Outer Approximation of the Spectrum of a Fractal Laplacian

Tyrus Berry Steven M. Heilman Robert S. Strichartz[1]

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Abstract

We present a new method to approximate the Neumann spectrum of a Laplacian on a fractal K in the plane as a renormalized limit of the Neumann spectra of the standard Laplacian on a sequence of domains that approximate K from the outside. The method allows a numerical approximation of eigenvalues and eigenfunctions for lower portions of the spectrum. We present experimental evidence that the method works by looking at examples where the spectrum of the fractal Laplacian is known (the unit interval and the Sierpinski Gasket (SG)). We also present a speculative description of the spectrum on the standard Sierpinski carpet (SC), where existence of a self-similar Laplacian is known, and also on nonsymmetric and random carpets and the octagasket, where existence of a self-similar Laplacian is not known. At present we have no explanation as to why the method should work. Nevertheless, we are able to prove some new results about the structure of the spectrum involving "miniaturization" of eigenfunctions that we discovered by examining the experimental results obtained using our method.

1 Introduction

Laplacians arise in many different mathematical contexts; three in particular that will interest us: manifolds, graphs and fractals. There are connections relating these different types of Laplacians. Manifold Laplacians may be obtained as limits of graph Laplacians for graphs arising from triangulations of the manifold ([Colin de Verdière 1998, Dodziuk and Patodi 1976]). Kigami's approach of construction Laplacians on certain fractals, such as the Sierpinski gasket (SG), also involves taking limits of graph Laplacians for graphs that approximate the fractal ([Kigami 2001, Strichartz 1999, Strichartz 2006]). In this paper we present another connection, where we approximate the fractal from without

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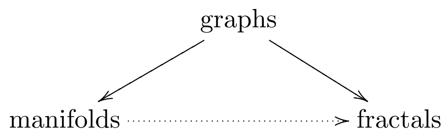
by planar domains, and attempt to capture spectral information about the fractal Laplacian from spectral information about the standard Laplacian on the domains. Thus we add an arrow to the diagram:



We should point out that the probabilistic approach to constructing Laplacians on fractals also involves approximating from without, but in that case it is the stochastic process generated by the Laplacian that is approximated, so it is not clear how to obtain spectral information.

We may describe our method succinctly as follows. Suppose we have a self-similar fractal K in the plane, determined by the identity

$$K = \bigcup F_i K \tag{1.1}$$

where $\{F_i\}$ is a finite set of contractive similarities (called an *iterated function system,* IFS). Choose a bounded open set Ω whose closure contains K, and form the sequence of domains

$$\begin{aligned} \Omega_0 &= \Omega \\ \Omega_m &= \bigcup F_i \Omega_{m-1} \qquad \text{for } m \geq 1 \end{aligned} \tag{1.2}$$

Consider the standard Laplacian Δ on Ω_m with Neumann boundary conditions (recall that such conditions make sense even for domains with rough boundary). Let $\{\lambda_n^{(m)}\}$ denote the eigenvalues in increasing order (repeated in case of nontrivial multiplicity) with eigenfunctions $\{u_n^{(m)}\}$ (L^2 normalized). So

$$-\Delta u_n^{(m)} = \lambda_n^{(m)} u_n^{(m)} \tag{1.3}$$

Of course $\lambda_0^m = 0$ with u_0^m constant. We then hope to find a renormalization factor r such that

$$\lim_{m\to\infty} r^m \lambda_n^{(m)} = \lambda_n \tag{1.4}$$

exists and

$$\lim_{m\to\infty} u_n^{(m)}|_K = u_n \tag{1.5}$$

exists. (We have to be careful in cases of nontrivial multiplicity, and we may have to adjust $u_n^{(m)}$ by a minus sign in general). If this is the case then we may simply define a self-adjoint operator Δ on K by

$$-\Delta u_n = \lambda_n u_n \tag{1.6}$$

Of course we would also like to identify Δ with a previously defined Laplacian, if such is possible, or at least show that Δ is a local operator satisfying some sort of self-similarity.

This may seem like wishful thinking, but it is not implausible. After all, many other types of structures on fractals can be obtained as limits of structures on Ω_m, so why not a Laplacian? After reading this paper, we hope the reader will agree that there is a lot of evidence that this method should work in many cases. We leave to the future the challenge of describing exactly when it works, and why.

We note one great advantage of our method: it not only approximates the Laplacian, but it gives information about the spectrum. Other methods of constructing Laplacians on fractals do not yield spectral information directly. Of course, not all spectral information is immediately available. In particular, asymptotic information must be lost, since we know from Weyl's law that $\lambda_n^{(m)} = O(n)$ for each fixed m, but for fractals Laplacians this is not the case. This means, in particular, that the limit (1.4) is not uniform in n. To get information about λ_n for large n requires taking a large value for m. In practice, our numerical calculations get stuck around $m = 4$. So we only see an approximation to a segment at the bottom of the spectrum. But this is already enough to reveal aspects of the spectrum that are provable. Briefly, if the fractal has a nontrivial finite group of symmetries, then every Neumann eigenfunction can be miniaturized, and so there is an eigenvalue renormalization factor R such that if λ is an eigenvalue then so is $R\lambda$. The argument for this works for the approximating domains and also for a self-similar Laplacian on the fractal. (In fact the argument could be presented on the fractal alone, so its validity is independent of the validity of the outer approximation method, but in fact it was discovered by examining the experimental data!)

So what is the evidence for the validity of the outer approximation method? First we show that it works for the case when K is the unit interval (embedded in the x-axis in the plane). In this case we can take $F_0(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right)$ and $F_1(x,y) = \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right)$. If we take Ω to be the unit square, then we can compute the spectra of Ω_m (rectangles) and verify everything by hand ($r = 1$ in this case). We do this in section 2, where we also look at different choices of Ω, producing sawtooth shaped domains, whose spectra are computed numerically.

In section 3 we look at the case of SG, where the spectrum is known exactly. Here we see numerically how the spectra of the approximating domains approaches the known spectra. This computation shows that the accuracy falls off rapidly as n increases. We are also able to compare the eigenfunctions of the approximating domains with the known eigenfunctions on SG. In this case it is natural to take Ω to be a triangle containing SG in its interior since this yields connected domains Ω_m. We examine how the size of the overlap influences the spectra. After the work reported in section 3 was completed, a different approach to outer approximation on SG was studied in [Blasiak et al. 2008]. In particular, different methods for choosing approximating domains are used, and a whole family of different Laplacians are studied.

In section 4 we examine numerical data for some fractals for which very little had been known about the spectrum of the Laplacian, and in some cases where even the existence of a Laplacian is unknown. These examples fall outside of the postcritically finite (PCF) category defined in [Kigami 2001]. The first example is the standard Sierpinski carpet SC (cut out the middle square in tic-tac-toe and iterate). Here it is known that a self-similar Laplacian exists [Barlow 1995], but the construction is indirect, and uniqueness is not known. (After this work was completed, uniqueness was established in [Barlow et al. 2008].) But we also examine some nonsymmetric variants of SC for which the existence of a Laplacian is unknown. We also examine a symmetric fractal, the octagasket, where existence of a Laplacian is unknown. In all cases the spectra of the approximating regions appear to converge when appropriately renormalized. We can identify features of the spectrum, such as multiple eigenvalues, and eigenvalue renormalization factors R, and we produce rough graphs of eigenfunctions on the fractal. In particular, there is no discernible difference between the behavior in the case of the standard SC and the other examples.

In section 5 we describe the miniaturization process that produces the eigenvalue renormalization factor. For this to work we need a dihedral group of symmetries of the fractal. We only deal with the examples at hand, but it is clear that it works quite generally (we also explain how it works on the square). For the approximating regions, this shows how $R'\lambda_n^{(m)}$ shows up in the spectrum on Ω_{m+1} (the factor R' is not the same as R).

In section 6 we examine numerical data of randomly constructed variants of SC, where the existence of Laplacians is unknown . To make these carpets, we modify the construction of SC. We fix the number of squares cut out at each recursive step, but we randomly determine which squares are removed. Then, we achieve connected domains Ω_m with a suitable change to the above algorithm and properly chosen parameters. Here we again see convergence of normalized eigenvalues. These random carpets are related to the Mandelbrot percolation process. See [Chayes et al. 1988] and [Broman and Camia 2008], for example.

How do we compute the spectrum of the Laplacian on the approximating domain? We use a finite element method solver in Matlab, Matlab's own `pdeeig` function. To do this we only need to describe the geometry of the polygonal domain Ω_m. Then we either choose a triangulation (exclusive to Section 6) or let Matlab's triangulation functions `decsg` and `initmesh` produce a triangulation and then use piecewise linear splines in the finite element method. Note that it would be preferable to use higher-order splines, at least piecewise cubic, since these increase accuracy dramatically for a fixed memory space and running time. As a concession, all of our triangulations may be further refined with the `refinemesh` function. The advantage of automating the triangulation is that it saves a tremendous amount of work; in particular it chooses nonregular triangulations that increase accuracy. The disadvantage is that the program usually does not pick a triangulation with the same symmetry as the domain. This means that the eigenspaces that have nontrivial multiplicity in the domain end up being split into clusters of eigenspaces with eigenvalues close but not quite equal. Since a lot of the structure of the spectrum we are trying to observe has to do with multiplicities, this forces us to make ad hoc judgements as to when we have close but unequal eigenvalues, versus multiple eigenvalues.

Why do we deal exclusively with Neumann spectra? The main reason is that Neumann boundary conditions on the approximating domains appears to lead to Neumann boundary conditions for the Laplacian on the fractal in the case of the interval and SG, while at the same time Dirichlet boundary conditions on the approximating domains do not lead to Dirichlet boundary conditions for the Laplacian on the fractal. For example, in the case of the interval you would need to use a mix of Dirichlet and Neumann boundary conditions on different portions of the boundary. It is not at all clear what to do for other fractals. Indeed for SC it is not even clear what to choose for the boundary. The advantage of Neumann boundary conditions is that one can dispense with all notions of boundary, and define eigenfunctions simply as stationary points of the Rayleigh quotient with no boundary restrictions. All our programs, as well as further numerical data may be found on the websites `www.math.cornell.edu/~thb9d` [and `www.math.cornell.edu/~smh82`].

Finally, we note that [Kuchment and Zeng 2001] have studied similar outer approximations in the context of quantum graphs.

2 The Unit Interval

For the unit interval I with the second derivative as Laplacian, the Neumann eigenfunctions are $\cos n\pi x$ with eigenvalues $(\pi n)^2$. If we take Ω to be the unit square, then Ω_m is the rectangle $[0,1]\times[0,2^{-m}]$, with Neumann eigenfunctions $\cos n\pi x \cos 2^m k\pi y$ and eigenvalues $(\pi n)^2+(\pi 2^m k)^2$. If we restrict attention to a fixed bottom segment of the spectrum, we will only see eigenvalues with $k=0$ for m large enough (specifically, eigenvalues up to L provided $L \le (\pi 2^m)^2$). So $\lambda_n^{(m)} = \lambda_n$ exactly for large enough m. Of course the corresponding eigenfunctions restricted to the interval give the exact eigenfunctions of the Laplacian on the interval. Note that for each m there are many other eigenfunctions on Ω_m (those with $k \neq 0$), but they are "blown away" in the limit. A similar analysis holds if we start with Ω equal to any rectangle with sides parallel to the axes. Note that we do not have to renormalize the spectrum, or equivalently, we can take $r=1$ in (1.4).

We also note how other structures on I may be approximated from corresponding structures on Ω_m. For example, Lebesgue measure on I is the limit of Lebesgue measure on Ω_m suitably renormalized in the sense that

$$\lim_{m\to\infty} 2^m \iint_{\Omega_m} u(x,y)dxdy = \int_0^1 u(x,0)dx \tag{2.1}$$

if $u(x,y)$ is continuous on Ω (the result is independent of the continuous extension $u(x,y)$ to Ω of $u(x,0)$ on I). A similar result holds for energy, provided we use the minimum energy extension. In other words, given $f \in H^1(I)$, let u be the minimum energy function with $u(x,0)=f(x)$. Then

$$\lim_{m\to\infty} 2^m \iint_{\Omega_m} |\nabla u(x,y)|^2 \, dxdy = \int_0^1 |f'(x)|^2 \, dx \tag{2.2}$$

In order to see this we expand f in a Fourier cosine series

$$f(x) = \sum_{k=0}^{\infty} a_k \cos \pi n x \tag{2.3}$$

for which we have

$$\int_0^1 |f'(x)|^2 \, dx = \frac{1}{2}\sum_{k=1}^{\infty} (\pi k)^2 |a_k|^2 \tag{2.4}$$

The minimum energy extension to Ω_m is easily seen to be

$$u(x,y) = a_0 + \sum_{k=1}^{\infty} a_k \cos \pi k x \frac{\cosh 2\pi k(2^{-m}-y)}{\cosh \pi k 2^{-m}} \tag{2.5}$$

with

$$\int_{\Omega_m} |\nabla u(x,y)|^2 \, dxdy = \sum_{k=1}^{\infty} |a_k|^2 \pi k \left(\frac{\sinh 2\pi k 2^{-m}}{4\cosh^2 \pi k 2^{-m}} \right) \tag{2.6}$$

Then (2.2) follows from (2.4) and (2.6). Note that we obtain the same result if we use the simpler extension $u(x, y) = f(x)$, although this extension does not minimize energy. (The energy minimizing extension must be harmonic on the interior and satisfy Neumann boundary conditions on the portion of the boundary of Ω_m disjoint from I, and this explains (2.5)). We also have a bilinear version: let

$$\mathcal{E}_I(f, g) = \int_0^1 f'(x)g'(x)dx \tag{2.7}$$

and

$$\mathcal{E}_m(u, v) = \int_{\Omega_m} (\nabla u \cdot \nabla v)dxdy \tag{2.8}$$

If u_m and v_m denote the minimum energy extensions of f and g to Ω_m, then

$$\lim_{m\to\infty} 2^m \mathcal{E}_m(u_m, v_m) = \mathcal{E}_I(f, g) \tag{2.9}$$

We can use this to "define" a Laplacian on I via the weak formulation

$$\mathcal{E}_I(f, g) = -\int_0^1 f''(x)g(x)dx \tag{2.10}$$

if g vanishes at 0 and 1. By the usual Gauss-Green formula

$$\mathcal{E}_m(u_m, v_m) = \int_{\partial\Omega_m} (\partial_n u_m)v_m, \tag{2.11}$$

and $\partial_n u_m = 0$ on all of $\partial\Omega_m$ except I, where $\partial_n u_m = -\frac{\partial}{\partial y} u_m$, so

$$\mathcal{E}_m(u_m, v_m) = -\int_0^1 \left(\frac{\partial u_m}{\partial y}\right) g dx. \tag{2.12}$$

Combining (2.9), (2.10) and (2.12) yields at least formally

$$f''(x) = \lim_{m\to\infty} 2^m \frac{\partial u_m}{\partial y}(x, 0) \tag{2.13}$$

We can verify this by differentiating (2.5) directly (assuming f is smooth enough) to obtain

$$\frac{\partial u_m}{\partial y}(x, 0) = -\sum_{k=1}^{\infty} (\pi k)^2 a_k \cos \pi k x \left(\frac{\pi k \sinh 2\pi k 2^{-m}}{\cosh \pi k 2^{-m}}\right) \tag{2.14}$$

and taking the limit to obtain

$$\lim_{m\to\infty} 2^m \frac{\partial u_m}{\partial y}(x, 0) = -\sum_{k=1}^{\infty} (\pi k)^2 a_k \cos \pi k x, \tag{2.15}$$

and this is the same value for $f''(x)$ that we obtain by differentiating (2.3) directly.

For a less trivial example we need only to take a geometrically more interesting Ω. In particular, let Ω be a triangle with vertices $(-\epsilon, 0)$, $(1+\epsilon, 0)$ and $\left(\frac{1}{2}, h\right)$ for some choice of positive parameters ϵ and h. Then Ω_m is a sawtooth region with 2^m teeth, maximum height $2^{-m}h$ and overlaps of length $2^{-m}\epsilon$. It is not feasible to compute the Neumann spectrum of the Ω_m exactly, so we use numerical methods. In Tables 2.1 and 2.3 we present the eigenvalues for several choices of parameters and level $m = 2, 3, 4$ (we also vary the number of refinements used in the FEM approximation). Actually the computations are done for a similar image of Ω_m so that the base is exactly I, but this makes no difference in the limit. The evidence suggests that we get $c(\epsilon, h)\frac{d^2}{dx^2}$ in the limit for some constant that depends on the parameters.

In Tables 2.2 and 2.4 we present the same data, but we normalize by dividing $\lambda_n^{(m)}$ by $\lambda_1^{(m)}$. This enables us to compare the normalized eigenvalues with the expected values n^2. Note that with level $m = 5$ we see about a 1% deviation already at $n = 6$.

In Figure 2.1 we show some graphs of eigenfunctions on Ω_m, that approximate eigenfunctions on I. In Figure 2.2 we show the graph of an eigenfunction on Ω_2 that does not approximate an eigenfunction on I. Indeed, this eigenfunction appears to be almost localized to one of the teeth. We will discuss this phenomenon in a forthcoming paper. Unfortunately, we do not know if we can define energy on I via (2.2) for a sawtooth region approximation. Indeed, we have no idea what the minimum energy extension looks like.

