Midterm Exam #1—Math 101, Section 205
January 30, 2015
Duration: 50 minutes

Name: ____________________________ Student Number: ____________________________

Do not open this test until instructed to do so! This exam should have 8 pages, including this cover sheet. No textbooks, notes, calculators, or other aids are allowed; phones, pencil cases, and other extraneous items cannot be on your desk. Turn off cell phones and anything that could make noise during the exam.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work. Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked. Continue on the back of the page if you run out of space.

UBC rules governing examinations:

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behavior be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (a) speaking or communicating with other examination candidates, unless otherwise authorized;
   (b) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (c) purposely viewing the written papers of other examination candidates;
   (d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)—(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Problem	**Out of**	**Score**	**Problem**	**Out of**	**Score**
1	6		5	8	
2	6		6	8	
3	6		7	8	
4	3		**Total**	45	

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

1a. **[3 pts]** Using 3 rectangles and right endpoints, approximate the area under the curve $y = x^2 + 1$ from $x = 0$ to $x = 3$. *Please simplify your answer.*

Solution: The interval has length 3 and since we are using 3 rectangles, $\Delta x = 1$ (**1 pt**). Therefore, the approximation with right endpoints is

$$\begin{aligned} ((f(x_1) + f(x_2) + f(x_3)))\,\Delta x &= f(1) + f(2) + f(3) \\ &= 2 + 5 + 10 = 17. \end{aligned}$$

(**2 pts**, one for identifying the values of x_i and the other for the answer).

1b. **[3 pts]** Determine by how much using 3 rectangles and right endpoints to approximate the area under the curve $y = x^2 + 1$ from $x = 0$ to $x = 3$ over approximates the actual area. *Please simplify your answer.*

Solution:

The actual area is given by the integral

$$\int_0^3 x^2 + 1\,\mathrm{d}x = \frac{x^3}{3}\Big|_0^3 + 3 = 12,$$

hence the approximation over approximates by $17 - 12 = 5$.

1 pt for realizing the answer is the difference between $\int_0^3 x^2 + 1\,\mathrm{d}x$ and the answer from $1a$).

1 pt for the correct antiderivative.

1 pt for the correct answer.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

2a. **[3 pts]** Calculate the following limit:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sqrt{3}}{n} \frac{1}{1+3\frac{i^2}{n^2}}$$

Please simplify your answer.

Solution:

This limit is an expression of the following definite integral:

$$\int_0^{\sqrt{3}} \frac{1}{1+x^2}\, \mathrm{d}x = \tan^{-1}\left(\sqrt{3}\right) - 0 = \frac{\pi}{3}.$$

1 pt for identifying the answer as the correct integral
1 pt for the correct antiderivative.
1 pt for the correct answer.
If endpoints are incorrect, but rest is OK award **2 pts** total.

2b. **[3 pts]** Calculate the derivative of the following function of x

$$f(x) = \int_{\cos(x)}^{\sin(x)} \mathrm{e}^t\, \mathrm{d}t$$

Solution:
e^t is its own antiderivative, therefore we have

$$f(x) = e^{\sin(x)} - e^{\cos(x)}.$$

(**1 pt** for correctly identifying this formula) By the chain rule (**1 pt** for invoking the chain rule), we then have its derivative is:

$$e^{\sin(x)}\cos(x) + e^{\cos(x)}\sin(x).$$

(**1 pt** for correct answer). Alternatively, one can use the fundamental theorem of calculus together with the chain rule.

For this solution:
1 pt for writing

$$\int_{\cos(x)}^{\sin(x)} \mathrm{e}^t\, \mathrm{d}t = \int_0^{\sin(x)} \mathrm{e}^t\, \mathrm{d}t - \int_0^{\cos(x)} \mathrm{e}^t\, \mathrm{d}t$$

1 pt for invoking the the fundamental theorem of calculus and the chain rule
1 pt for the correct answer.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

3a. **[3 pts]** Calculate the following definite integral

$$\int_{-1}^{1} \sin\left(\sin\left(\sin\left(x\right)\right)\right)\ \mathrm{d}x.$$

Solution:
The integrand is odd and the interval is symmetric about the x-axis, so the integral is zero. (**1 pt** for mentioning the integrand is odd, if this is not mentioned but a graphical argument is given, award a total of **2 pts**).

