1.1 The Real Number system

- To get a deeper understanding of calculus, we need a precise understanding of what is the set of real numbers.
- We define the real numbers to be a set satisfying a specific set of axioms.
- First we define what is an "Archimedean Ordered Field".
- There is more than one number system that is an Archimedean ordered field, but after adding one more axiom, the Completeness Axiom, we will have a set that in some sense uniquely satisfies all of the axioms, **and we will call that set the real numbers**.

Archimedean Ordered Field

An Archimedean Ordered Field is any set $\mathbb{F}$ with two binary operations $+$ and $\cdot$ and an order relation $<$ satisfying the following ten properties:

(1) *Closure*:

- (i) $(\forall a, b \in \mathbb{F})[a + b \in \mathbb{F}]$
- (ii) $(\forall a, b \in \mathbb{F})[a \cdot b \in \mathbb{F}]$

(2) *Commutativity*:

- (i) $(\forall a, b \in \mathbb{F})[a + b = b + a]$
- (ii) $(\forall a, b \in \mathbb{F})[a \cdot b = b \cdot a]$

(3) *Associativity*:

- (i) $(\forall a, b, c \in \mathbb{F})[a + (b + c) = (a + b) + c]$
- (ii) $(\forall a, b, c \in \mathbb{F})[a \cdot (b \cdot c) = (a \cdot b) \cdot c]$

(4) *Distributivity*: $(\forall a, b, c \in \mathbb{F})[a \cdot (b + c) = (a \cdot b) + (a \cdot c)]$

(5) *Identity*: $(\exists 0, 1 \in \mathbb{F})[(0 \neq 1) \wedge ((\forall a \in \mathbb{F})(a + 0 = a \text{ and } a \cdot 1 = a))]$

(6) *Inverses*:

- (i) (Additive inverses) $(\forall a \in \mathbb{F})[(\exists - a \in \mathbb{F})[a + (-a) = 0]$
- (ii) (Multiplicative inverse) $(\forall a \in \mathbb{F})[a \neq 0 \Rightarrow (\exists a^{-1} \in \mathbb{F})[a \cdot a^{-1} = 1])]$

(7) *Transitivity*: $(\forall a, b, c \in \mathbb{F})[(a < b \text{ and } b < c) \Rightarrow a < c]$

(8) *Preservation of Order*:

- (i) $(\forall a, b, c \in \mathbb{F})[a < b \Rightarrow a + c < b + c]$
- (ii) $(\forall a, b, c \in \mathbb{F})[(a < b \text{ and } c > 0) \Rightarrow a \cdot c < b \cdot c]$

(9) *Trichotomy*: $(\forall a, b \in \mathbb{F})[$ exactly one of the following occurs: $a < b, a = b, a > b]$

(10) *Archimedean Property*: Define $\mathbb{N}$ to be the smallest subset of $\mathbb{F}$ such that $1 \in \mathbb{F}$ and is closed under $+$. Then $(\forall \varepsilon > 0)(\forall M > 0)(\exists n \in \mathbb{N})[n\varepsilon > M]$

Exercise

Which of the above axioms are satisfied by each of the following sets (with the usual operations of addition and multiplication)?

- the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \dots\}$
- the set of integers, $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$
- the set of rational numbers, $\mathbb{Q} = \{m/n : m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\}\}$
- the set of irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$
- the set of complex numbers, $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$
- the set of real numbers, $\mathbb{R}$

Exercise

Is there more than one example of an Archimedian Ordered Field?

- $\mathbb{N}$: All except the axioms with 0 and the existence of additive and multiplicative inverses.

- $\mathbb{Z}$: All except the axioms with multiplicative inverses
- $\mathbb{Q}$: All the axioms hold.
- $\mathbb{R} \setminus \mathbb{Q}$: - Don't have either closure axiom
  (e.g. $\sqrt{2} + (-\sqrt{2}) = 0 \notin \mathbb{R} \setminus \mathbb{Q}$; $\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1 \notin \mathbb{R} \setminus \mathbb{Q}$

- Don't have 0 or 1 in $\mathbb{R}-\mathbb{Q}$.
- Without closure, the associative laws have no meaning.

$\underline{\mathbb{C}}$: All the "field laws" ①–⑥, but the order properties do not. We don't define "$<$" at all for complex numbers.

$\underline{\mathbb{R}}$: All the axioms hold.

Both $\mathbb{R}$ and $\mathbb{Q}$ are examples of Archimedean ordered fields.

Exercise

There are lots of familiar properties of arithmetic which don't appear in Axioms 1-10 of an Archimedean Ordered Field. Prove that each of the following properties hold in any field.

(f-i) Uniqueness of the identity elements 0 and 1.

(f-ii) Uniqueness of additive and multiplicative inverses.