Equilateral Triangles Sawtooth Region (height determined by requirement that triangles are equilateral, overlaps set to $(2^{-m})/10$)

Level:	2	2	2	2	3	3	3	3	4	4	4	4
Refinement:	1	2	3	4	1	2	3	4	1	2	3	4
n												
1	4.905	4.823	4.790	4.777	4.868	4.789	4.756	4.743	4.828	4.749	4.717	4.703
2	17.980	17.662	17.535	17.483	19.097	18.782	18.655	18.602	19.218	18.903	18.776	18.724
3	33.418	32.790	32.53	32.436	41.513	40.808	40.525	40.408	42.900	42.191	41.906	41.787
4	246.809	243.909	243.176	242.991	69.950	68.724	68.232	68.029	75.398	74.140	73.635	73.425
5	246.809	243.910	243.176	242.991	100.984	99.165	98.436	98.134	116.012	114.066	113.283	112.958
6	248.850	246.991	246.524	246.407	129.818	127.424	126.463	126.064	163.836	161.053	159.935	159.471
7	250.833	248.743	248.218	248.087	150.737	147.863	146.713	146.238	217.674	213.915	212.408	211.783
8	253.564	251.508	250.992	250.863	959.592	952.139	950.177	949.677	276.002	271.161	269.220	268.417
9	337.235	332.179	330.654	330.157	959.592	952.139	950.177	949.677	336.966	330.967	328.563	327.569
10	389.324	382.371	380.228	379.513	970.250	963.501	961.797	961.369	398.337	391.162	388.285	387.094
11	449.038	440.249	437.449	436.483	971.305	964.578	962.895	962.474	457.640	449.268	445.913	444.524
12	782.622	760.213	754.319	752.824	973.587	966.696	964.959	964.524	512.105	502.535	498.708	497.126
13	817.310	787.079	779.191	777.161	977.148	970.044	968.246	967.794	558.576	548.015	543.791	542.045
14	884.992	851.631	843.276	841.121	982.108	974.799	972.955	972.492	594.214	582.929	578.409	576.538
15	931.268	900.120	892.271	890.247	987.564	980.415	978.615	978.163	616.898	605.004	600.254	598.291
16	1022.576	984.632	974.972	972.543	991.919	985.129	983.418	982.989				
17	1022.620	984.636	974.973	972.544	1253.731	1237.690	1232.817	1231.219				
18	1023.447	995.768	988.646	986.841	1315.232	1297.342	1291.832	1290.003				
19	1042.893	1004.686	995.410	993.103	1407.013	1386.000	1379.452	1377.254				
20	1050.656	1012.066	1002.876	1000.611	1518.109	1493.410	1485.623	1482.974				
21	1269.022	1219.852	1206.751	1203.162	1636.329	1608.281	1599.289	1596.175				
22	1288.306	1235.661	1221.617	1217.694	1747.569	1716.437	1706.278	1702.698				
23	1304.295	1255.860	1242.652	1238.793	1833.745	1798.417	1786.867	1782.782				
24	1855.667	1749.382	1712.980	1703.675								
25	1878.164	1749.398	1712.983	1703.676								
26	1883.733	1766.473	1743.511	1737.727								
27	1883.965	1785.448	1761.326	1755.233								
28	1909.401	1821.564	1798.312	1792.391								
29		1948.299	1915.997	1907.606								

Table 2.1: Sawtooth Unnormalized Eigenvalues, built with Equilateral Triangles

There is yet another outer approximation approach to I, in which we regard

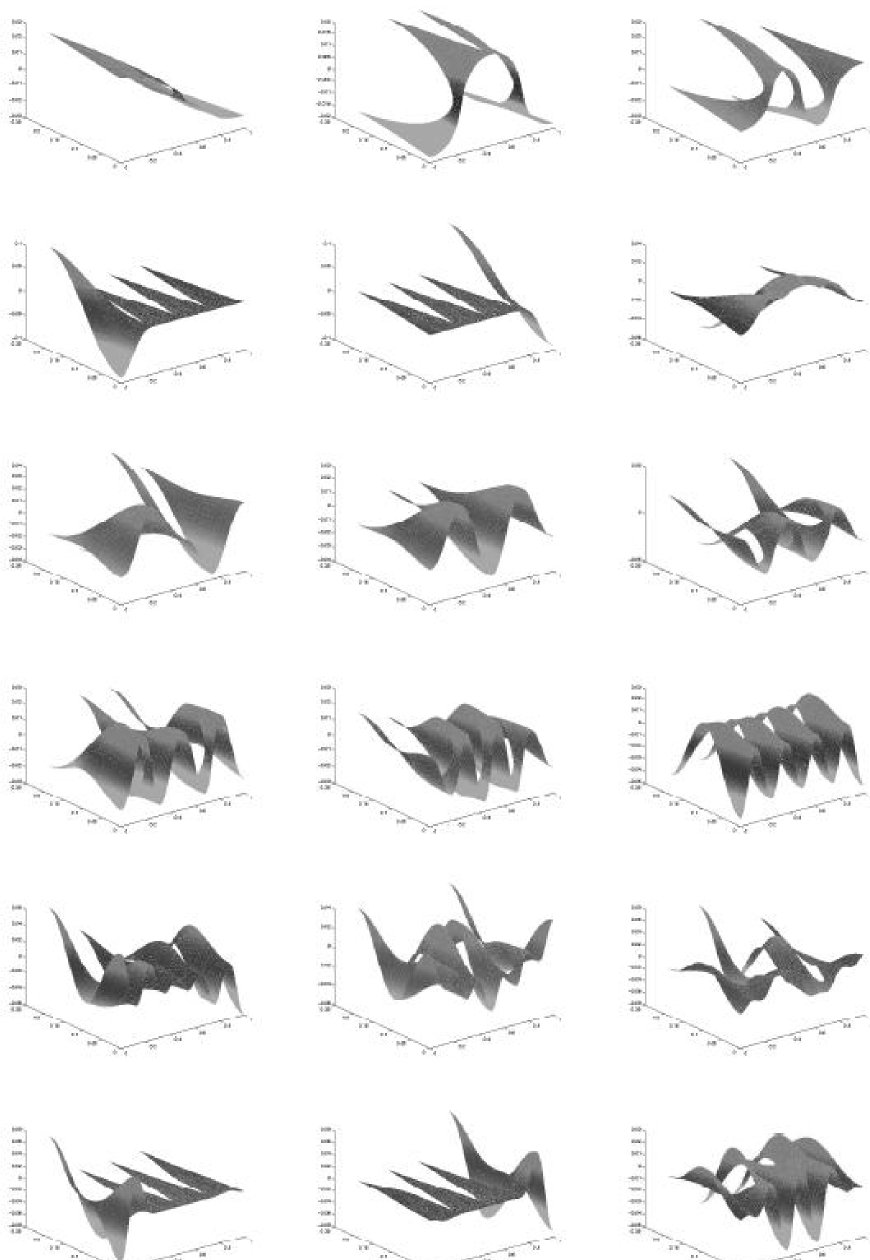


Figure 2.1: Sawtooth Eigenfunctions, m=2

Equilateral Triangles Sawtooth Region (height determined by requirement that triangles are equilateral, overlaps set to $(2^{-m})/10$)

Level:	2	2	2	2	3	3	3	3	4	4	4	4
Refinement:	1	2	3	4	1	2	3	4	1	2	3	4
n												
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	3.665	3.662	3.661	3.660	3.923	3.922	3.922	3.922	3.981	3.981	3.981	3.981
3	6.813	6.798	6.793	6.790	8.527	8.522	8.520	8.519	8.886	8.885	8.884	8.884
4	50.316	50.570	50.764	50.870	14.368	14.352	14.345	14.343	15.618	15.613	15.612	15.611
5	50.316	50.570	50.764	50.870	20.742	20.709	20.695	20.690	24.030	24.021	24.017	24.016
6	50.732	51.209	51.463	51.585	26.665	26.610	26.588	26.578	33.936	33.916	33.908	33.905
7	51.136	51.572	51.816	51.936	30.962	30.878	30.845	30.832	45.088	45.048	45.033	45.027
8	51.693	52.145	52.395	52.518	197.104	198.834	199.766	200.222	57.170	57.104	57.078	57.068
9	68.750	68.871	69.025	69.118	197.104	198.834	199.766	200.222	69.797	69.698	69.659	69.644
10	79.370	79.277	79.374	79.450	199.293	201.207	202.209	202.687	82.510	82.375	82.321	82.299
11	91.543	91.277	91.319	91.377	199.510	201.432	202.440	202.920	94.793	94.611	94.539	94.509
12	159.549	157.615	157.466	157.602	199.978	201.874	202.874	203.352	106.075	105.829	105.732	105.693
13	166.621	163.185	162.658	162.697	200.710	202.573	203.564	204.042	115.701	115.406	115.290	115.243
14	180.419	176.568	176.036	176.087	201.729	203.566	204.554	205.032	123.083	122.759	122.630	122.576
15	189.853	186.622	186.264	186.372	202.849	204.739	205.745	206.228	127.781	127.408	127.261	127.201
16	208.468	204.144	203.528	203.600	203.744	205.723	206.754	207.245				
17	208.477	204.144	203.528	203.600	257.521	258.466	259.188	259.580				
18	208.645	206.452	206.383	206.593	270.153	270.923	271.595	271.974				
19	212.610	208.301	207.795	207.904	289.006	289.437	290.017	290.369				
20	214.192	209.831	209.353	209.476	311.825	311.867	312.338	312.658				
21	258.709	252.912	251.913	251.880	336.108	335.856	336.235	336.525				
22	262.641	256.189	255.016	254.922	358.957	358.442	358.729	358.983				
23	265.900	260.377	259.407	259.339	376.658	375.562	375.672	375.867				
24	378.306	362.699	357.590	356.661								
25	382.892	362.702	357.590	356.661								
26	384.028	366.242	363.963	363.790								
27	384.075	370.176	367.682	367.455								
28	389.260	377.664	375.403	375.234								
29		403.940	399.970	399.354								

Table 2.2: Sawtooth Normalized Eigenvalues, built with Equilateral Triangles

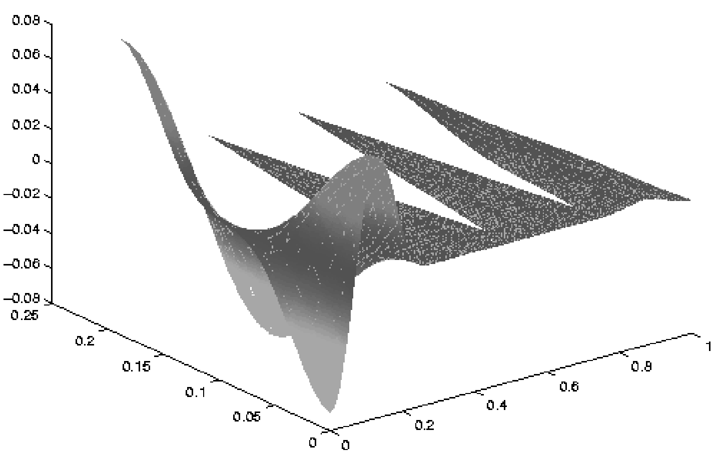


Figure 2.2: Almost Localized Sawtooth Eigenfunction, m=2

Eigenvalue Data for Sawtooth Regions with different Parameters

Level:	2	2	2	2	3	3	3	3	4	4	4	4	5	5	5	4	4	4	4	5
Height:	0.100	0.010	0.001	0.001	0.100	0.010	0.001	0.001	0.100	0.010	0.001	0.001	0.100	0.010	0.001	0.100	0.010	0.001	0.001	0.001
Refinement:	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3
n																				
1	6.256	7.155	7.550	8.017	5.093	6.996	7.229	7.326	3.836	6.782	7.060	7.155	2.867	6.341	6.981	3.795	6.768	7.011	7.061	6.931
2	23.467	26.916	29.096	31.996	20.023	27.616	28.606	29.123	15.210	27.041	28.154	28.543	11.388	25.345	27.902	15.046	26.987	27.957	28.160	27.704
3	45.256	52.154	56.017	60.827	43.675	60.656	63.040	64.508	33.701	60.510	63.022	63.921	25.325	56.948	62.698	33.341	60.387	62.567	63.035	62.251
4	294.835	327.674	331.873	333.511	73.960	103.746	108.178	111.071	58.625	106.725	111.207	112.857	44.286	101.047	111.258	58.004	106.506	110.372	111.227	110.461
5	319.781	356.983	368.346	377.480	107.475	152.707	159.642	163.925	89.037	164.994	172.017	174.690	67.757	157.493	173.423	88.094	164.649	170.667	172.045	172.175
6	388.559	435.688	465.781	504.639	139.164	200.594	210.056	214.981	123.691	234.348	244.478	248.465	95.054	226.090	248.988	122.397	233.848	242.467	244.511	247.185
7	478.443	535.783	582.431	651.190	162.497	237.152	248.566	253.227	161.172	313.478	327.260	332.860	125.431	306.595	337.691	159.508	312.794	324.432	327.290	335.229
8	696.381	851.228	1007.893	1106.376	1017.767	1277.229	1293.087	1295.630	199.921	400.647	418.579	426.082	158.038	398.709	439.215	197.867	399.751	414.783	418.596	435.987
9	766.306	930.457	1254.199	1439.666	1027.972	1304.205	1326.104	1331.648	238.235	493.580	516.082	525.745	192.104	502.074	553.181	235.820	492.443	511.178	516.074	549.076
10	890.920	1092.418	1677.158		1068.070	1393.716	1422.250	1432.226	274.559	589.334	616.709	628.727	226.653	616.269	679.138	271.805	587.934	610.581	616.660	674.045
11	1003.561	1266.292	1819.137		1121.065	1527.962	1568.308	1587.383	307.455	684.176	716.553	731.030	260.952	740.802	816.560	304.396	682.499	709.130	716.448	810.365
12	1557.709	1812.385			1175.215	1694.892	1751.443	1784.010	335.656	773.516	810.781	827.688	294.369	875.099	964.834	332.378	771.560	802.054	810.605	957.419
13	1561.255	1931.647			1220.465	1882.831	1958.367		358.389	851.985	893.700	912.842	326.301	1018.500	1123.252	354.918	849.767	883.758	893.449	1114.496
14	1608.660				1252.193				375.031	913.808	959.146	980.122	356.303	1170.244	1290.993	371.402	911.372	948.192	958.826	1280.776
15	1616.961				1270.474				385.025	953.571	1001.298	1023.492	384.206	1329.457	1467.118	381.364	950.989	989.668	1000.930	1455.315
16	1618.303				1277.740				1427.143				409.710	1495.142	1650.547	1425.511				1637.034
17	1865.746				1592.815				1429.230				432.971	1666.162	1840.045	1427.493				1824.701
18	1996.687				1610.143				1437.825				453.826	1841.225		1436.042				
19					1641.624				1452.086				472.532			1450.237				
20					1689.362				1472.750				489.103			1470.701				
21					1751.504				1500.282				503.710			1497.967				
22					1818.223				1535.318				516.571			1532.667				
23					1871.686				1578.626				527.762			1575.480				
24									1630.678				537.315			1626.922				
25									1691.729				545.583			1687.201				
26									1761.543				552.515			1755.965				
27									1838.629				558.237			1831.784				
28									1919.939				562.760			1911.655				
29													566.286			1990.465				
30													568.765							
31													570.211							
32													1466.956							
33													1469.178							

Table 2.3: Sawtooth Unnormalized Eigenvalues

Eigenvalue Data for Sawtooth Regions with different Parameters

Level:	2	2	2	2	3	3	3	3	4	4	4	4	5	5	5	4	4	4	4	5
Height:	0.100	0.010	0.001	0.001	0.100	0.010	0.001	0.001	0.100	0.010	0.001	0.001	0.100	0.010	0.001	0.100	0.010	0.001	0.001	0.001
Refinement:	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3
n																				
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	3.751	3.762	3.854	3.991	3.931	3.947	3.957	3.975	3.965	3.987	3.988	3.989	3.973	3.997	3.997	3.965	3.987	3.987	3.988	3.997
3	7.234	7.289	7.419	7.587	8.575	8.670	8.721	8.805	8.786	8.922	8.927	8.933	8.834	8.980	8.981	8.786	8.922	8.924	8.927	8.981
4	47.131	45.799	43.955	41.602	14.521	14.829	14.965	15.160	15.283	15.737	15.752	15.773	15.448	15.935	15.937	15.285	15.736	15.742	15.752	15.936
5	51.119	49.895	48.786	47.086	21.102	21.827	22.084	22.375	23.211	24.328	24.366	24.414	23.636	24.836	24.842	23.214	24.326	24.342	24.365	24.840
6	62.113	60.896	61.690	62.948	27.323	28.671	29.059	29.343	32.245	34.554	34.630	34.725	33.158	35.654	35.666	32.253	34.550	34.583	34.627	35.661
7	76.481	74.886	77.140	81.228	31.905	33.897	34.386	34.564	42.015	46.222	46.356	46.520	43.754	48.349	48.373	42.032	46.214	46.274	46.350	48.364
8	111.320	118.976	133.491	138.007	199.828	182.557	178.882	176.844	52.117	59.075	59.291	59.548	55.128	62.875	62.915	52.140	59.061	59.161	59.281	62.900
9	122.498	130.049	166.113	179.581	201.832	186.413	183.449	181.760	62.105	72.778	73.102	73.477	67.011	79.175	79.240	62.140	72.756	72.909	73.085	79.215
10	142.418	152.687	222.132		209.704	199.207	196.750	195.488	71.574	86.896	87.356	87.870	79.063	97.184	97.283	71.623	86.864	87.087	87.330	97.245
11	160.424	176.989	240.936		220.109	218.395	216.955	216.666	80.149	100.881	101.499	102.167	91.027	116.822	116.968	80.211	100.836	101.143	101.461	116.912
12	249.007	253.316			230.741	242.255	242.290	243.504	87.501	114.054	114.846	115.676	102.684	138.000	138.208	87.585	113.994	114.397	114.796	138.127
13	249.574	269.985			239.626	269.118	270.915		93.427	125.624	126.591	127.577	113.823	160.614	160.900	93.524	125.548	126.050	126.528	160.789
14	257.152				245.855				97.765	134.740	135.862	136.980	124.288	184.544	184.929	97.868	134.650	135.241	135.786	184.778
15	258.479				249.444				100.371	140.603	141.833	143.041	134.022	209.651	210.158	100.493	140.503	141.156	141.749	209.959
16	258.694				250.871				372.037				142.918	235.779	236.433	375.635				236.175
17	298.248				312.733				372.581				151.032	262.748	263.578	376.157				263.250
18	319.180				316.135				374.821				158.307	290.355		378.409				
19					322.316				378.539				164.832			382.150				
20					331.689				383.926				170.613			387.542				
21					343.890				391.103				175.708			394.727				
22					356.989				400.236				180.194			403.871				
23					367.486				411.526				184.098			415.153				
24									425.095				187.430			428.708				
25									441.010				190.314			444.592				
26									459.210				192.732			462.712				
27									479.305				194.728			482.691				
28									500.502				196.306			503.738				
29													197.536			524.505				
30													198.401							
31													198.905							
32													511.714							
33													512.490							

Table 2.4: Sawtooth Normalized Eigenvalues

it as the bottom line in SG. So we take $\Omega = \text{SG}$ and $\Omega_{m+1} = F_1\Omega_m \bigcup F_2\Omega_m$. Then Ω_m is a fractafold in the sense of [Strichartz 2003] consisting of 2^m cells of level m along the bottom of SG. The bottom 2^m Neumann eigenfunctions of the fractal Laplacian on Ω_m are obtained by the method of spectral decimation as follows. Fix a parameter j satisfying $0 \leq j < 2^m$. Let $x_k = \frac{k}{2^m}$ for $0 \leq k \leq 2^m$ denote the points along I where the cells of Ω_m intersect, and let y_k for $1 \leq k \leq 2^m$ denote the top vertices of the cells (so cell number k has vertices x_{k-1}, x_k, y_k). Then u_j restricted to these points is defined by

$$\begin{cases} u_j(x_k) = \frac{1}{2}(\cos \pi j x_k + \cos \pi j x_{k+1}) \\ u_j(y_k) = \cos \pi j x_k \end{cases} \tag{2.16}$$

One can check that for a graph Laplacian Δ_m on the graph $\{x_k, y_k\}$ we have

$$-\Delta_m u_y = \left(2 - 2\cos \frac{\pi j}{2^m}\right) u_j \tag{2.17}$$

with the appropriate Neumann conditions at the boundary points x_0, x_{2m}. Let

$$\phi_-(t) = \frac{5 - \sqrt{25 - 4t}}{2} \tag{2.18}$$

and

$$\Phi(t) = \lim_{n\to\infty} 5^n \phi_-^{(n)}(t), \tag{2.19}$$

where $\phi_-^{(n)}(t)$ denotes the n-fold composition. In particular, Φ is a smooth function with $\Phi(0) = 0$ and $\Phi'(0) = 1$. Then the method of spectral decimation (See [Strichartz 2006] for a detailed explanation) says that u_j may be extended to eigenfunctions of the fractal Laplacian on Ω_m with eigenvalue

$$\lambda_j^{(m)} = \frac{3}{2} \lim_{n\to\infty} 5^{m+n} \phi_-^{(n)} \left(2 - 2\cos \frac{\pi j}{2^m}\right) = \frac{3}{2} 5^m \Phi \left(2 - 2\cos \frac{\pi j}{2^m}\right) \tag{2.20}$$

Now observe that $2 - 2\cos \frac{\pi j}{2^m} \approx (\frac{\pi j}{2^m})^2$ for large m so

$$\lim_{m\to\infty} \left(\frac{4}{5}\right)^m \lambda_j^{(m)} = \frac{3}{2}(\pi j)^2. \tag{2.21}$$

Of course $(\pi j)^2$ is the correct eigenvalue for the eigenfunction $\cos \pi j x$ on I, which is clearly the limit of u_j as $u \to \infty$.