3b. **[3 pts]** Calculate the following indefinite integral:

$$\int \frac{1}{x} \cos\left(\ln|x|\right)\ \mathrm{d}x.$$

Solution:
We can use u-substitution with $u = \ln|x|$ (**1 pt**). We have that $du = \frac{1}{x}dx$ (**1 pt**). Hence

$$\begin{aligned} \int \frac{1}{x}\cos\left(\ln|x|\right)\ \mathrm{d}x &= \int \cos\left(u\right)\ \mathrm{d}u \\ &= \sin\left(u\right) + C \\ &= \sin\left(\ln|x|\right) + C. \end{aligned}$$

(**1 pt**)- subtract a point if no constant added.

Problems 1–4 are short-answer questions: put a box around your final answer, but no credit will be given for the answer without the correct accompanying work.

4. **[3 pts]** A force of 10 lb is required to hold a spring stretched 4 in.= $\frac{1}{3}$ ft. beyond its natural length. Assuming that the spring obeys Hooke's law, how much *work* is done in stretching it from its natural length to 6 in.= $\frac{1}{2}$ ft. beyond its natural length? **Please give the answer in foot pounds and simplify your answer.**

Solution:

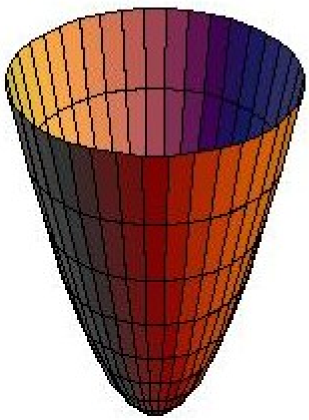
By Hooke's law the force necessary to stretch the spring x units past its natural length is kx for some constant. From the data given, it takes $10lb$ of force to stretch the spring $4in.$ therefore $k = \frac{5}{2}lb/in = 30lb/ft$ (**1 pt**). The total work done in stretching the spring to $6in. = \frac{1}{2}ft.$ past its natural length is hence

$$\int_0^{\frac{1}{2}} 30x \, \mathrm{d}x = 15x^2|_0^{\frac{1}{2}} = \frac{15}{4} ft.lb.$$

(**1 pt** for the correct integral including end points, **1 pt** for the correct answer).

Problems 5–7 are long-answer: give complete arguments and explanations for all your calculations—answers without justifications will not be marked.

5. **[8 pts]** An underground container has the same shape as the surface obtained by rotating the portion of the curve $y = 2x^2$ between $x = 0$ to $x = 1$ about the y-axis, where one unit is one meter. Assume that the ground is perpendicular to the y-axis and the very top of the container is at ground level and has width 2 meters. Suppose that the container is completely full with a liquid of constant density $600kg/m^3$. Calculate the total mass of the liquid contained in the container. *Please simplify your answer.*



Solution:

To calculate the mass, we can calculate the volume of the container and multiply it by the density (**1 pt**). The volume is the volume of the solid obtained by rotating the portion of the curve $y = 2x^2$ between $x = -1$ to $x = 1$ about the y-axis. The cross section at a certain height y is given by a disk of radius $x = \sqrt{\frac{y}{2}}$ and hence has area $A(y) = \pi x^2 = \frac{\pi y}{2}$ (**1 pt** for taking cross-sections with respect to y, **1 pt** for realizing the cross sectional area is a circle, and **1 pt** for the correct radius). Since the height of the container is $2 \cdot 1^2 = 2$ meters, the volume in cubic meters is therefore

$$\int_0^2 \frac{\pi y}{2}\,\mathrm{d}y = \pi \frac{y^2}{4}\Big|_0^2 = \pi.$$

(**1 pt** for the correct form of the integral, **1 pt** the correct end points, **1 pt** for the correct antiderivative). The mass is therefore 600π kg (**1 pt** for correct answer).