(f-iii) $(\forall a \in \mathbb{F})[a \cdot 0 = 0]$

(f-iv) $(\forall a, b, c \in \mathbb{F})[a \neq 0 \text{ and } b \neq 0 \Rightarrow a \cdot b \neq 0]$

(f-v) $(\forall a \in \mathbb{F})[-1 \cdot a = -a]$

(f-vi) $(\forall a \in \mathbb{F})[-(-a) = a]$

(f-vii) $(\forall a, b \in \mathbb{F})[(-a) \cdot (-b) = a \cdot b]$ *(Hint: Begin by proving that $(-1) \cdot (-1) = 1$.)*

(f-viii) $(\forall a, b \in \mathbb{F})[-(a \cdot b) = (-a) \cdot b = a \cdot (-b)]$

Exercise

Prove that the following properties hold in any ordered field.

(o-i) $(\forall a, b, c, d \in \mathbb{F})[a < b \text{ and } c < d \Longrightarrow a + c < b + d]$

(o-ii) $(\forall a, b \in \mathbb{F})[a < b \Longleftrightarrow -b < -a]$

(o-iii) $(\forall a \in \mathbb{F})[a > 0 \Longleftrightarrow -a < 0]$

(o-iv) $(\forall a, b, c \in \mathbb{F})[a < b \text{ and } c < 0 \Longrightarrow a \cdot c > b \cdot c]$

(o-v) $(\forall a \in \mathbb{F} \setminus \{0\} : a^2 > 0]$

(o-vi) $1 > 0$

(o-vii) $(\forall a, b \in \mathbb{F} \setminus \{0\})[a < 0 \text{ and } b > 0 \Longrightarrow ab < 0]$

(o-viii) $(\forall a, b \in \mathbb{F} \setminus \{0\})[a < 0 \text{ and } b < 0 \Longrightarrow ab > 0$

Exercise

Let $\varepsilon > 0$ be a given positive number in an Archimedean ordered field $\mathbb{F}$. Prove that there exists $n \in \mathbb{N}$ such that $0 < 1/n < \varepsilon$.

We apply the Archimedean axiom with the given ε and $M = 1$. In an earlier exercise we showed $1 > 0$. Thus there exists $n \in \mathbb{N}$ such that

$$n\varepsilon > 1.$$

We claim $n^{-1} > 0$. If not, then $n^{-1} < 0$ or $n^{-1} = 0$. If $n^{-1} = 0$, then $1 = n \cdot n^{-1} = n \cdot 0 = 0$, contradicting that $1 \neq 0$. If $n^{-1} < 0$, then $1 = n \cdot n^{-1} < 0$, contradicting the exercise showing $1 > 0$. Thus $n^{-1} > 0$, so

$$\varepsilon = n^{-1}(n\varepsilon) > n^{-1}.$$

This proves $0 < \frac{1}{n} < \varepsilon$.

$\square$

Definition (Absolute value)

Let a be an element of an Archimedean ordered field $\mathbb{F}$. Then $|a|$, the absolute value of a, is defined by

$$|a| = \begin{cases} a & \text{if a } \geq 0 \\ -a & \text{if a } < 0 \end{cases}$$

Exercise

Prove that for all $a \in \mathbb{F}$ each of the following hold:

a) $|a| = |-a|$.

b) $a \leq |a|$

(a) If $a \geq 0$, then $|a| = a$; in this case $-a \leq 0$, so $|-a| = -(-a) = a = |a|$.

If $a < 0$, then $|a| = -a$; in this case

$-a > 0$, so $|-a| = -a$. Thus $|a| = |-a|$.

(b) Case 1: $a \geq 0$: Then $|a| = a$

Case 2: $a < 0$: Then $|a| = -a > 0 > a$;

i.e. $|a| > a$.

Exercise

Let $x \in \mathbb{F}$ such that the following statement is true:

$$(\forall \varepsilon > 0)[|x| < \varepsilon].$$

Prove that $x = 0$.

We argue by contradiction. Suppose $x \neq 0$. Then $|x| > 0$. Thus taking $\varepsilon = |x|$ we have $\varepsilon > 0$ and $\varepsilon = |x| \leq |x|$. This is a contradiction. Thus $x = 0$.

$\square$

The Triangle Inequality

Theorem (Triangle Inequality) For any two elements a, b of an Archimedean Ordered Field $\mathbb{F}$, we have

$$|a + b| \leq |a| + |b|.$$

Exercise

Prove the triangle inequality.

Pf case 1: $a+b \geq 0$.

Then $|a+b| = a+b \leq |a| + |b|$.

case 2 $a+b < 0$.

Then $|a+b| = -(a+b) = (-a) + (-b)$

$\leq |-a| + |-b|$

$= |a| + |b|$. $\square$

Exercise

Let a, b be any elements of an Archimedean Ordered Field $\mathbb{F}$. Deduce each of the following from the triangle inequality.

a) $|a - b| \le |a| + |b|$

b) (reverse triangle inequality) $|a - b| \ge |a| - |b|$ and $|a + b| \ge |a| - |b|$

(a) $|a-b| = |a+(-b)| \le |a| + |-b| = |a| + |b|$

(b) $|a| = |(a-b)+b| \le |a-b| + |b|$

which is same as $|a-b| \geq |a| - |b|$.

other is similar