3 The Sierpinski Gasket

Let $\{q_0, q_1, q_2\}$ denote the vertices of a unit length equilateral triangle in the plane, and let $F_i x = \frac{1}{2}(x + q_i)$ for $i = 0, 1, 2$. Then SG is the invariant set for this IFS. We take Ω to be the equilateral triangle dilated by a factor $1 + \epsilon$. Then Ω_m is a union of 3^m triangles of size 2^{-m} that overlap in triangles of size $(1 + \epsilon)2^{-m}$.

In Tables 3.1 and 3.2 we present the same data as in Tables 2.1 through 2.4 for this example. The multiplicities and normalized eigenvalues agree with the known values for the Neumann spectrum of the standard Laplacian on SG [Strichartz 2003]. For example, the first six distinct normalized eigenvalues on SG are $1, 5, 8.103, 10.305, 25, 31.784$. So the numerical accuracy improves as we decrease ϵ but the error remains significant. (Much better accuracy is achieved in [Blasiak et al. 2008]). Nevertheless, the qualitative features of the spectrum, including high multiplicities and large gaps, are already apparent. In Figure 3.1 we show some graphs of eigenfunctions. Actual graphs of Dirichlet eigenfunctions on SG may be found in [Dalrymple et al. 1999].

In this case we know the eigenfunction renormalization factor $R = 5$, so we expect $r = 1.25$ in (1.4). The data is not inconsistent with this expectation, but it is impossible to deduce these values from the data alone.

We also look at the case $\epsilon = 0$, where the 3^m triangles in Ω_m intersect at single points. Thus the interior of Ω_m consists of 3^m disjoint triangles, and if we interpret the Neumann Laplacian on Ω_m in the usual way, the spectrum would just be 3^m copies of the spectrum of Ω. This is nothing like the spectrum of SG, and also it is not what we get when we use the FEM. The reason is that the spline space chosen consists of continuous functions, and this effectively couples the disjoint triangles at their junction points. Effectively this means that we are not looking at the entire Sobolev space $H^1(\Omega_m)$, but only the subspace $H_0^1(\Omega_m)$ defined to be the closure of continuous functions in $H^1(\Omega_m)$ in the Sobolev norm. In fact, functions in $H_0^1(\Omega_m)$ do not have to be continuous (or even bounded), since H^1 does not embed in continuous functions on $\mathbb{R}^2$. They do have to satisfy some integral continuity condition (see [Strichartz 1967] for analogous results for $H^{1/2}$ on a half-line). The conclusion is that the Neumann eigenvalues (and eigenfunctions) that the FEM approximates are the stationary values (and associated functions) for the Rayleigh quotient

$$R(u) = \frac{\int_{\Omega_m} |\nabla u|^2 \, dx}{\int_{\Omega_m} |u|^2 \, dx} \tag{3.1}$$

for some $u \in H_0^1(\Omega_m)$. Of course some of these eigenfunctions restrict to Neumann eigenfunctions on each triangle in Ω_m and are continuous functions at the junction points, but it is easy to see that there are not enough of these (in fact the smallest such eigenvalue must be on the order of magnitude 4^m). We claim that all the other eigenfunctions have poles at some junction points. Indeed, consider the restriction of an eigenfunction to a triangle. Because it is a Neumann eigenfunction, it must have vanishing normal derivatives along the side of

Sierpinski Gasket Eigenvalue Data

Level:	2	2	2	2	2	3	3	3	3	4	5
Refinement:	0	1	2	3	4	0	1	2	3	0	0
n											
1	5.0727	4.8920	4.8223	4.7946	4.7832	4.1689	4.0255	3.9697	3.9473	3.3372	2.6327
2	5.0729	4.8924	4.8226	4.7948	4.7833	4.1690	4.0255	3.9697	3.9473	3.3376	2.6333
3	20.6394	19.9346	19.6622	19.5531	19.5080	18.2283	17.6218	17.3890	17.2965	15.0372	12.2130
4	20.6560	19.9498	19.6783	19.5698	19.5251	18.2452	17.6387	17.4059	17.3134	15.0492	12.2253
5	20.6657	19.9529	19.6796	19.5704	19.5254	18.2457	17.6389	17.4060	17.3135	15.0518	12.2256
6	35.4198	34.0098	33.4700	33.2558	33.1678	32.1806	31.0512	30.6141	30.4394	26.2223	20.8931
7	35.4331	34.0165	33.4733	33.2574	33.1685	32.1839	31.0522	30.6144	30.4394	26.2245	20.8956
8	43.3830	41.5793	40.8896	40.6160	40.5037	41.3292	39.8556	39.2856	39.0578	33.7524	26.8513
9	271.4544	266.9576	265.7778	265.4780	265.4017	83.0086	80.1965	79.1253	78.7016	71.6959	58.5692
10	271.5749	266.9740	265.7838	265.4848	265.4100	83.0336	80.2024	79.1260	78.7019	71.6980	58.5772
11	272.0985	267.1555	265.8297	265.4951	265.4123	83.0497	80.2096	79.1293	78.7031	71.7027	58.5823
12	272.4539	268.5083	267.5057	267.2524	267.1889	83.2255	80.4029	79.3294	78.9054	71.9215	58.7675
13	272.6653	268.5884	267.5274	267.2580	267.1903	83.2406	80.4093	79.3318	78.9064	71.9232	58.7823
14	273.1340	269.0642	268.1061	267.8663	267.8062	83.3102	80.4900	79.4147	78.9892	72.0256	58.8670
15	299.2086	293.1155	291.4105	290.9176	290.7663	110.1633	106.0977	104.5471	103.9332	96.1550	78.0084
16	316.7469	309.9446	307.9827	307.3882	307.1950	119.9714	115.5265	113.8296	113.1570	106.2227	86.4177
17	316.9947	310.0058	307.9991	307.3930	307.1966	120.0147	115.5398	113.8336	113.1582	106.2293	86.4276
18	344.9089	336.4442	333.9342	333.1492	332.8849	130.5284	125.6595	123.8065	123.0738	118.5657	97.4488
19	345.3793	337.1256	334.6768	333.9079	333.6483	130.5647	125.7034	123.8531	123.1212	118.6246	97.5097
20	345.4605	337.1269	334.6778	333.9091	333.6491	130.5856	125.7091	123.8549	123.1219	118.6412	97.5150
21	427.1256	414.3260	410.2927	408.9367	408.4465	158.4240	152.2677	149.9313	149.0087	150.7050	124.3511
22	427.2329	414.3685	410.3104	408.9440	408.4495	158.4260	152.2679	149.9313	149.0087	150.7129	124.3577
23	427.7069	414.5074	410.3490	408.9552	408.4531	158.4307	152.2696	149.9319	149.0089	150.7148	124.3641
24	437.5025	423.7340	419.2998	417.7744	417.2121	179.2826	171.6045	168.6461	167.4636	159.7685	127.3438
25	437.6399	423.7789	419.3077	417.7749	417.2131	179.3092	171.6125	168.6488	167.4646	159.7975	127.3581
26	462.1666	446.9485	441.9660	440.2315	439.5866	184.3542	176.4395	173.3946	172.1788	166.8725	133.4426
27	851.9381	817.0677	807.8753	805.5351	804.9468	1071.5212	1057.8350	1053.1274	1051.9382	335.710	288.0482
28	888.8229	848.6904	837.8213	834.9749	834.2263	1071.7714	1057.8753	1053.1396	1051.9413	335.737	288.0994
29	891.2254	849.3639	837.9969	835.0193	834.2374	1071.8135	1058.3988	1053.2748	1051.975	335.770	288.1694
30	992.8633	939.1885	923.5286	919.3524	918.2220	1071.8613	1059.0201	1055.7633	1054.939	335.892	288.1905
31	993.9410	941.2736	927.4429	923.7731	922.7715	1072.3634	1059.1477	1055.7990	1054.949	335.896	288.2102
32	999.8911	941.9922	927.5956	923.8111	922.7812	1072.6331	1059.1996	1055.8151	1054.953	335.921	288.2306
33	1052.0007	999.5010	985.4672	981.7155	980.6827	1073.2824	1059.2529	1055.8720	1054.965	335.937	288.2510
34	1054.0103	999.9568	985.5786	981.7422	980.6887	1073.5674	1059.3885	1055.8825	1054.968	335.939	288.2655
35	1088.8728	1032.6001	1017.3610	1013.3103	1012.2129	1073.9888	1059.4116	1055.9128	1054.975	335.953	288.2700
36	1144.7398	1084.5089	1067.2785	1062.9436	1061.8501	1076.4724	1059.4875	1056.0661	1055.239	335.960	288.2956
37	1150.1233	1084.9562	1067.3690	1062.9740	1061.8796	1076.7105	1059.5993	1056.0783	1055.244	336.002	288.3556
38	1154.9383	1087.7050	1068.1575	1063.1712	1061.9241	1078.5174	1059.6378	1056.0826	1055.246	336.036	288.3577
39	1156.8863	1089.5572	1073.8220	1069.7891	1068.7754	1089.0059	1075.0220	1071.3853	1070.461	338.630	291.1745
40	1160.0776	1090.2141	1074.2545	1069.9123	1068.8069	1089.4789	1075.1464	1071.4168	1070.469	338.695	291.3120
41	1170.7057	1092.5885	1075.6761	1071.8657	1070.9267	1090.4701	1076.1214	1072.4046	1071.462	338.855	291.4274
42	1291.1778	1215.5339	1192.9766	1186.5037	1184.6079	1174.0043	1154.0648	1148.4476	1146.813	425.376	371.3145
43	1293.3068	1217.2739	1196.5209	1190.5539	1188.8129	1174.1797	1154.1186	1148.4619	1146.817	425.398	371.3541
44	1293.9845	1217.7969	1196.5696	1190.5670	1188.8201	1174.6064	1154.2334	1148.4912	1146.825	425.429	371.3888
45	1297.6383	1219.9161	1199.2968	1193.7045	1192.0944	1181.4177	1162.3500	1156.8737	1155.237	437.903	379.8021
46	1303.1037	1221.8710	1199.8517	1193.9074	1192.2241	1197.4002	1177.4067	1171.6223	1169.878	447.009	388.8530
47	1306.6105	1222.1916	1200.0086	1193.9475	1192.2329	1197.4897	1177.4369	1171.6306	1169.880	447.057	388.9220
48	1340.1246	1241.9893	1216.9260	1210.4150	1208.6378	1244.6256	1220.8687	1213.9726	1211.877	465.355	410.6224
49	1343.7226	1242.7779	1217.0955	1210.4525	1208.6464	1246.8784	1223.3028	1216.4472	1214.360	465.569	410.9257
50	1358.2827	1245.7145	1217.7373	1210.6107	1208.6875	1247.1785	1223.3827	1216.4669	1214.365	465.642	410.9518
51	1383.4011	1296.6487	1273.2079	1266.5890	1264.6036	1292.5390	1266.7158	1259.0430	1256.642	493.288	436.0154
52	1384.0920	1296.8827	1273.3351	1266.6775	1264.6515	1292.7459	1266.7771	1259.0591	1256.646	493.357	436.0628
53	1408.7496	1302.9443	1274.8446	1267.0475	1264.7924	1318.1064	1290.8384	1282.6980	1280.138	503.268	447.3366
54		1931.2295	1881.3515	1867.0606	1863.3648	1366.3125	1335.7326	1326.6378	1323.763	518.856	466.2302
55		1932.7859	1881.4113	1867.1235	1863.4643	1366.8361	1335.8772	1326.6733	1323.778	518.868	466.2960
56		1936.4743	1882.9905	1867.5235	1863.5513	1367.4013	1336.0234	1326.7149	1323.791	518.968	466.3212
57		1942.3175	1897.9042	1886.8888	1884.1212	1373.4237	1344.3897	1335.6043	1332.801	519.650	467.2563
58		1943.1835	1898.2372	1886.9745	1884.1428	1373.5194	1344.4135	1335.6090	1332.802	519.664	467.3472
59		1951.4159	1905.0061	1894.3829	1891.7581	1376.6480	1347.7240	1338.9737	1336.176	520.014	467.7506
60		1986.5116	1939.1087	1926.7835	1923.6106	1602.4894	1559.1123	1545.2809	1540.591	622.030	572.2704

Table 3.1: SG Unnormalized Eigenvalues

	Sierpinski Gasket Eigenvalue Data											Spectral Decimation Eigenvalues	
Level:	2	2	2	2	2	3	3	3	3	4	5	Actual	Actual
Refinement:	0	1	2	3	4	0	1	2	3	0	0	Normalized	Unormalized
n													
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	27.1144
2	1.0000	1.0001	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0001	1.0002	1.0000	27.1144
3	4.0687	4.0750	4.0773	4.0781	4.0784	4.3724	4.3775	4.3805	4.3819	4.5059	4.6390	5.0000	135.5721
4	4.0720	4.0781	4.0807	4.0816	4.0820	4.3764	4.3817	4.3847	4.3862	4.5095	4.6437	5.0000	135.5721
5	4.0739	4.0787	4.0809	4.0818	4.0820	4.3766	4.3818	4.3847	4.3862	4.5103	4.6437	5.0000	135.5721
6	6.9824	6.9522	6.9407	6.9361	6.9342	7.7191	7.7136	7.7120	7.7115	7.8575	7.9360	8.1039	219.7332
7	6.9850	6.9535	6.9414	6.9364	6.9343	7.7199	7.7139	7.7121	7.7115	7.8582	7.9370	8.1039	219.7332
8	8.5522	8.4995	8.4793	8.4712	8.4678	9.9136	9.9008	9.8964	9.8949	10.1139	10.1992	10.3056	279.4291
9	53.5124	54.5705	55.1142	55.3701	55.4858	19.9112	19.9221	19.9324	19.9381	21.4837	22.2469	25.0000	677.8606
10	53.5361	54.5739	55.1155	55.3715	55.4876	19.9172	19.9236	19.9326	19.9382	21.4843	22.2499	25.0000	677.8606
11	53.6394	54.6110	55.1250	55.3737	55.4880	19.9210	19.9254	19.9334	19.9385	21.4857	22.2519	25.0000	677.8606
12	53.7094	54.8875	55.4726	55.7402	55.8595	19.9632	19.9734	19.9838	19.9898	21.5513	22.3222	25.0000	677.8606
13	53.7511	54.9039	55.4771	55.7413	55.8598	19.9668	19.9750	19.9844	19.9900	21.5518	22.3278	25.0000	677.8606
14	53.8435	55.0012	55.5971	55.8682	55.9885	19.9835	19.9951	20.0053	20.0110	21.5825	22.3600	25.0000	677.8606
15	58.9836	59.9177	60.4297	60.6760	60.7887	26.4248	26.3564	26.3364	26.3303	28.8128	29.6307	51.5278	1397.1457
16	62.4410	63.3578	63.8663	64.1112	64.2233	28.7774	28.6987	28.6747	28.6670	31.8296	32.8248	35.1398	952.7966
17	62.4898	63.3703	63.8697	64.1122	64.2236	28.7878	28.7020	28.6757	28.6673	31.8316	32.8286	35.1398	952.7966
18	67.9926	68.7747	69.2478	69.4841	69.5941	31.3097	31.2159	31.1880	31.1794	35.5282	37.0149	40.5196	1098.6658
19	68.0854	68.9140	69.4018	69.6424	69.7537	31.3184	31.2268	31.1998	31.1913	35.5458	37.0380	40.5196	1098.6658
20	68.1014	68.9143	69.4020	69.6426	69.7539	31.3234	31.2282	31.2002	31.1915	35.5508	37.0400	40.5196	1098.6658
21	84.2002	84.6951	85.0823	85.2909	85.3913	38.0010	37.8258	37.7691	37.7496	45.1587	47.2335	8.1039	219.7331
22	84.2213	84.7038	85.0859	85.2925	85.3919	38.0015	37.8259	37.7691	37.7497	45.1611	47.2360	8.1039	219.7331
23	84.3148	84.7322	85.0939	85.2948	85.3927	38.0026	37.8263	37.7693	37.7497	45.1617	47.2384	31.7847	861.8226
24	86.2458	86.6182	86.9501	87.1342	87.2239	43.0044	42.6294	42.4836	42.4250	47.8746	48.3702	31.7847	861.8226
25	86.2729	86.6274	86.9517	87.1343	87.2241	43.0107	42.6314	42.4842	42.4253	47.8833	48.3756	31.7847	861.8226
26	91.1079	91.3637	91.6503	91.8180	91.9016	44.2209	43.8305	43.6797	43.6195	50.0033	50.6868	31.7847	861.8226
27	167.9444	167.0221	167.5288	168.0085	168.2851	257.0248	262.7838	265.2927	266.4966	100.5955	109.4121	202.5980	5493.3291
28	175.2155	173.4863	173.7387	174.1487	174.4064	257.0848	262.7938	265.2958	266.4974	100.6038	109.4315	202.5980	5493.3291
29	175.6892	173.6240	173.7751	174.1580	174.4087	257.0949	262.9239	265.3299	266.5060	100.6136	109.4581	202.5980	5493.3291
30	195.7253	191.9856	191.5118	191.7471	191.9668	257.1063	263.0782	265.9567	267.2570	100.6502	109.4661	202.5980	5493.3291
31	195.9377	192.4119	192.3235	192.6691	192.9179	257.2268	263.1099	265.9657	267.2593	100.6512	109.4736	202.5980	5493.3291
32	197.1107	192.5588	192.3552	192.6771	192.9200	257.2915	263.1228	265.9698	267.2604	100.6589	109.4813	202.5980	5493.3291
33	207.3831	204.3145	204.3560	204.7540	205.0251	257.4472	263.1360	265.9841	267.2634	100.6636	109.4891	202.5980	5493.3291
34	207.7793	204.4077	204.3791	204.7596	205.0263	257.5156	263.1697	265.9868	267.2643	100.6643	109.4946	202.5980	5493.3291
35	214.6518	211.0805	210.9698	211.3437	211.6169	257.6167	263.1754	265.9944	267.2660	100.6685	109.4963	202.5980	5493.3291
36	225.6650	221.6915	221.3211	221.6956	221.9942	258.2124	263.1943	266.0330	267.3330	100.6706	109.5060	202.5980	5493.3291
37	226.7263	221.7830	221.3399	221.7019	222.0004	258.2695	263.2221	266.0361	267.3342	100.6830	109.5288	202.5980	5493.3291
38	227.6754	222.3449	221.5034	221.7431	222.0097	258.7029	263.2316	266.0372	267.3347	100.6934	109.5296	202.5980	5493.3291
39	228.0595	222.7235	222.6781	223.1233	223.4420	261.2188	267.0533	269.8921	271.1893	101.4705	110.5995	202.5980	5493.3291
40	228.6886	222.8578	222.7678	223.1490	223.4486	261.3323	267.0843	269.9000	271.1913	101.4899	110.6518	202.5980	5493.3291
41	230.7837	223.3431	223.0626	223.5565	223.8918	261.5700	267.3264	270.1488	271.4429	101.5379	110.6956	202.5980	5493.3291
42	254.5326	248.4752	247.3871	247.4662	247.6584	281.6073	286.6889	289.3048	290.5322	127.4639	141.0399	158.9233	4309.1129
43	254.9523	248.8309	248.1221	248.3110	248.5375	281.6494	286.7022	289.3084	290.5332	127.4704	141.0549	158.9233	4309.1129
44	255.0859	248.9378	248.1322	248.3137	248.5390	281.7517	286.7308	289.3158	290.5351	127.4799	141.0681	158.9233	4309.1129
45	255.8062	249.3710	248.6978	248.9681	249.2236	283.3855	288.7471	291.4274	292.6663	131.2177	144.2638	158.9233	4309.1129
46	256.8836	249.7706	248.8128	249.0104	249.2507	287.2192	292.4874	295.1427	296.3753	133.9464	147.7017		
47	257.5749	249.8362	248.8453	249.0188	249.2525	287.2407	292.4949	295.1448	296.3758	133.9608	147.7279		
48	264.1816	253.8831	252.3535	252.4533	252.6822	298.5472	303.2841	305.8112	307.0152	139.4436	155.9706		
49	264.8909	254.0443	252.3887	252.4612	252.6840	299.0875	303.8888	306.4345	307.6444	139.5079	156.0858		
50	267.7611	254.6446	252.5217	252.4942	252.6926	299.1595	303.9086	306.4395	307.6455	139.5297	156.0957		
51	272.7128	265.0564	264.0247	264.1694	264.3826	310.0401	314.6733	317.1648	318.3560	147.8138	165.6158		
52	272.8490	265.1042	264.0510	264.1879	264.3926	310.0897	314.6885	317.1689	318.3570	147.8345	165.6338		
53	277.7098	266.3433	264.3641	264.2651	264.4221	316.1729	320.6657	323.1237	324.3086	150.8042	169.9161		
54		394.7752	390.1352	389.4083	389.5618	327.7361	331.8182	334.1926	335.3604	155.4753	177.0926		
55		395.0934	390.1476	389.4215	389.5826	327.8617	331.8541	334.2015	335.3642	155.4788	177.1176		
56		395.8473	390.4751	389.5049	389.6008	327.9972	331.8904	334.2120	335.3674	155.5087	177.1272		
57		397.0418	393.5677	393.5439	393.9012	329.4418	333.9688	336.4513	337.6500	155.7132	177.4824		
58		397.2188	393.6368	393.5617	393.9057	329.4648	333.9747	336.4525	337.6503	155.7173	177.5169		
59		398.9017	395.0404	395.1069	395.4978	330.2152	334.7970	337.3001	338.5050	155.8221	177.6701		
60		406.0758	402.1123	401.8646	402.1570	384.3876	387.3094	389.2708	390.2912	186.3913	217.3709		