6. **[8 pts]** Calculate the total work necessary to pump the liquid out of the top of the container from problem 5. To simplify your calculations, you can assume that the acceleration due to gravity is constant at $g = 10m/s^2$ rather than $9.8m/s^2$. *Please simplify your answer.*

Solution:

We can approximate the work done by approximating the container by n circular cylinders of height Δy (**1 pt** for this approach) . The i^{th} cylinder is at a height y_i has volume $A(y_i)\Delta y = \frac{\pi y_i}{2}\Delta y$ (**1 pt**) and hence mass $600\frac{\pi y_i}{2}\Delta y$ kg. (**1 pt** for realizing mass is density×volume). The i^{th} such cylinder must be moved a distance $2 - y_i$ meters by applying a constant force of $600\frac{\pi y_i}{2}\Delta y \cdot g$ (**1 pt** for realizing force is mass×g) which with $g = 10m/s^2$ becomes $3000\pi y_i \Delta y$ Newtons. It follows that the total work done in Newton meters is approximately

$$\sum_{i=1}^{n} 3000\pi y_i \Delta y\,(2 - y_i)$$

(**1 pt** for realizing that with constant force, work is force×distance) which in the limit that $n \to \infty$ becomes

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n} 3000\pi y_i \Delta y\,(2 - y_i) &= \int_0^2 3000\pi y\,(2-y)\,\mathrm{d}y \\ &= 2940\pi y^2|_0^2 - 1000\pi y^3|_0^2 \\ &= 12000\pi - 8000\pi = 4000\pi \text{ Nm.}\end{aligned}$$

(**1 pt** for writing the correct integral, **1 pt** for the correct antiderivative, **1 pt** for the correct answer).

7.

(a) **[4 pts]** Show that the indefinite integral of the function $f(x) = x\cos(x)$ is

$$\int x\cos(x)\,\mathrm{d}x = x\sin(x) + \cos(x) + C.$$

Solution:

By the Fundamental Theorem of Calculus, we can prove this by differentiating the function $F(x) = x\sin(x) + \cos(x) + C$ (**1 pt**). By the product rule (**1 pt**)we have

$$F'(x) = x\cos(x) + \sin(x) - \sin(x) = x\cos(x).$$

It follows that $F(x)$ is an antiderivative for $f(x)$, which is what we wanted to show (**2 pts**).

If using integration by parts instead:

1 pt $u = x$, $dv = \cos(x)\,dx$
1 pt $v = \sin(x)$
1 pt for $\int udv = uv - \int vdu$
1 pt for the correct answer.

(b) **[4 pts]** Calculate the area enclosed by the curves $y = \cos(x)$ and $y = x\cos(x)$ between $x = 0$ and $x = 1$. You can use the result from part a) even if you have not solved it. *You do not need to evaluate the trigonometric functions involved on non-zero angles, but otherwise simplify your answer.*

Solution:

On the interval $[0,1]$, $x < 1$ and $\cos(x) > 0$, hence

$$\cos(x) \geq x\cos(x)$$

(**2 pts** for this justification, or a reasonably correct graph). Therefore, the area enclosed by the curves is given by

$$\begin{aligned}\int_0^1 (\cos(x) - x\cos(x))\,\mathrm{d}x &= (\sin(x) - x\sin(x) - \cos(x))\,\Big|_0^1 \\ &= 1 - \cos(1).\end{aligned}$$

(**1 pt** for the correct form of the integral, **1 pt** for the correct answer. If no justification is given for why $\cos(x) \geq x\cos(x)$ on the interval, but rest is correct, award a total of **3pts**.)