarrow (1+y)^n \to \infty.$

Binomial Thm

$$(1+y)^n = 1 + \binom{n}{1} y + \binom{n}{2} y^2 + \binom{n}{3} y^3 + \cdots + \binom{n}{n} y^n$$

where $\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$

Note $\binom{n}{1} = \frac{n!}{1!\,(n-1)!} = n$.

$(1+y)^n > ny$

want $r^n < \varepsilon \iff \frac{1}{(1+y)^n} < \varepsilon$

$\iff (1+y)^n > \frac{1}{\varepsilon}$

So we're ok if $ny > \frac{1}{\varepsilon}$

So choose $N > \frac{1}{\varepsilon y}$

© Proof Let $\varepsilon > 0$. Define $y = \frac{1}{r} - 1$. Since $0 < r < 1$ it follows $y > 0$. Choose $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon y}$. Let $n \in \mathbb{N}$. Suppose $n \geq N$.

Then

$$\frac{1}{r^n} = (1+y)^n = \sum_{j=0}^{n} \binom{n}{j} y^j \quad \text{(binomial theorem)}$$

$$> \binom{n}{1} y$$

$$= ny$$

$$\geq Ny$$

$$> \frac{1}{\varepsilon})$$

So $|r^n| < \varepsilon$. $\square$