Table 3.2: SG Normalized Eigenvalues

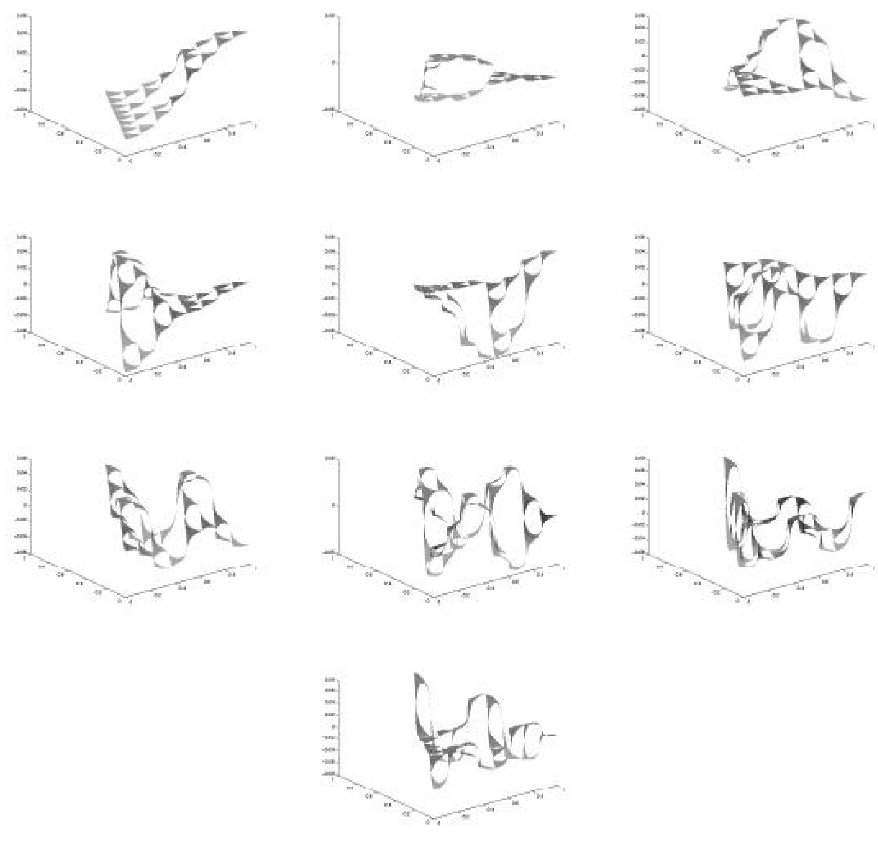


Figure 3.1: Sierpinski Gasket (SG) Eigenfunctions, Level 3

the triangle. Choose a vertex of the triangle and reflect the eigenfunction evenly six times around. This yields an eigenfunction in a deleted neighborhood of the vertex. The removable singularities theorem yields the following dichotomy: either the function is unbounded or it satisfies the eigenvalue equation at the vertex. If it satisfies the eigenvalue equation at all three vertices of the triangle, then the restriction to the triangle is a Neumann eigenvalue, contrary to our assumption. It is not difficult to see that the singularities must be logarithmic poles.

With this in mind, we look at the eigenvalue data in Tables 3.3 and 3.4. In contrast to our preceding computations, we do not see an apparent convergence of eigenvalues on a fixed Ω_m when we increase the refinement of the triangulation. In particular the numerical values in Table 3.4 are even better than the data in Table 3.2. In other words, the poor approximations by the FEM to the actual eigenvalues on Ω_m yield very good approximations to the relative eigenvalues on SG. We can even extract rather decent estimates for $r = 1.25$ from the data in Table 3.3 if we pair off corresponding refinements at different levels. For example, if we compute $\lambda_n^{(3)}/\lambda_n^{(4)}$ using 3 refinements on level 3 and 4 refinements on level 4, the first six distinct eigenvalues yield ratios $1.246, 1.233, 1.223, 1.158, 1.128, 1.112$.

Of course the eigenfunctions on Ω_m cannot approximate the eigenfunctions on SG, since the latter are bounded. Since we are already getting more information than we deserve, we might speculate that the eigenfunction approximation might be accurate in the complement of a small neighborhood of the junction points.

Sierpinski Gasket, No Overlap, Unnormalized

Level:	1	1	1	2	2	2	3	3	4	4
Refinement:	2	3	4	2	3	4	3	4	3	4
n										
1	3.650	3.113	2.713	3.689	3.103	2.677	2.773	2.364	2.687	2.224
2	3.721	3.164	2.752	3.689	3.103	2.677	2.773	2.364	2.687	2.224
3	70.334	70.221	70.193	17.179	14.285	12.220	13.684	11.644	13.409	11.091
4	70.356	70.227	70.195	17.179	14.285	12.220	13.684	11.644	13.409	11.091
5	70.362	70.228	70.195	17.179	14.285	12.220	13.684	11.644	13.409	11.091
6	82.579	80.476	79.025	25.961	21.402	18.196	21.944	18.643	21.699	17.941
7	82.850	80.663	79.163	25.961	21.402	18.196	21.944	18.643	21.699	17.941
8	96.289	91.743	88.598	31.117	25.509	21.604	27.689	23.498	27.564	22.783
9	212.039	210.923	210.645	282.881	281.270	280.869	63.390	53.398	66.354	54.735
10	235.397	230.816	227.972	282.881	281.270	280.869	63.390	53.398	66.354	54.735
11	236.040	231.194	228.236	282.881	281.270	280.869	63.390	53.398	66.354	54.735
12	283.145	281.338	280.886	282.881	281.270	280.869	63.390	53.398	66.354	54.735
13	283.383	281.397	280.901	282.881	281.270	280.869	63.390	53.398	66.354	54.735
14	283.654	281.464	280.918	282.881	281.270	280.869	63.390	53.398	66.354	54.735
15	303.428	296.864	293.464	302.450	297.037	294.131	78.165	65.625	84.053	69.271
16	304.318	297.305	293.740	316.088	308.066	303.425	85.035	71.278	92.754	76.407
17	310.855	303.820	299.961	316.088	308.066	303.425	85.035	71.278	92.754	76.407
18	497.970	492.957	491.705	342.012	329.003	321.049	95.415	79.778	106.636	87.776
19	499.255	493.281	491.786	342.012	329.003	321.049	95.415	79.778	106.636	87.776
20	500.036	493.472	491.833	342.012	329.003	321.049	95.415	79.778	106.636	87.776
21	526.723	515.905	511.065	391.489	368.308	353.783	110.130	91.738	128.376	105.547
22	528.063	516.482	511.415	391.489	368.308	353.783	110.130	91.738	128.376	105.547
23	568.769	547.592	536.643	408.152	381.181	364.323	114.069	94.922	134.759	110.756
24	642.681	634.411	632.344	408.152	381.181	364.323	114.069	94.922	134.759	110.756
25	644.193	634.780	632.435	408.152	381.181	364.323	114.069	94.922	134.759	110.756
26	644.399	634.848	632.454	408.152	381.181	364.323	114.069	94.922	134.759	110.756
27	646.983	635.458	632.604	861.618	847.027	843.410	1127.762	1124.145	311.734	253.559
28	677.294	661.371	654.619	887.555	867.807	860.994	1127.762	1124.145	311.734	253.559
29	681.117	662.685	655.217	887.555	867.807	860.994	1127.762	1124.145	311.734	253.559
30	864.923	847.924	843.639	973.982	938.678	921.579	1127.762	1124.145	311.734	253.559
31	888.687	867.510	860.707	973.982	938.678	921.579	1127.762	1124.145	311.734	253.559
32	890.096	868.076	861.020	973.982	938.678	921.579	1127.762	1124.145	311.734	253.559
33	938.355	918.889	914.016	1028.867	984.272	960.390	1127.762	1124.145	311.734	253.559
34	941.159	919.527	914.172	1028.867	984.272	960.390	1127.762	1124.145	311.734	253.559
35	944.371	920.288	914.359	1063.734	1013.093	984.352	1127.762	1124.145	311.734	253.559
36	990.412	956.321	943.405	1157.651	1131.525	1125.082	1127.762	1124.145	311.734	253.559
37	995.980	958.075	944.225	1157.651	1131.525	1125.082	1127.762	1124.145	311.734	253.559
38	1007.032	969.717	955.860	1157.651	1131.525	1125.082	1127.762	1124.145	311.734	253.559
39	1161.208	1132.507	1125.335	1157.651	1131.525	1125.082	1127.762	1124.145	311.734	253.559
40	1163.930	1133.179	1125.504	1157.651	1131.525	1125.082	1127.762	1124.145	311.734	253.559
41	1167.498	1134.086	1125.730	1157.651	1131.525	1125.082	1127.762	1124.145	311.734	253.559
42	1184.368	1147.961	1137.349	1230.800	1174.160	1154.340	1198.824	1182.609	386.255	312.662
43	1186.310	1148.403	1137.598	1247.495	1189.429	1167.359	1198.824	1182.609	386.255	312.662
44	1203.984	1162.140	1148.937	1247.495	1189.429	1167.359	1198.824	1182.609	386.255	312.662
45	1384.681	1346.242	1336.679	1267.855	1209.137	1185.248	1198.824	1182.609	386.255	312.662
46	1392.804	1348.470	1337.250	1267.855	1209.137	1185.248	1209.019	1191.017	394.037	318.795
47	1398.054	1349.707	1337.549	1267.855	1209.137	1185.248	1209.019	1191.017	394.037	318.795
48	1418.418	1369.638	1355.732	1289.621	1231.534	1206.796	1248.438	1223.543	421.196	340.141
49	1424.922	1371.372	1356.319	1289.621	1231.534	1206.796	1248.438	1223.543	421.196	340.141
50	1488.198	1415.530	1390.277	1294.363	1236.613	1211.852	1248.438	1223.543	421.196	340.141
51	1540.507	1490.281	1477.951	1294.363	1236.613	1211.852	1277.622	1247.622	439.088	354.151
52	1544.858	1491.604	1478.303	1294.363	1236.613	1211.852	1277.622	1247.622	439.088	354.151
53	1547.571	1492.089	1478.407	1294.363	1236.613	1211.852	1297.625	1264.115	450.532	363.091
54	1562.171	1495.591	1479.271		1991.484	1971.705	1342.696	1301.218	474.374	381.660
55	1628.290	1554.758	1529.753		1991.484	1971.705	1342.696	1301.218	474.374	381.660
56	1648.776	1560.742	1531.934		1991.484	1971.705	1342.696	1301.218	474.374	381.660
57	1847.262	1777.702	1760.374		1991.484	1971.705	1342.696	1301.218	474.374	381.660
58	1852.666	1779.553	1760.911		1991.484	1971.705	1342.696	1301.218	474.374	381.660
59	1862.961	1782.208	1761.547		1991.484	1971.705	1342.696	1301.218	474.374	381.660
60	1872.896	1793.161	1771.954			1999.469	1465.787	1401.719	529.475	424.272

Table 3.3: Sierpinski Gasket, No Overlap, Unnormalized

Sierpinski Gasket, No Overlap, Normalized

Level:	1	1	1	2	2	2	3	3	4	4
Refinement:	2	3	4	2	3	4	3	4	3	4
n										
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	1.019	1.016	1.014	1.000	1.000	1.000	1.000	1.000	1.000	1.000
3	19.268	22.557	25.875	4.657	4.604	4.564	4.935	4.925	4.990	4.987
4	19.274	22.559	25.876	4.657	4.604	4.564	4.935	4.925	4.990	4.987
5	19.275	22.559	25.876	4.657	4.604	4.564	4.935	4.925	4.990	4.987
6	22.622	25.851	29.131	7.037	6.897	6.796	7.914	7.886	8.075	8.067
7	22.697	25.911	29.182	7.037	6.897	6.796	7.914	7.886	8.075	8.067
8	26.378	29.470	32.660	8.435	8.221	8.069	9.986	9.939	10.257	10.244
9	58.087	67.754	77.650	76.680	90.646	104.909	22.861	22.586	24.692	24.612
10	64.486	74.144	84.038	76.680	90.646	104.909	22.861	22.586	24.692	24.612
11	64.662	74.266	84.135	76.680	90.646	104.909	22.861	22.586	24.692	24.612
12	77.566	90.373	103.543	76.680	90.646	104.909	22.861	22.586	24.692	24.612
13	77.632	90.392	103.549	76.680	90.646	104.909	22.861	22.586	24.692	24.612
14	77.706	90.414	103.555	76.680	90.646	104.909	22.861	22.586	24.692	24.612
15	83.123	95.361	108.180	81.985	95.727	109.862	28.189	27.758	31.278	31.148
16	83.367	95.502	108.282	85.682	99.281	113.334	30.667	30.149	34.517	34.357
17	85.158	97.595	110.575	85.682	99.281	113.334	30.667	30.149	34.517	34.357
18	136.417	158.351	181.258	92.709	106.029	119.916	34.410	33.744	39.682	39.469
19	136.769	158.455	181.287	92.709	106.029	119.916	34.410	33.744	39.682	39.469
20	136.983	158.517	181.305	92.709	106.029	119.916	34.410	33.744	39.682	39.469
21	144.294	165.723	188.394	106.121	118.696	132.143	39.717	38.804	47.772	47.459
22	144.661	165.908	188.523	106.121	118.696	132.143	39.717	38.804	47.772	47.459
23	155.812	175.901	197.823	110.638	122.844	136.080	41.138	40.150	50.148	49.802
24	176.060	203.790	233.102	110.638	122.844	136.080	41.138	40.150	50.148	49.802
25	176.474	203.908	233.135	110.638	122.844	136.080	41.138	40.150	50.148	49.802
26	176.530	203.930	233.142	110.638	122.844	136.080	41.138	40.150	50.148	49.802
27	177.238	204.126	233.197	233.558	272.973	315.026	406.713	475.493	116.005	114.013
28	185.542	212.450	241.313	240.589	279.670	321.593	406.713	475.493	116.005	114.013
29	186.589	212.872	241.533	240.589	279.670	321.593	406.713	475.493	116.005	114.013
30	236.942	272.376	310.991	264.016	302.510	344.223	406.713	475.493	116.005	114.013
31	243.452	278.668	317.283	264.016	302.510	344.223	406.713	475.493	116.005	114.013
32	243.838	278.850	317.399	264.016	302.510	344.223	406.713	475.493	116.005	114.013
33	257.059	295.172	336.935	278.894	317.203	358.719	406.713	475.493	116.005	114.013
34	257.827	295.377	336.992	278.894	317.203	358.719	406.713	475.493	116.005	114.013
35	258.707	295.621	337.061	288.345	326.492	367.669	406.713	475.493	116.005	114.013
36	271.319	307.196	347.768	313.804	364.659	420.234	406.713	475.493	116.005	114.013
37	272.845	307.760	348.071	313.804	364.659	420.234	406.713	475.493	116.005	114.013
38	275.872	311.499	352.360	313.804	364.659	420.234	406.713	475.493	116.005	114.013
39	318.108	363.792	414.834	313.804	364.659	420.234	406.713	475.493	116.005	114.013
40	318.854	364.008	414.896	313.804	364.659	420.234	406.713	475.493	116.005	114.013
41	319.831	364.299	414.979	313.804	364.659	420.234	406.713	475.493	116.005	114.013
42	324.453	368.756	419.262	333.632	378.399	431.163	432.341	500.222	143.736	140.589
43	324.985	368.898	419.354	338.157	383.320	436.025	432.341	500.222	143.736	140.589
44	329.826	373.311	423.534	338.157	383.320	436.025	432.341	500.222	143.736	140.589
45	379.328	432.449	492.741	343.676	389.671	442.707	432.341	500.222	143.736	140.589
46	381.553	433.165	492.952	343.676	389.671	442.707	436.018	503.779	146.633	143.347
47	382.991	433.562	493.062	343.676	389.671	442.707	436.018	503.779	146.633	143.347
48	388.570	439.965	499.765	349.576	396.889	450.756	450.234	517.537	156.739	152.945
49	390.352	440.522	499.981	349.576	396.889	450.756	450.234	517.537	156.739	152.945
50	407.686	454.706	512.499	350.862	398.526	452.644	450.234	517.537	156.739	152.945
51	422.016	478.719	544.819	350.862	398.526	452.644	460.759	527.721	163.397	159.245
52	423.208	479.144	544.948	350.862	398.526	452.644	460.759	527.721	163.397	159.245
53	423.951	479.299	544.986	350.862	398.526	452.644	467.972	534.698	167.656	163.265
54	427.951	480.424	545.305		641.800	736.460	484.227	550.392	176.528	171.614
55	446.064	499.430	563.915		641.800	736.460	484.227	550.392	176.528	171.614
56	451.676	501.353	564.718		641.800	736.460	484.227	550.392	176.528	171.614
57	506.050	571.046	648.928		641.800	736.460	484.227	550.392	176.528	171.614
58	507.530	571.641	649.126		641.800	736.460	484.227	550.392	176.528	171.614
59	510.351	572.493	649.361		641.800	736.460	484.227	550.392	176.528	171.614
60	513.073	576.012	653.197			746.830	528.618	592.902	197.033	190.775

Table 3.4: Sierpinski Gasket, No Overlap, Normalized

4 Non-PCF Fractals

Our first example is the octagasket, generated by eight contractive homotheties with contraction ratio $1 - \sqrt{2}/2$ and fixed points $\{q_i\}$ the vertices of a regular octagon. Then the consecutive images F_iK and $F_{i+1}K$ intersect along a Cantor set. As yet, there has been no construction of a self-similar Laplacian on this fractal, although it is reasonable to expect the probabilistic methods in [Barlow 1995] will work, given the high symmetry in this example. It is natural to approximate from without by taking Ω to be the interior of the octagon with vertices $\{q_i\}$. Then Ω_m consists of the interior of the union of 8^m octagons that meet along edges.

In table 4.1 we give the eigenvalues on Ω_m for $m = 0, 1, 2, 3$ along with level-to-level ratios, suggesting a renormalization factor of about $r = 1.2$. In Table 4.2 we normalize the eigenvalues by dividing by $\lambda_1^{(m)}$. This suggests an eigenvalue renormalization factor of about $R = 14.9476$ (the table indicates when a new eigenvalue appears that is approximately $R\lambda_n$ for an earlier value of n). In the next section we will explain why this happens. The tables show eigenvalues of multiplicities 1 and 2, but no higher multiplicities. The D_8 symmetry forces multiplicity 2, since there are three irreducible representations of dimension 2. There are a number of close coincidences (for example 910.5058 and 910.8645, each with multiplicity 2), but not close enough to be regarded as the same, in our judgement. There is some evidence of large gaps in the spectrum, for example (66.45202,122.0411), (162.1709,223.2267) and (253.6123,336.1848). However, there is not enough data to guess whether or not there are infinitely many gaps ($\lambda_{j+1}/\lambda_j \geq 1 + \epsilon$ for fixed ϵ). In Figure 4.1 we display the graphs of some eigenfunctions, and in Figure 4.2 we show the Weyl ratios.

The Weyl ratio is defined to be $W(x) = N(x)/x^\alpha$, where $N(x) = \#\{\lambda_j \leq x\}$ is the eigenvalue counting function, and x^α is its approximate growth rate. We determine α experimentally as the slope of the line of best fit to a log-log plot of $N(x)$. The Weyl ratio gives a nice "snapshot" of the spectrum. A question of interest is whether it tends to a limit, or shows periodic behavior for large x. Our experimental data does not give an indication of what answer to expect.

The next example we consider is the standard SC generated by eight contractions of ratio $\frac{1}{3}$ (omitting the middle tic-tac-toe square). Here the existence of a self-similar Laplacian is known, and as stated above, uniqueness is established in [Barlow et al. 2008]. Here it is natural to choose Ω to be the interior of the square that just contains SC, so Ω_m contains 8^m squares of side length 3^{-m} intersecting along edges. In Tables 4.3 and 4.4 we report unnormalized and normalized eigenvalue data, as before. In Table 4.5 we describe the D_4 representation type associated to the eigenspace. There is one 2-dimensional representation (denoted 2) and four 1-dimensional representations ($1++, 1+-, 1-+$, and $1--$) described in more detail in the next section. Again we only see eigenvalue multiplicities of 1 or 2. There is an apparent eigenvalue renormalization factor of about $R = 10.0081$, which is consistent with computations in [Barlow et al. 1990]. In the next section we will give an explanation of this behavior. Spectral gaps are consistent with the data. Figure 4.3 shows some

Octagasket Unnormalized Eigenvalues												Ratios $\lambda_n^{(j)}/\lambda_n^{(j+1)}$, highest refinements used		
Level:	1	1	2	2	2	2	3	3	3	3	4			
Refinement:	0	1	1	2	3	4	1	2	3	4	1			
n														
1	12.87	12.81	6.28	6.14	6.08	6.06	5.07	4.86	4.78	4.74	3.95	2.11	1.28	1.20
2	12.87	12.81	6.30	6.15	6.09	6.06	5.07	4.86	4.78	4.74	3.95	2.11	1.28	1.20
3	35.55	35.14	23.98	23.37	23.14	23.04	19.15	18.37	18.04	17.90	14.93	1.53	1.29	1.20
4	35.56	35.15	24.00	23.38	23.14	23.04	19.15	18.37	18.04	17.90	14.93	1.53	1.29	1.20
5	57.06	55.93	47.97	46.60	46.08	45.88	38.75	37.15	36.47	36.20	30.19	1.22	1.27	1.20
6	67.47	66.00	48.24	46.70	46.12	45.90	38.75	37.15	36.47	36.20	30.19	1.44	1.27	1.20
7	67.49	66.00	62.46	60.28	59.47	59.17	53.10	50.89	49.96	49.58	41.32	1.12	1.19	1.20
8	99.03	95.78	151.54	149.80	149.30	149.17	75.96	72.75	71.40	70.86	59.09	0.64	2.11	1.20
9	112.63	108.51	151.61	149.81	149.31	149.17	75.96	72.75	71.40	70.86	59.09	0.73	2.11	1.20
10	112.78	108.55	154.94	152.84	152.22	152.04	78.99	75.65	74.25	73.68	61.42	0.71	2.06	1.20
11	124.95	120.29	155.27	152.92	152.24	152.04	78.99	75.65	74.25	73.68	61.42	0.79	2.06	1.20
12	166.13	157.78	161.09	158.27	157.40	157.14	84.90	81.30	79.79	79.18	65.96	1.00	1.98	1.20
13	166.21	157.80	161.44	158.37	157.43	157.14	84.90	81.30	79.79	79.18	65.96	1.00	1.98	1.20
14	179.85	169.80	162.62	159.42	158.46	158.17	85.54	81.91	80.39	79.78	66.45	1.07	1.98	1.20
15	180.29	169.87	163.28	159.63	158.52	158.19	85.54	81.91	80.39	79.78	66.45	1.07	1.98	1.20
16	201.34	188.62	223.56	213.46	210.19	209.10	157.34	150.33	147.46	146.31	122.04	0.90	1.43	1.20
17	237.11	220.91	225.86	217.24	214.55	213.67	157.34	150.33	147.46	146.31	122.04	1.03	1.46	1.20
18	237.65	221.00	227.27	217.60	214.66	213.72	159.54	152.43	149.51	148.34	123.73	1.03	1.44	1.20
19	274.23	251.90	251.17	242.41	239.73	238.90	168.29	160.76	157.67	156.43	130.46	1.05	1.53	1.20
20	274.43	251.95	251.31	242.44	239.75	238.90	168.29	160.76	157.67	156.43	130.46	1.05	1.53	1.20
21	289.57	265.40	290.09	278.88	275.54	274.51	193.38	184.60	181.03	179.60	149.71	0.97	1.53	1.20
22	290.74	265.65	292.73	279.78	275.84	274.61	193.38	184.60	181.03	179.60	149.71	0.97	1.53	1.20
23	335.78	304.52	314.75	299.70	295.20	293.80	209.59	199.98	196.08	194.53	162.17	1.04	1.51	1.20
24	336.40	304.62	428.94	413.83	409.52	408.37	290.88	277.07	271.53	269.34	223.23	0.75	1.52	1.21
25	360.47	322.62	431.35	414.35	409.64	408.40	290.88	277.07	271.53	269.34	223.23	0.79	1.52	1.21
26	379.58	339.03	448.72	432.14	427.30	425.95	309.23	294.38	288.46	286.11	237.13	0.80	1.49	1.21
27	400.07	356.68	452.24	432.96	427.49	425.99	309.23	294.38	288.46	286.11	237.13	0.84	1.49	1.21
28	428.17	381.56	453.90	433.38	427.60	426.02	310.63	295.70	289.75	287.39	238.19	0.90	1.48	1.21
29	441.65	390.22	459.24	434.84	427.99	426.12	310.63	295.70	289.75	287.39	238.19	0.92	1.48	1.21
30	445.04	390.90	479.16	456.52	450.02	448.16	331.23	315.14	308.75	306.23	253.61	0.87	1.46	1.21
31	486.74	425.77	488.40	458.88	450.63	448.32	331.23	315.14	308.75	306.23	253.61	0.95	1.46	1.21
32	500.94	434.79	562.07	535.94	528.39	526.24	437.21	414.29	405.37	401.88	336.18	0.83	1.31	1.20
33	503.98	435.36	565.73	537.94	529.17	526.46	437.21	414.29	405.37	401.88	336.18	0.83	1.31	1.20
34	517.82	450.69	571.43	539.01	530.14	527.96	441.53	418.44	409.46	405.95	338.61	0.85	1.30	1.20
35	519.43	451.06	573.83	539.58	530.53	528.06	441.53	418.44	409.46	405.95	338.61	0.85	1.30	1.20
36	607.09	518.07	584.89	549.09	539.29	536.57	459.56	435.33	425.94	422.28	351.80	0.97	1.27	1.20
37	611.05	518.98	643.32	605.90	595.21	592.22	493.31	466.56	456.26	452.25	379.14	0.88	1.31	1.19
38	638.35	538.29	651.28	607.91	595.75	592.37	493.31	466.56	456.26	452.25	379.14	0.91	1.31	1.19
39	644.93	539.72	700.98	661.74	651.18	648.45	557.35	525.93	513.96	509.33	428.58	0.83	1.27	1.19
40	659.58	557.54	749.55	703.59	690.24	686.48	588.09	554.32	541.50	536.55	451.74	0.81	1.28	1.19
41	665.39	558.99	759.91	706.48	690.87	686.62	588.09	554.32	541.50	536.55	451.74	0.81	1.28	1.19
42	669.03	569.71	764.98	710.86	695.42	689.38	616.83	580.78	567.16	561.91	473.92	0.83	1.23	1.19
43	671.06	569.92	768.33	711.94	696.68	692.85	616.83	580.78	567.16	561.91	473.92	0.82	1.23	1.19
44	752.63	614.76	780.01	716.52	696.98	692.92	649.02	610.27	595.70	590.10	499.69	0.89	1.17	1.18
45	799.56	664.07	790.04	724.17	709.42	705.55	649.02	610.27	595.70	590.10	499.69	0.94	1.20	1.18
46	802.16	664.43	792.39	727.47	710.39	705.82	673.48	632.54	617.22	611.34	519.17	0.94	1.15	1.18
47	815.03	679.96	806.00	740.88	723.51	718.84	673.48	632.54	617.22	611.34	519.17	0.95	1.18	1.18
48	832.53	685.05	847.00	785.98	769.26	764.87	712.51	668.11	651.58	645.25	551.07	0.90	1.19	1.17
49	840.23	686.13	849.40	786.33	769.32	764.88	712.51	668.11	651.58	645.25	551.07	0.90	1.19	1.17
50	870.15	712.75	889.09	821.77	803.21	798.22	737.80	690.37	672.83	666.14	576.94	0.89	1.20	1.15
51	895.83	718.89	899.51	824.78	803.97	798.40	737.80	690.37	672.83	666.14	576.94	0.90	1.20	1.15
52	904.07	722.76	951.70	867.27	844.89	838.85	749.87	701.93	684.17	677.40	582.26	0.86	1.24	1.16
53	951.97	779.21	956.27	869.23	845.43	838.99	749.87	701.93	684.17	677.40	582.26	0.93	1.24	1.16
54	977.47	790.31	984.78	894.52	869.56	862.77	763.18	713.92	695.71	688.77	593.76	0.92	1.25	1.16
55	983.56	791.59	987.21	895.16	869.79	862.84	769.50	718.12	699.26	692.09	619.00	0.92	1.25	1.12
56	1008.08	813.30	1099.26	990.90	961.30	953.18	968.08	909.14	887.65	879.56	668.25	0.85	1.08	1.32
58	1035.98	828.30	1101.07	996.03	962.98	953.67	1005.76	941.74	918.58	909.87	712.82	0.87	1.05	1.28
59	1059.87	841.04	1103.77	999.31	971.51	964.09	1005.76	941.74	918.58	909.87	712.82	0.87	1.06	1.28
60	1131.15	901.70	1117.52	1000.43	971.72	964.14	1078.51	1005.57	979.42	969.63	769.87	0.94	0.99	1.26

Table 4.1: Octagasket Unnormalized Eigenvalues and Ratios

Octagasket Normalized Eigenvalues

Level:	1	1	2	2	2	2	3	3	3	3	4
Refinement:	0	1	1	2	3	4	1	2	3	4	1
n											
1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	**1.00**	1.00
2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	**1.00**	1.00
3	2.76	2.74	3.82	3.81	3.80	3.80	3.78	3.78	3.78	**3.78**	3.78
4	2.76	2.74	3.82	3.81	3.80	3.80	3.78	3.78	3.78	**3.78**	3.78
5	4.43	4.37	7.64	7.59	7.57	7.57	7.64	7.64	7.64	**7.64**	7.64
6	5.24	5.15	7.68	7.60	7.58	7.57	7.64	7.64	7.64	**7.64**	7.64
7	5.25	5.15	9.95	9.82	9.77	9.76	10.48	10.46	10.46	10.46	10.45
8	7.70	7.48	24.13	24.39	24.54	24.61	14.99	14.96	14.95	14.95	**14.95**
9	8.75	8.47	24.14	24.39	24.54	24.61	14.99	14.96	14.95	14.95	**14.95**
10	8.77	8.47	24.67	24.89	25.02	25.08	15.58	15.55	15.54	15.54	15.54
11	9.71	9.39	24.72	24.90	25.02	25.08	15.58	15.55	15.54	15.54	15.54
12	12.91	12.31	25.65	25.77	25.87	25.92	16.75	16.71	16.70	16.70	16.69
13	12.92	12.32	25.70	25.79	25.87	25.92	16.75	16.71	16.70	16.70	16.69
14	13.98	13.25	25.89	25.96	26.04	26.09	16.88	16.84	16.83	16.83	16.81
15	14.01	13.26	26.00	25.99	26.05	26.09	16.88	16.84	16.83	16.83	16.81
16	15.65	14.72	35.59	34.76	34.54	34.49	31.04	30.91	30.87	30.86	30.87
17	18.43	17.24	35.96	35.37	35.26	35.25	31.04	30.91	30.87	30.86	30.87
18	18.47	17.25	36.18	35.43	35.28	35.25	31.47	31.34	31.30	31.29	31.30
19	21.31	19.66	39.99	39.47	39.40	39.41	33.20	33.05	33.01	32.99	33.00
20	21.33	19.66	40.01	39.48	39.40	39.41	33.20	33.05	33.01	32.99	33.00
21	22.51	20.71	46.19	45.41	45.28	45.28	38.15	37.95	37.90	37.88	37.87
22	22.60	20.73	46.61	45.56	45.33	45.30	38.15	37.95	37.90	37.88	37.87
23	26.10	23.77	50.11	48.80	48.51	48.46	41.35	41.11	41.05	41.03	41.02
24	26.15	23.78	68.29	67.39	67.30	67.36	57.38	56.96	56.84	56.81	**56.47**
25	28.02	25.18	68.68	67.47	67.32	67.37	57.38	56.96	56.84	56.81	**56.47**
26	29.50	26.46	71.44	70.37	70.22	70.26	61.00	60.52	60.39	60.35	59.98
27	31.09	27.84	72.00	70.50	70.26	70.27	61.00	60.52	60.39	60.35	59.98
28	33.28	29.78	72.27	70.57	70.27	70.27	61.28	60.79	60.66	60.62	60.25
29	34.33	30.46	73.12	70.81	70.34	70.29	61.28	60.79	60.66	60.62	60.25
30	34.59	30.51	76.29	74.34	73.96	73.92	65.34	64.79	64.64	64.59	64.15
31	37.83	33.23	77.76	74.72	74.06	73.95	65.34	64.79	64.64	64.59	64.15
32	38.93	33.93	89.49	87.27	86.84	86.80	86.25	85.17	84.86	84.76	85.04
33	39.17	33.98	90.07	87.59	86.96	86.84	86.25	85.17	84.86	84.76	85.04
34	40.25	35.18	90.98	87.77	87.12	87.09	87.10	86.02	85.72	85.62	85.65
35	40.37	35.20	91.36	87.86	87.19	87.10	87.10	86.02	85.72	85.62	85.65
36	47.18	40.43	93.12	89.41	88.63	88.51	90.66	89.50	89.17	89.07	88.99
37	47.49	40.51	102.42	98.66	97.82	97.69	97.32	95.92	95.52	95.39	95.91
38	49.61	42.01	103.69	98.99	97.91	97.71	97.32	95.92	95.52	95.39	95.91
39	50.13	42.12	111.60	107.75	107.02	106.96	109.95	108.12	107.60	107.43	108.41
40	51.26	43.52	119.34	114.57	113.44	113.23	116.01	113.96	113.36	113.17	**114.27**
41	51.72	43.63	120.99	115.04	113.54	113.26	116.01	113.96	113.36	113.17	**114.27**
42	52.00	44.47	121.79	115.75	114.29	113.71	121.68	119.40	118.73	118.52	119.88
43	52.16	44.48	122.33	115.93	114.49	114.29	121.68	119.40	118.73	118.52	119.88
44	58.50	47.98	124.19	116.67	114.54	114.30	128.03	125.46	124.71	124.46	126.40
45	62.14	51.83	125.78	117.92	116.59	116.38	128.03	125.46	124.71	124.46	126.40
46	62.35	51.86	126.16	118.46	116.75	116.42	132.86	130.04	129.21	128.94	131.33
47	63.34	53.07	128.32	120.64	118.90	118.57	132.86	130.04	129.21	128.94	131.33
48	64.70	53.47	134.85	127.98	126.42	126.16	140.56	137.35	136.40	136.09	139.40
49	65.30	53.55	135.23	128.04	126.43	126.17	140.56	137.35	136.40	136.09	139.40
50	67.63	55.63	141.55	133.81	132.00	131.67	145.55	141.93	140.85	140.50	145.94
51	69.62	56.11	143.21	134.30	132.13	131.70	145.55	141.93	140.85	140.50	145.94
52	70.27	56.41	151.52	141.22	138.85	138.37	147.93	144.30	143.23	142.87	147.29
53	73.99	60.82	152.25	141.54	138.94	138.39	147.93	144.30	143.23	142.87	147.29
54	75.97	61.68	156.79	145.66	142.90	142.31	150.55	146.77	145.64	145.27	150.20
55	76.44	61.78	157.17	145.76	142.94	142.32	151.80	147.63	146.39	145.97	156.58
56	78.35	63.48	175.01	161.35	157.98	157.23	190.97	186.90	185.83	185.51	169.04
57	80.52	64.65	175.30	162.19	158.26	157.31	198.41	193.61	192.30	191.91	180.32
58	82.37	65.64	175.73	162.72	159.66	159.03	198.41	193.61	192.30	191.91	180.32
59	87.91	70.38	177.92	162.90	159.69	159.03	212.76	206.73	205.04	204.51	194.75
60	88.87	70.49	195.56	174.63	169.25	167.91	212.76	206.73	205.04	204.51	194.75

Table 4.2: Octagasket Normalized Eigenvalues. Eigenvalues in boldface on level 4 are approximately R (14.95) times the eigenvalues in boldface on level 3

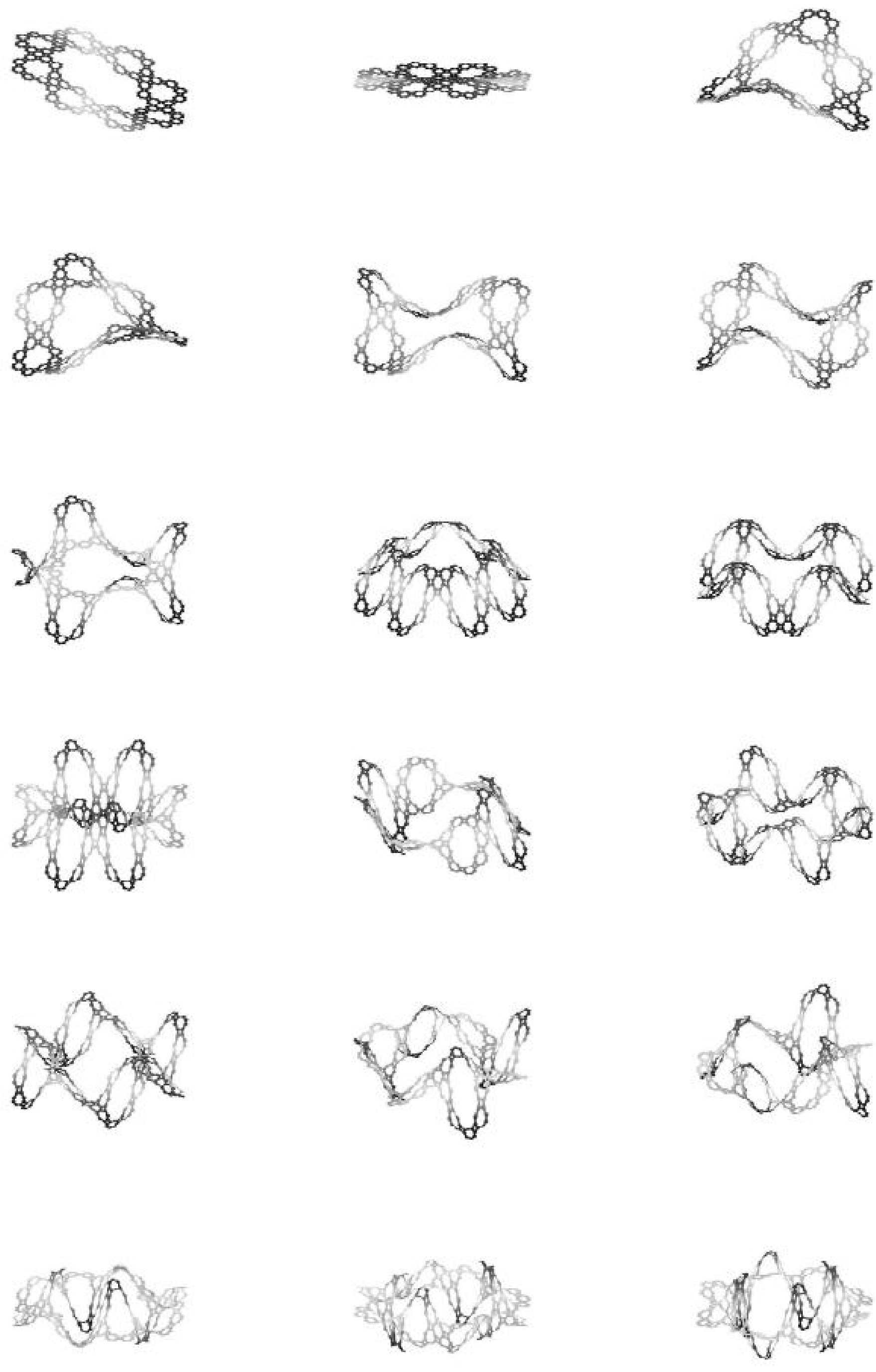


Figure 4.1: Octagasket Eigenfunctions, Level 3

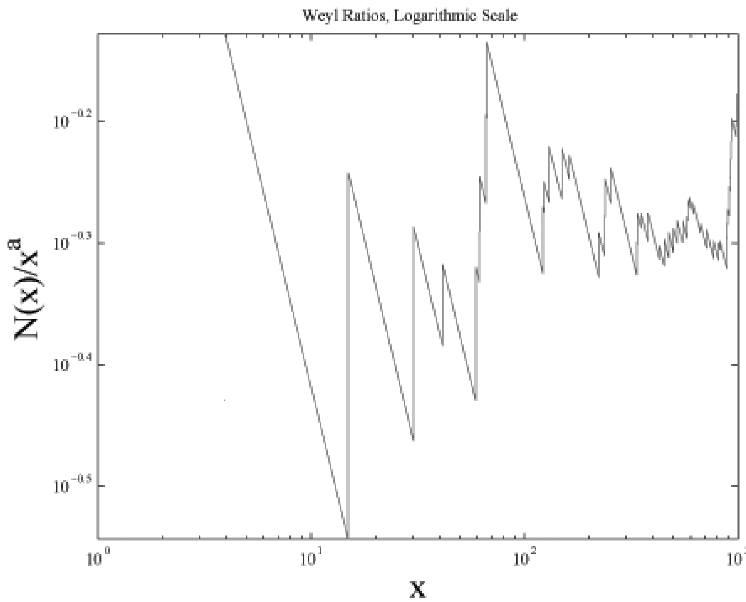


Figure 4.2: Octagasket Weyl Ratios, Level 4, 1 Refinement, $\alpha = .71938$

eigenfunctions and Figure 4.4 shows the Weyl ratios.

The last two examples we consider are alternate carpets. We subdivide the unit square into 16 subsquares of side length $\frac{1}{4}$, and retain all but the 4 inner squares ($\frac{12}{16}$ carpet) or all but 3 of the inner squares ($\frac{13}{16}$ carpet). The $\frac{12}{16}$ carpet has D_4 symmetry and it is known that a self-similar Laplacian exists. The $\frac{13}{16}$ carpet has no symmetry, and the methods used to construct a Laplacian on SC do not work on this example. So the situation is even worse than for the octagasket.

In tables 4.6 and 4.7 we present unnormalized and normalized eigenvalues for the $\frac{12}{16}$ carpet, and in Tables 4.8 and 4.9 the same data for the $\frac{13}{16}$ carpet. (Again we use the interior of the square for Ω). In Figure 4.5 we show the Weyl ratios for the $\frac{12}{16}$ carpet, and in Figure 4.6 we show those for the $\frac{13}{16}$ carpet. The evidence for convergence is strong in both cases. But the nature of the spectrum is quite different. In the symmetric $\frac{12}{16}$ carpet, we see multiplicities of 1 or 2, and an eigenvalue renormalization factor of about $R = 20.123$. For the $\frac{13}{16}$ carpet we do not see any multiplicities above 1, and there is no apparent eigenvalue renormalization factor. The evidence for spectral gaps is also weaker for the $\frac{13}{16}$ carpet, but this is not conclusive.

SC Unnormalized Eigenvalue Data

Level:	1	1	1	1	1	2	2	2	2	3	3	3	3
Refinement:	0	1	2	3	4	0	1	2	3	0	1	2	3
n													
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	6.9095	6.8043	6.7653	6.7505	6.7449	6.4313	6.2251	6.1354	6.10	5.87	5.6310	5.5325	5.4936
3	6.9151	6.8070	6.7664	6.7510	6.7451	6.4323	6.2254	6.1355	6.10	5.87	5.6310	5.5325	5.4936
4	18.9925	18.8600	18.8243	18.8150	18.8127	17.0488	16.5574	16.3441	16.26	15.57	14.9425	14.6841	14.5823
5	34.3954	33.3153	32.9506	32.8218	32.7746	32.1097	30.7611	30.2096	30.00	28.99	27.7633	27.2621	27.0651
6	47.0332	45.9328	45.6120	45.5142	45.4829	42.8446	41.1639	40.4941	40.24	38.71	37.0938	36.4341	36.1754
7	47.1243	45.9546	45.6175	45.5157	45.4834	42.8729	41.1729	40.4965	40.24	38.71	37.0938	36.4341	36.1754
8	52.1193	51.1387	50.8862	50.8221	50.8059	44.8902	43.1147	42.3923	42.12	40.48	38.7875	38.0972	37.8262
9	92.8584	89.8444	89.0822	88.8905	88.8425	66.1417	62.7984	61.4619	60.97	58.99	56.4159	55.3699	54.9598
10	93.0125	89.8865	89.0933	88.8933	88.8432	66.2982	62.8473	61.4753	60.97	58.99	56.4159	55.3699	54.9598
11	99.0749	95.5391	94.6410	94.4145	94.3576	71.9960	68.3327	66.9090	66.39	64.16	61.3551	60.2173	59.7724
12	103.3472	99.2335	98.1700	97.8905	97.8154	75.6967	71.6324	70.0686	69.50	67.31	64.3456	63.1414	62.6712
13	116.5071	111.4375	110.1583	109.8362	109.7554	90.1313	85.5167	83.8352	83.24	80.14	76.6848	75.2891	74.7457
14	138.4478	129.5749	127.0433	126.2799	126.0367	108.9329	102.2554	99.8578	99.01	96.14	91.8300	90.0930	89.4182
15	139.6143	132.0182	130.0016	129.4432	129.2825	109.1286	102.3168	99.8748	99.01	96.14	91.8300	90.0930	89.4182
16	140.8159	132.2970	130.0662	129.4575	129.2854	113.3883	104.7730	101.6793	100.57	98.96	94.3752	92.5219	91.7984
17	176.7080	165.3371	162.4409	161.6916	161.4947	161.7668	152.1424	148.7128	147.54	141.59	135.1528	132.6025	131.6229
18	177.1511	165.4399	162.4692	161.6997	161.4971	162.4681	152.3584	148.7706	147.56	141.59	135.1528	132.6025	131.6229
19	186.3468	174.2256	171.1654	170.3923	170.1977	168.6568	155.5422	151.3127	149.93	145.60	138.8378	136.1567	135.1289
20	196.9752	182.4076	178.8404	177.9499	177.7272	178.1670	171.8544	169.9944	169.49	156.72	150.0903	147.4883	146.4968
21	207.4874	192.1334	188.2268	187.2187	186.9544	185.7692	174.5895	170.8667	169.66	161.04	153.7589	150.8882	149.7908
22	255.1933	232.4075	226.4589	224.8390	224.3803	224.8288	207.6838	200.9126	198.63	193.66	184.0351	180.2588	178.8197
23	258.1854	233.1381	226.6316	224.8783	224.3886	225.7046	211.4197	207.1909	205.92	194.18	185.2580	181.7748	180.4537
24	281.0056	253.6079	246.3701	244.3644	243.7824	228.0738	211.4581	207.2012	205.93	194.18	185.2580	181.7748	180.4537
25	294.4730	267.9779	261.2521	259.5384	259.1060	267.3355	245.9265	238.5531	236.14	226.58	215.4968	211.1887	209.5562
26	296.0104	268.7574	261.9548	260.2517	259.8246	268.9097	246.0942	238.5881	236.15	226.58	215.4968	211.1887	209.5562
27	299.1848	269.4117	262.0976	260.2854	259.8329	287.3782	265.0656	258.6422	256.75	242.91	231.2213	226.7026	225.0039
28	331.3359	297.2510	288.6697	286.5081	285.9648	308.4704	283.9907	277.1404	275.23	256.13	244.2915	239.7481	238.0328
29	375.1390	328.3141	316.2114	312.9576	312.0478	327.3493	299.1712	290.6243	288.04	273.39	259.6175	254.3202	252.3353
30	411.4908	363.1051	350.9457	347.8623	347.0751	358.6256	317.3075	302.3844	297.35	296.49	279.3908	272.8143	270.3526
31	415.5502	364.2623	351.2403	347.9359	347.0933	366.8938	332.0219	321.4484	318.28	302.48	286.9005	280.9431	278.7202
32	422.8709	372.1289	359.5109	356.3576	355.5688	370.3870	332.2614	321.5045	318.29	302.48	286.9005	280.9431	278.7202
33	428.1018	373.1788	359.7477	356.4153	355.5831	402.0958	361.9981	350.5678	347.27	328.13	311.5196	305.2009	302.8469
34	436.6960	380.8205	367.3222	363.9749	363.1377	426.2507	379.0875	363.7003	358.88	346.65	327.5608	320.3207	317.6237
35	439.9719	381.6613	367.5250	364.0248	363.1501	431.0532	380.0237	363.9265	358.94	346.65	327.5608	320.3207	317.6237
36	451.1346	393.6453	379.2330	375.6239	374.7192	469.0618	415.1885	398.1174	392.86	378.00	357.2698	349.4391	346.5403
37	483.1916	416.8176	400.4065	396.3120	395.2878	481.9094	429.2890	414.4408	409.86	390.66	369.4654	361.4892	358.5420
38	493.4493	422.8265	405.5962	401.2680	400.1626	496.4015	433.4022	416.1715	411.27	394.69	373.5203	365.5479	362.6043
39	517.9593	439.4004	419.9247	414.9014	413.5579	504.2032	436.6132	416.8702	411.39	394.69	373.5203	365.5479	362.6043
40	521.0245	440.3721	420.1850	414.9666	413.5741	515.6273	445.0090	422.9119	416.05	404.24	380.8049	372.0055	368.7444
41	554.2167	471.2427	450.8932	445.8230	444.5550	536.0207	478.3656	461.6796	455.99	412.24	390.5909	382.4977	379.4817
42	561.4218	472.5244	451.1818	445.8931	444.5724	552.9508	482.3672	461.9683	456.00	425.16	402.1807	393.5960	390.4050
43	605.5574	504.9343	479.8486	473.4384	471.7517	558.1539	482.9689	463.2601	458.71	425.16	402.1807	393.5960	390.4050
44	616.5028	518.5646	493.7396	487.4812	485.9106	567.3412	488.3777	465.6960	459.52	434.77	411.2604	402.4804	399.2239
45	628.0957	524.1471	498.5358	492.2513	490.6859	574.2506	492.2321	468.0827	460.80	454.40	428.3600	418.6438	415.0663
46	634.2437	531.7247	505.9813	499.5675	497.9617	598.2522	519.5769	494.7583	487.00	466.46	438.7119	428.4159	424.6321
47	641.5337	532.7246	506.1940	499.6197	497.9748	608.7209	521.6171	495.0149	487.05	466.46	438.7119	428.4159	424.6321
48	667.4687	547.9734	517.8253	510.1141	508.0784	630.2738	531.2947	504.3023	496.86	473.55	445.8369	435.5522	431.7830
49	698.1628	573.9857	543.1162	535.4598	533.5479	664.4216	562.5890	529.0856	518.77	503.18	473.2766	462.2661	458.2523
50	700.7259	574.8348	543.7180	535.8214	533.7649	670.4290	576.8794	551.2608	543.26	514.10	483.1208	471.5134	467.2072
51	752.4778	618.6779	584.2646	575.3579	572.9848	683.4854	579.9359	551.3653	543.30	514.10	483.1208	471.6932	467.5056
52	772.0233	624.3244	585.6444	575.6898	573.0624	701.0202	589.9260	560.3374	553.09	514.95	483.2307	471.6932	467.5056
53	818.5900	672.0718	631.6665	621.5085	618.9558	803.3161	678.8308	641.9561	631.51	585.09	547.0977	533.1663	528.0563
54	833.2754	676.1826	632.6191	621.7422	619.0143	805.8113	679.9199	642.3952	631.64	590.98	552.3832	538.3562	533.2348
55	846.5291	681.5713	641.2196	631.1787	628.6572	832.7409	709.9328	668.0265	655.94	592.82	554.2646	540.1749	535.0612
56	861.5834	694.2754	650.1452	639.0805	636.2458	848.3769	711.2532	670.9435	659.33	592.82	554.2646	540.1749	535.0612
57	868.5235	702.4581	659.4855	648.8736	646.2178	866.9036	733.0618	700.4748	690.67	610.41	569.6585	554.8341	549.3760
58	869.8747	703.6618	659.8472	649.0127	646.3092	881.1987	754.4762	717.1605	706.72	610.41	569.6585	554.8341	549.3760
59	874.0871	704.4428	659.9817	649.0347	646.3133	885.7117	755.2010	718.9525	708.32	611.55	570.8047	555.9232	550.4958
60	969.8144	782.3749	728.5923	715.1103	711.7366	901.5093	767.8076	720.6769	708.45	611.66	572.3659	558.0957	552.8390

Table 4.3: SC Unnormalized Eigenvalues

SC Normalized Eigenvalue Data

Level:	1	1	1	1	1	2	2	2	2	3	3	3	3
Refinement:	0	1	2	3	4	0	1	2	3	0	1	2	3
n													
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3	1.0008	1.0004	1.0002	1.0001	1.0000	1.0002	1.0001	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4	2.7488	2.7718	2.7825	2.7872	2.7892	2.6509	2.6598	2.6639	2.6657	2.6530	2.6536	2.6541	2.6544
5	4.9780	4.8962	4.8706	4.8621	4.8592	4.9927	4.9414	4.9238	4.9181	4.9383	4.9305	4.9276	4.9267
6	6.8070	6.7505	6.7421	6.7423	6.7433	6.6619	6.6126	6.6001	6.5973	6.5950	6.5875	6.5855	6.5850
7	6.8202	6.7537	6.7429	6.7425	6.7434	6.6663	6.6140	6.6005	6.5974	6.5950	6.5875	6.5855	6.5850
8	7.5431	7.5156	7.5217	7.5286	7.5325	6.9800	6.9259	6.9095	6.9052	6.8963	6.8882	6.8861	6.8855
9	13.4392	13.2040	13.1676	13.1679	13.1719	10.2844	10.0879	10.0176	9.9946	10.0497	10.0189	10.0081	10.0043
10	13.4615	13.2102	13.1692	13.1684	13.1720	10.3087	10.0958	10.0198	9.9951	10.0497	10.0189	10.0081	10.0043
11	14.3389	14.0409	13.9892	13.9862	13.9895	11.1947	10.9769	10.9055	10.8833	10.9307	10.8960	10.8843	10.8804
12	14.9572	14.5839	14.5109	14.5012	14.5022	11.7701	11.5070	11.4205	11.3931	11.4684	11.4271	11.4128	11.4080
13	16.8618	16.3774	16.2829	16.2708	16.2724	14.0146	13.7374	13.6643	13.6459	13.6534	13.6184	13.6085	13.6059
14	20.0373	19.0430	18.7787	18.7067	18.6863	16.9380	16.4263	16.2758	16.2307	16.3794	16.3081	16.2843	16.2768
15	20.2061	19.4020	19.2160	19.1753	19.1675	16.9684	16.4361	16.2786	16.2314	16.3794	16.3081	16.2843	16.2768
16	20.3800	19.4430	19.2256	19.1774	19.1680	17.6308	16.8307	16.5727	16.4879	16.8602	16.7600	16.7233	16.7100
17	25.5746	24.2988	24.0110	23.9524	23.9433	25.1532	24.4401	24.2386	24.1877	24.1225	24.0017	23.9679	23.9593
18	25.6387	24.3139	24.0152	23.9536	23.9437	25.2622	24.4748	24.2481	24.1901	24.1225	24.0017	23.9679	23.9593
19	26.9696	25.6051	25.3006	25.2413	25.2337	26.2245	24.9863	24.6624	24.5786	24.8067	24.6561	24.6103	24.5975
20	28.5078	26.8075	26.4351	26.3609	26.3500	27.7032	27.6067	27.7073	27.7851	26.7006	26.6545	26.6585	26.6668
21	30.0292	28.2369	27.8225	27.7340	27.7180	28.8853	28.0460	27.8495	27.8134	27.4371	27.3060	27.2730	27.2664
22	36.9336	34.1558	33.4737	33.3069	33.2668	34.9587	33.3623	32.7467	32.5627	32.9950	32.6827	32.5818	32.5505
23	37.3666	34.2631	33.4993	33.3127	33.2680	35.0949	33.9624	33.7700	33.7589	33.0824	32.8999	32.8558	32.8479
24	40.6694	37.2715	36.4169	36.1993	36.1434	35.4633	33.9686	33.7716	33.7593	33.0824	32.8999	32.8558	32.8479
25	42.6185	39.3834	38.6166	38.4471	38.4153	41.5681	39.5056	38.8817	38.7131	38.6022	38.2700	38.1724	38.1455
26	42.8410	39.4979	38.7205	38.5528	38.5218	41.8128	39.5325	38.8874	38.7146	38.6022	38.2700	38.1724	38.1455
27	43.3004	39.5941	38.7416	38.5578	38.5230	44.6845	42.5801	42.1560	42.0914	41.3849	41.0625	40.9765	40.9574
28	47.9536	43.6855	42.6693	42.4424	42.3974	47.9642	45.6202	45.1710	45.1209	43.6369	43.3836	43.3345	43.3291
29	54.2931	48.2507	46.7404	46.3605	46.2645	50.8997	48.0588	47.3687	47.2201	46.5777	46.1053	45.9684	45.9325
30	59.5542	53.3637	51.8746	51.5312	51.4576	55.7628	50.9722	49.2855	48.7473	50.5131	49.6169	49.3112	49.2122
31	60.1417	53.5338	51.9181	51.5421	51.4603	57.0484	53.3359	52.3928	52.1778	51.5339	50.9505	50.7805	50.7354
32	61.2012	54.6899	53.1406	52.7896	52.7169	57.5916	53.3744	52.4019	52.1804	51.5339	50.9505	50.7805	50.7354
33	61.9583	54.8442	53.1756	52.7982	52.7190	62.5220	58.1513	57.1389	56.9307	55.9042	55.3226	55.1651	55.1271
34	63.2021	55.9673	54.2952	53.9180	53.8391	66.2778	60.8965	59.2794	58.8346	59.0590	58.1714	57.8980	57.8170
35	63.6762	56.0909	54.3252	53.9254	53.8409	67.0246	61.0469	59.3162	58.8439	59.0590	58.1714	57.8980	57.8170
36	65.2918	57.8521	56.0558	55.6437	55.5562	72.9346	66.6958	64.8890	64.4045	64.4013	63.4474	63.1611	63.0807
37	69.9313	61.2576	59.1856	58.7083	58.6057	74.9322	68.9609	67.5495	67.1924	66.5581	65.6132	65.3392	65.2653
38	71.4159	62.1407	59.9527	59.4425	59.3284	77.1856	69.6217	67.8316	67.4236	67.2439	66.3333	66.0728	66.0048
39	74.9632	64.5765	62.0706	61.4621	61.3144	78.3987	70.1375	67.9455	67.4430	67.2439	66.3333	66.0728	66.0048
40	75.4068	64.7193	62.1091	61.4718	61.3168	80.1751	71.4862	68.9302	68.2071	68.8714	67.6269	67.2400	67.1225
41	80.2106	69.2562	66.6482	66.0427	65.9101	83.3460	76.8446	75.2490	74.7537	70.2347	69.3648	69.1365	69.0770
42	81.2534	69.4446	66.6908	66.0531	65.9127	85.9785	77.4874	75.2960	74.7554	72.4361	71.4231	71.1425	71.0653
43	87.6411	74.2077	70.9282	70.1336	69.9423	86.7875	77.5840	75.5066	75.2006	72.4361	71.4231	71.1425	71.0653
44	89.2252	76.2108	72.9815	72.2138	72.0415	88.2161	78.4529	75.9036	75.3333	74.0737	73.0355	72.7483	72.6706
45	90.9030	77.0313	73.6904	72.9205	72.7495	89.2904	79.0721	76.2926	75.5423	77.4183	76.0722	75.6699	75.5544
46	91.7928	78.1449	74.7910	74.0043	73.8282	93.0224	83.4647	80.6405	79.8375	79.4725	77.9106	77.4362	77.2957
47	92.8479	78.2919	74.8224	74.0120	73.8301	94.6502	83.7925	80.6823	79.8453	79.4725	77.9106	77.4362	77.2957
48	96.6014	80.5329	76.5417	75.5666	75.3281	98.0015	85.3471	82.1960	81.4535	80.6806	79.1759	78.7261	78.5974
49	101.0437	84.3558	80.2800	79.3212	79.1042	103.3111	90.3742	86.2354	85.0454	85.7282	84.0489	83.5546	83.4156
50	101.4146	84.4806	80.3690	79.3748	79.1364	104.2452	92.6698	89.8498	89.0616	87.5885	85.7972	85.2261	85.0456
51	108.9046	90.9240	86.3623	85.2316	84.9512	106.2754	93.1608	89.8668	89.0675	87.5885	85.7972	85.2586	85.0999
52	111.7333	91.7538	86.5663	85.2808	84.9627	109.0018	94.7656	91.3292	90.6730	87.7330	85.8167	85.2586	85.0999
53	118.4728	98.7710	93.3690	92.0682	91.7669	124.9079	109.0473	104.6322	103.5291	99.6834	97.1588	96.3698	96.1220
54	120.5982	99.3752	93.5097	92.1028	91.7756	125.2959	109.2222	104.7037	103.5495	100.6863	98.0975	97.3079	97.0646
55	122.5164	100.1671	94.7810	93.5007	93.2052	129.4831	114.0435	108.8814	107.5333	101.0009	98.4316	97.6366	97.3971
56	124.6952	102.0342	96.1004	94.6712	94.3303	131.9144	114.2556	109.3568	108.0895	101.0009	98.4316	97.6366	97.3971
57	125.6996	103.2367	97.4810	96.1220	95.8088	134.7951	117.7589	114.1701	113.2266	103.9967	101.1654	100.2863	100.0028
58	125.8952	103.4137	97.5344	96.1426	95.8223	137.0179	121.1989	116.8897	115.8587	103.9967	101.1654	100.2863	100.0028
59	126.5048	103.5284	97.5543	96.1458	95.8229	137.7196	121.3154	117.1818	116.1207	104.1921	101.3689	100.4831	100.2066
60	140.3592	114.9817	107.6959	105.9340	105.5226	140.1760	123.3405	117.4628	116.1421	104.2096	101.6462	100.8758	100.6332

Table 4.4: SC Normalized Eigenvalues

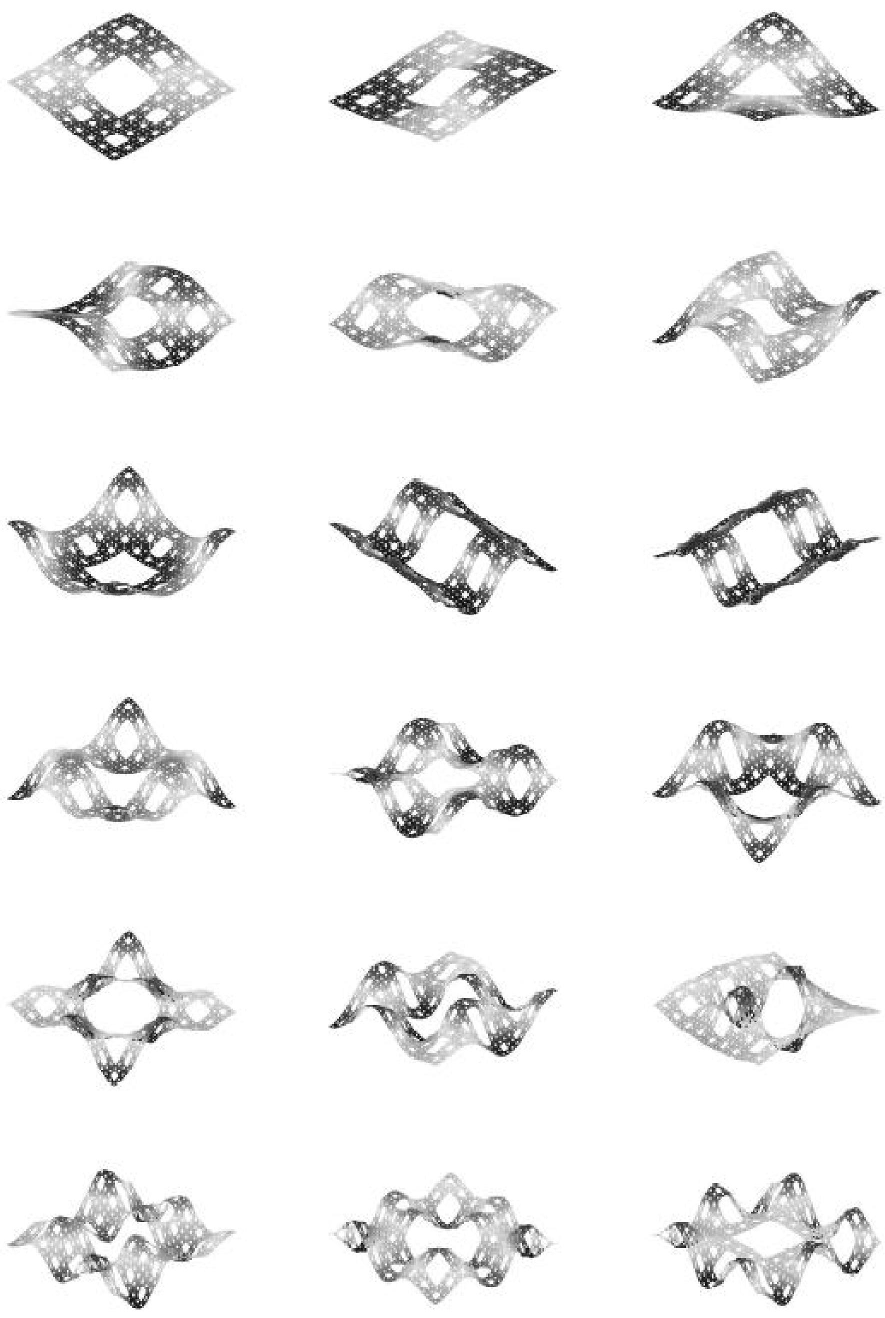


Figure 4.3: Sierpinski Carpet (SC) Eigenfunctions, Level 4

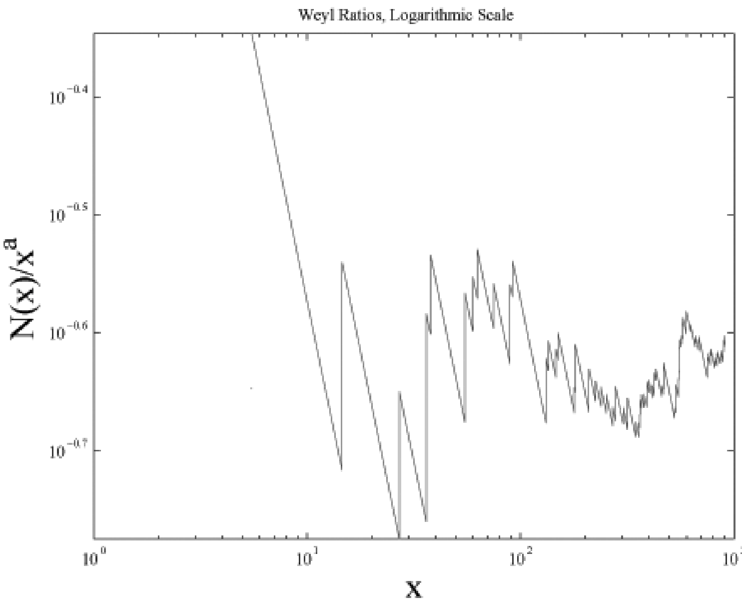


Figure 4.4: SC Weyl Ratios, Level 3, 3 Refinements, $\alpha = .87392$

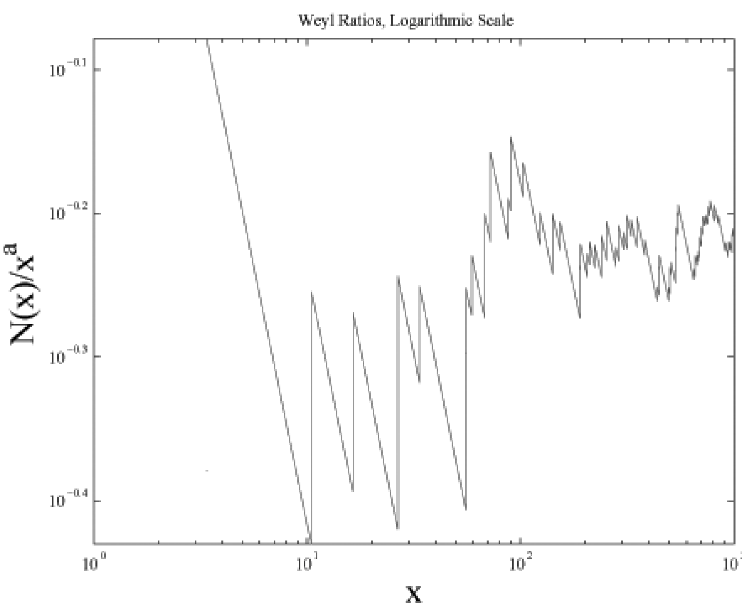


Figure 4.5: $\frac{12}{16}$ Carpet Weyl Ratios, Level 3, 0 Refinements, $\alpha = .71738$

Sierpinski Carpet, Level 3, 3 Refinements, D4 Representation Type

Number	Eigenvalue	Eigenfunction Type	Number	Eigenvalue	Eigenfunction Type
1	5.4936	2	48	458.2523	1+ -
2	5.4936	2	49	467.2072	1- +
3	14.5823	1+ -	50	467.5056	2
4	27.0651	1- +	51	467.5056	2
5	36.1754	2	52	528.0563	1- +
6	36.1754	2	53	533.2348	1+ +
7	37.8262	1+ +	54	535.0612	2
8	54.9598	2	55	535.0612	2
9	54.9598	2	56	549.3760	2
10	59.7724	1+ -	57	549.3760	2
11	62.6712	1- -	58	550.4958	1+ -
12	74.7457	1+ +	59	552.8390	1+ +
13	89.4182	2	60	553.6998	2
14	89.4182	2			
15	91.7984	1- +			
16	131.6229	2			
17	131.6229	2			
18	135.1289	1- +			
19	146.4968	1+ -			
20	149.7908	1+ +			
21	178.8197	1- -			
22	180.4537	2			
23	180.4537	2			
24	209.5562	2			
25	209.5562	2			
26	225.0039	1+ +			
27	238.0328	1+ -			
28	252.3353	1- -			
29	270.3526	1- +			
30	278.7202	2			
31	278.7202	2			
32	302.8469	1+ -			
33	317.6237	2			
34	317.6237	2			
35	346.5403	1+ +			
36	358.5420	1- +			
37	362.6043	2			
38	362.6043	2			
39	368.7444	1- -			
40	379.4817	1+ +			
41	390.4050	2			
42	390.4050	2			
43	399.2239	1+ -			
44	415.0663	1+ +			
45	424.6321	2			
46	424.6321	2			
47	431.7830	1- -			

Table 4.5: D_4 Representation Type

12/16 Symmetric Carpet Unnormalized Eigenvalues

Level:	1	1	1	2	2	2	3
Refinement:	0	1	2	0	1	2	0
n							
1	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	5.184	5.108	5.079	4.246	4.119	4.065	3.38
3	5.194	5.113	5.081	4.246	4.119	4.065	3.38
4	16.788	16.699	16.673	13.176	12.805	12.649	10.47
5	25.029	24.128	23.805	20.638	19.903	19.593	16.37
6	43.231	42.181	41.846	33.584	32.459	31.992	26.58
7	43.301	42.210	41.858	33.584	32.459	31.992	26.58
8	58.008	56.942	56.648	42.636	41.255	40.689	33.70
9	93.773	89.292	87.948	70.623	68.005	66.881	55.56
10	101.704	98.004	96.965	70.623	68.005	66.961	55.56
11	102.074	98.095	96.985	70.890	68.023	66.961	55.62
12	107.886	104.162	103.184	75.068	72.356	71.284	59.05
13	168.673	160.716	158.630	86.447	83.122	81.802	67.90
14	168.747	160.725	158.632	86.472	83.124	81.803	67.90
15	175.179	166.385	164.087	92.260	88.575	87.127	72.27
16	175.369	166.449	164.106	92.260	88.575	87.127	72.27
17	195.384	183.514	180.284	112.216	107.801	106.110	87.76
18	195.913	183.653	180.315	115.148	110.400	108.576	90.08
19	198.538	186.390	183.291	115.148	110.400	108.576	90.08
20	226.387	208.148	203.133	132.107	125.817	123.405	102.89
21	250.061	230.500	225.375	159.110	152.124	149.550	123.50
22	258.904	237.074	231.163	183.241	173.226	169.527	141.49
23	275.083	252.049	245.674	183.241	173.226	169.527	141.49
24	276.210	252.346	245.734	199.505	186.143	181.305	153.30
25	293.995	270.945	264.860	251.553	235.159	229.326	189.94
26	301.675	274.485	267.363	251.553	235.159	229.326	189.94
27	303.099	274.895	267.479	253.119	235.612	229.495	191.29
28	357.036	327.150	318.681	274.092	255.344	248.670	205.68
29	390.429	332.905	318.781	276.411	269.536	267.090	210.72
30	411.828	361.426	347.970	299.289	275.422	267.421	223.12
31	414.711	366.359	353.561	318.345	301.347	294.251	239.50
32	416.004	366.831	353.679	318.345	301.347	294.251	239.50
33	434.933	389.600	377.634	341.808	315.583	308.149	253.11
34	468.215	408.316	392.905	341.808	315.583	308.149	253.11
35	476.368	409.745	393.225	382.105	347.140	334.977	279.87
36	540.090	462.398	442.531	387.080	356.422	342.840	288.43
36	588.083	490.445	465.059	396.599	366.152	359.525	288.50
37	598.813	516.054	493.601	410.635	385.344	373.006	304.18
38	605.365	517.742	493.954	437.436	389.213	373.006	315.47
40	619.523	521.657	496.133	437.436	389.213	377.237	315.47
41	621.643	532.628	510.347	461.115	405.563	387.438	329.91
42	677.697	568.112	538.900	479.305	446.788	436.606	350.60
43	685.080	569.402	539.138	479.305	446.788	436.606	350.60
44	744.709	619.345	585.659	525.984	488.838	477.254	381.79
45	806.313	678.293	641.135	611.631	562.499	546.976	437.50
46	817.160	681.832	643.705	622.996	575.895	561.102	445.75
47	825.398	684.435	644.671	623.144	575.895	561.102	445.75
48	830.539	688.865	650.019	623.144	580.403	566.833	447.11
49	842.319	689.219	650.609	688.848	646.061	633.294	494.74
50	856.140	694.608	653.557	699.940	653.837	639.759	501.09
51	864.861	696.009	654.689	699.940	653.837	639.759	501.09
52	882.370	724.078	681.934	716.995	663.525	646.620	509.19
53	899.459	729.338	683.412	752.776	692.509	673.230	532.66
54	913.961	730.414	685.271	759.692	694.576	673.230	532.94
55	949.231	751.437	701.637	759.692	694.576	674.760	532.94
56	957.747	754.911	702.461	766.267	699.657	677.267	536.83
57		802.767	748.028	771.554	700.249	677.322	536.83
58		804.804	748.705	782.463	709.971	687.690	544.75
59		821.999	758.507	782.463	709.971	687.690	544.75
60		839.997	780.140	789.713	715.217	692.308	548.14

Table 4.6: 12/16 Carpet Unnormalized Eigenvalues

12/16 Symmetric Carpet Normalized Eigenvalues

Level:	1	1	1	2	2	2	3
Refinement:	0	1	2	0	1	2	0
n							
1	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	1.000	1.000	1.000	1.000	1.000	1.000	1.000
3	1.002	1.001	1.000	1.000	1.000	1.000	1.000
4	3.239	3.269	3.283	3.103	3.109	3.112	3.097
5	4.829	4.724	4.687	4.861	4.831	4.820	4.841
6	8.340	8.258	8.239	7.910	7.879	7.870	7.859
7	8.354	8.263	8.242	7.910	7.879	7.870	7.859
8	11.191	11.147	11.154	10.042	10.015	10.009	9.967
9	18.091	17.481	17.317	16.633	16.508	16.453	16.431
10	19.621	19.186	19.092	16.633	16.508	16.472	16.431
11	19.692	19.204	19.096	16.696	16.513	16.472	16.449
12	20.813	20.392	20.316	17.680	17.565	17.536	17.463
13	32.540	31.463	31.233	20.360	20.178	20.123	20.081
14	32.555	31.465	31.234	20.366	20.178	20.124	20.081
15	33.795	32.573	32.308	21.729	21.502	21.433	21.373
16	33.832	32.586	32.312	21.729	21.502	21.433	21.373
17	37.693	35.927	35.497	26.429	26.169	26.103	25.953
18	37.795	35.954	35.503	27.120	26.800	26.710	26.640
19	38.302	36.490	36.089	27.120	26.800	26.710	26.640
20	43.674	40.749	39.996	31.114	30.542	30.358	30.428
21	48.242	45.125	44.375	37.474	36.928	36.789	36.523
22	49.948	46.412	45.515	43.157	42.051	41.704	41.843
23	53.069	49.344	48.372	43.157	42.051	41.704	41.843
24	53.286	49.402	48.384	46.988	45.186	44.601	45.338
25	56.717	53.043	52.150	59.246	57.085	56.414	56.173
26	58.199	53.736	52.642	59.246	57.085	56.414	56.173
27	58.474	53.816	52.665	59.615	57.195	56.456	56.572
28	68.879	64.046	62.747	64.554	61.985	61.173	60.827
29	75.321	65.173	62.766	65.101	65.430	65.704	62.317
30	79.450	70.756	68.514	70.489	66.859	65.786	65.987
31	80.006	71.722	69.614	74.977	73.152	72.386	70.830
32	80.255	71.815	69.638	74.977	73.152	72.386	70.830
33	83.907	76.272	74.354	80.503	76.608	75.805	74.855
34	90.328	79.936	77.361	80.503	76.608	75.805	74.855
35	91.901	80.216	77.424	89.993	84.268	82.405	82.767
36	104.194	90.524	87.132	91.165	86.521	84.339	85.300
37	113.453	96.015	91.568	93.407	88.884	88.443	85.321
38	115.523	101.028	97.188	96.713	93.542	91.760	89.958
39	116.787	101.359	97.257	103.025	94.482	91.760	93.297
40	119.518	102.125	97.686	103.025	94.482	92.801	93.297
41	119.927	104.273	100.485	108.602	98.451	95.310	97.568
42	130.741	111.219	106.107	112.886	108.458	107.405	103.688
43	132.165	111.472	106.154	112.886	108.458	107.405	103.688
44	143.669	121.249	115.313	123.880	118.666	117.405	112.911
45	155.554	132.790	126.236	144.052	136.547	134.556	129.385
46	157.646	133.482	126.742	146.728	139.799	138.032	131.825
47	159.236	133.992	126.933	146.763	139.799	138.032	131.825
48	160.227	134.859	127.986	146.763	140.893	139.441	132.228
49	162.500	134.929	128.102	162.238	156.832	155.791	146.313
50	165.166	135.984	128.682	164.850	158.719	157.381	148.192
51	166.849	136.258	128.905	164.850	158.719	157.381	148.192
52	170.226	141.753	134.269	168.867	161.071	159.069	150.587
53	173.523	142.783	134.560	177.294	168.107	165.615	157.527
54	176.321	142.993	134.927	178.923	168.609	165.615	157.610
55	183.125	147.109	138.149	178.923	168.609	165.991	157.610
56	184.768	147.789	138.311	180.472	169.842	166.608	158.761
57		157.158	147.283	181.717	169.986	166.622	158.761
58		157.557	147.416	184.286	172.346	169.172	161.104
59		160.923	149.346	184.286	172.346	169.172	161.104
60		164.446	153.606	185.994	173.619	170.308	162.107

Table 4.7: 12/16 Carpet Normalized Eigenvalues

13/16 Alternate Carpet Unnormalized Eigenvalues

Level:	1	1	1	2	2	2	3
Refinement:	0	1	2	0	1	2	0
n							
1	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	8.141	8.025	7.981	7.777	7.590	7.512	7.261
3	8.328	8.201	8.151	7.920	7.724	7.643	7.375
4	18.904	18.728	18.674	18.046	17.656	17.498	16.889
5	36.040	35.189	34.898	33.783	32.859	32.488	31.352
6	41.026	40.392	40.209	38.618	37.681	37.307	36.009
7	47.908	46.956	46.686	45.136	44.028	43.592	42.108
8	51.147	50.262	50.022	47.827	46.543	46.037	44.385
9	78.834	76.085	75.308	72.746	70.651	69.849	67.394
10	83.837	79.862	78.600	77.387	74.736	73.703	71.197
11	98.313	95.303	94.500	91.043	88.502	87.534	84.498
12	101.603	97.886	96.879	93.070	90.247	89.174	86.066
13	110.047	105.754	104.599	98.936	95.459	94.134	90.796
14	130.410	123.766	121.928	118.044	113.854	112.296	108.495
15	158.249	149.670	147.378	137.776	132.030	129.857	125.544
16	168.441	160.642	158.609	138.092	132.224	130.016	125.671
17	168.531	160.653	158.611	140.785	135.943	134.176	129.506
18	177.769	168.614	166.217	145.902	139.753	137.470	132.850
19	180.457	170.836	168.310	147.908	141.626	139.298	134.689
20	212.057	199.474	196.150	176.061	168.656	165.974	160.416
21	218.578	203.366	199.309	180.743	173.034	170.242	164.492
22	224.839	208.408	204.094	187.949	180.161	177.368	171.216
23	244.328	220.798	214.229	202.900	192.952	189.318	183.189
24	276.284	252.641	245.740	223.491	212.361	208.388	201.776
25	279.689	253.095	246.152	227.445	216.001	211.926	204.606
26	282.172	260.464	254.842	254.339	244.861	241.567	232.848
27	284.558	261.350	255.511	254.348	245.378	242.312	233.531
28	296.378	271.155	264.532	266.098	253.406	249.096	240.709
29	315.919	285.291	277.333	269.589	258.570	254.766	244.795
30	345.168	315.927	308.136	306.237	287.886	281.712	271.460
31	356.577	326.898	318.697	309.963	300.347	297.247	286.438
32	387.016	335.984	322.765	327.873	315.925	311.747	299.839
33	412.864	370.586	359.404	355.748	337.374	331.115	319.702
34	419.313	374.241	362.670	356.922	341.061	335.720	323.243
35	429.690	377.769	364.163	361.714	344.265	338.169	325.064
36	431.820	383.436	371.330	372.504	353.870	347.561	333.930
37	449.266	398.314	384.230	379.569	356.686	349.061	336.157
38	473.522	412.916	397.013	385.570	365.766	359.457	346.095
39	476.703	413.305	397.076	385.668	368.320	363.064	349.714
40	493.685	424.342	407.726	419.014	399.057	392.766	377.470
41	519.345	450.491	432.843	429.524	414.512	410.168	394.171
42	543.384	466.699	447.185	456.574	433.048	423.859	408.458
43	585.649	501.099	478.186	467.753	439.744	432.360	416.050
44	591.644	514.912	489.882	491.355	464.059	455.055	437.295
45	601.176	517.679	496.841	498.651	470.180	461.081	443.572
46	620.159	520.471	497.666	506.206	476.228	466.560	448.465
47	622.937	532.290	509.206	509.421	485.929	478.706	460.467
48	651.834	544.330	515.901	513.937	489.455	481.930	463.403
49	660.203	555.040	530.133	539.091	507.854	498.336	479.947
50	675.397	567.572	538.424	541.151	512.129	503.378	483.585
51	694.404	583.198	554.867	557.873	526.650	516.944	497.031
52	753.195	627.341	589.310	604.534	567.907	556.668	534.699
53	771.635	629.002	595.643	617.337	576.543	563.962	543.660
54	804.767	672.510	629.396	631.441	587.316	572.032	548.612
55	811.516	677.324	643.350	637.710	596.322	583.937	561.722
56	836.030	680.492	644.141	660.452	620.202	606.507	581.113
57	844.566	691.307	652.911	663.900	621.142	608.571	585.293
58	850.130	694.214	653.524	669.942	629.911	617.199	591.844
59	866.855	704.415	661.306	685.466	639.886	625.486	600.271
60	873.875	716.879	674.548	686.218	640.602	626.171	601.676

Table 4.8: 13/16 Carpet Unnormalized Eigenvalues

13/16 Alternate Carpet Normalized Eigenvalues

Level:	1	1	1	2	2	2	3
Refinement:	0	1	2	0	1	2	0
n							
1	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	1.000	1.000	1.000	1.000	1.000	1.000	1.000
3	1.023	1.022	1.021	1.018	1.018	1.017	1.016
4	2.322	2.334	2.340	2.320	2.326	2.329	2.326
5	4.427	4.385	4.373	4.344	4.329	4.325	4.318
6	5.040	5.033	5.038	4.966	4.965	4.966	4.960
7	5.885	5.851	5.850	5.804	5.801	5.803	5.799
8	6.283	6.263	6.268	6.150	6.132	6.128	6.113
9	9.684	9.481	9.436	9.354	9.309	9.298	9.282
10	10.299	9.952	9.849	9.951	9.847	9.811	9.806
11	12.077	11.876	11.841	11.706	11.661	11.652	11.638
12	12.481	12.198	12.139	11.967	11.891	11.870	11.854
13	13.518	13.178	13.107	12.721	12.577	12.531	12.505
14	16.020	15.423	15.278	15.178	15.001	14.948	14.943
15	19.440	18.651	18.467	17.715	17.396	17.286	17.291
16	20.692	20.018	19.874	17.756	17.421	17.307	17.309
17	20.703	20.020	19.874	18.102	17.911	17.861	17.837
18	21.837	21.012	20.827	18.760	18.413	18.299	18.297
19	22.168	21.289	21.090	19.018	18.660	18.543	18.551
20	26.049	24.857	24.578	22.638	22.222	22.094	22.094
21	26.851	25.342	24.974	23.240	22.798	22.662	22.655
22	27.620	25.970	25.574	24.167	23.738	23.610	23.581
23	30.014	27.514	26.843	26.089	25.423	25.201	25.230
24	33.939	31.482	30.792	28.737	27.980	27.740	27.790
25	34.358	31.539	30.844	29.245	28.460	28.211	28.180
26	34.663	32.457	31.932	32.703	32.262	32.156	32.070
27	34.956	32.568	32.016	32.704	32.330	32.255	32.164
28	36.408	33.790	33.147	34.215	33.388	33.158	33.153
29	38.808	35.551	34.751	34.664	34.068	33.913	33.715
30	42.401	39.369	38.610	39.376	37.931	37.500	37.388
31	43.803	40.736	39.934	39.855	39.573	39.568	39.451
32	47.542	41.868	40.443	42.158	41.625	41.498	41.297
33	50.717	46.180	45.034	45.742	44.451	44.076	44.032
34	51.509	46.635	45.444	45.893	44.937	44.689	44.520
35	52.784	47.075	45.631	46.510	45.359	45.015	44.771
36	53.046	47.781	46.529	47.897	46.625	46.266	45.992
37	55.189	49.635	48.145	48.805	46.996	46.465	46.299
38	58.168	51.455	49.747	49.577	48.192	47.849	47.667
39	58.559	51.503	49.755	49.589	48.529	48.329	48.166
40	60.645	52.879	51.089	53.877	52.579	52.283	51.989
41	63.797	56.137	54.236	55.229	54.615	54.600	54.289
42	66.750	58.157	56.034	58.707	57.057	56.422	56.256
43	71.942	62.444	59.918	60.144	57.939	57.554	57.302
44	72.679	64.165	61.384	63.179	61.143	60.575	60.228
45	73.850	64.510	62.256	64.117	61.949	61.377	61.093
46	76.182	64.858	62.359	65.088	62.746	62.106	61.767
47	76.523	66.331	63.805	65.502	64.025	63.723	63.420
48	80.073	67.831	64.644	66.082	64.489	64.152	63.824
49	81.101	69.165	66.427	69.317	66.913	66.336	66.103
50	82.967	70.727	67.466	69.582	67.477	67.007	66.604
51	85.302	72.674	69.526	71.732	69.390	68.813	68.456
52	92.524	78.175	73.842	77.732	74.826	74.101	73.644
53	94.789	78.382	74.636	79.378	75.964	75.072	74.878
54	98.859	83.804	78.865	81.191	77.383	76.146	75.560
55	99.688	84.404	80.614	81.997	78.570	77.731	77.365
56	102.700	84.798	80.713	84.922	81.716	80.735	80.036
57	103.748	86.146	81.812	85.365	81.840	81.010	80.612
58	104.432	86.508	81.888	86.142	82.995	82.159	81.514
59	106.486	87.780	82.863	88.138	84.309	83.262	82.675
60	107.349	89.333	84.523	88.235	84.404	83.353	82.868

Table 4.9: 13/16 Carpet Normalized Eigenvalues

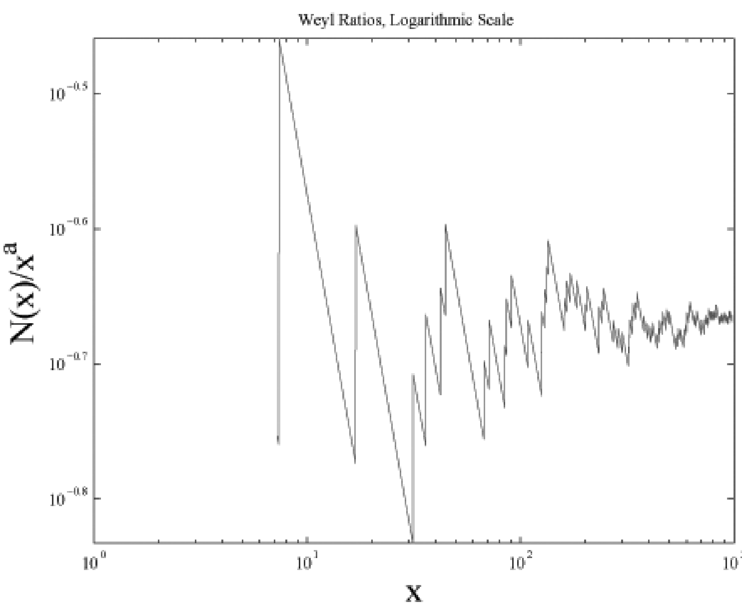


Figure 4.6: $\frac{13}{16}$ Carpet Weyl Ratios, Level 3, 0 Refinements, $\alpha = .87537$

5 Miniaturization

In order to make the ideas clear, we begin by explaining the method of miniaturization on the unit interval I. Here we have a two element group of symmetries consisting of the identity and the reflection $\rho(x) = 1 - x$ about the midpoint. Every Neumann eigenfunction is of the form $\cos \pi kx$. When k is even, the function is even under ρ, namely $u \circ \rho = u$, while if k is odd then the function is odd, namely $u \circ \rho = -u$. In this way all eigenspaces are sorted corresponding to the two irreducible representations of the symmetry group. For every even eigenfunction u (except the constant), we can miniaturize it by defining u_+ to be

$$u_+(x) = \begin{cases} u \circ F_0^{-1} \text{ on } F_0 I \\ u \circ F_1^{-1} \text{ on } F_1 I \end{cases} \tag{5.1}$$

Note that $u \circ F_0^{-1}(\frac{1}{2}) = u \circ F_1^{-1}(\frac{1}{2})$ because u is even, and the derivative vanishes at $\frac{1}{2}$ because u is a Neumann eigenfunction. This shows that u_+ is also a Neumann eigenfunction, and indeed $u_+(x) = \cos 2\pi kx$. On the other hand, if u is an odd eigenfunction, then define u_- by

$$u_+(x) = \begin{cases} \quad u \circ F_0^{-1} \text{ on } F_0 I \\ -u \circ F_1^{-1} \text{ on } F_1 I \end{cases} \tag{5.2}$$

Again $u \circ F_0^{-1}(\frac{1}{2}) = -u \circ F_1^{-1}(\frac{1}{2})$ because u is odd, so u_- is also a Neumann eigenfunction, and again $u_-(x) = \cos 2\pi kx$. We call u_+ or u_- the miniaturization of u. Note that the representation type of the miniaturization is always even. The eigenvalue of u_+ or u_- is always 4 times the eigenvalue of u. Thus $R = 4$ is an eigenvalue renormalization factor. (Of course I has other eigenvalue renormalization factors, namely any square integer, but such luxuries do not generalize to other fractals).

Now consider a self-similar fractal with a finite group of symmetries G, and suppose the Laplacian is G invariant. Then each eigenspace splits according to the irreducible representations of G. We seek to find a set of recipes, analogous to (5.1) and (5.2), to miniaturize eigenfunctions according to the corresponding irreducible representations of G. In fact, our goal is to obtain recipes that make sense on the fractal and also on the outer approximating domains. In the latter case the miniaturization of an eigenfunction on Ω_m will be an eigenfunction on Ω_{m+1}.

It is by no means clear that this goal is always attainable. We will show explicitly that it is possible for SC, the $\frac{12}{16}$ carpet, and the octagasket. In the first two examples the symmetry group is D_4 (the dihedral symmetry group of the square), and in the last example it is D_8. In contrast to the interval, the representation type of the miniaturized eigenfunctions is the same as the original one.

The referee has pointed out that it is also possible to explain miniaturization on carpets using local reflection maps introduced in [Barlow and Bass 1989] and [Barlow and Bass 1999] (see also Definition 2.12 in [Barlow et al. 2008]).

We mention in passing that a version of miniaturization is valid for SG, but the recipes are more complicated. In particular, the multiplicities increase. This is part of the story of spectral decimation (see [Strichartz 2006] for a description). On the other hand, it is not clear how to extend the recipes for the approximating domains Ω_m with a positive ϵ overlap, although they are presumably valid in the zero overlap case.

The symmetry group D_4 has five irreducible representations. Let ρ_H and ρ_V denote the reflections about the horizontal and vertical axes in D_4, and ρ'_D and ρ''_D denote the two diagonal reflections. The four one-dimensional representations $1++, 1+-, 1-+$, and $1--$ are characterized by parity with respect to these reflections. (Strictly speaking, we describe functions that transform according to the representations, rather than the abstract representations, since we are interested in eigenfunctions that transform according to representations). Functions transforming according to $1++$ are even with respect to all reflections, and those transforming according to $1--$ are odd with respect to all reflections. The $1+-$ functions are odd with respect to ρ_H and ρ_V and even with respect to ρ'_D and ρ''_D, while for $1-+$ the reverse holds.

Now suppose u is a Neumann eigenfunction on Ω_m of $1++$ or $1-+$ type. Define the miniaturization

$$u_+ = \{u \circ F_i^{-1} \text{ on } F_i\Omega_m\} \text{ on } \Omega_{m+1} \tag{5.3}$$

for either the SC or $\frac{12}{16}$ carpet. On the other hand, for an eigenfunction of $1+-$ or $1--$ type define

$$u_- = \{\pm u \circ F_i^{-1} \text{ on } F_i\Omega_m\} \text{ on } \Omega_{m+1} \tag{5.4}$$

where we alternate the choice of $\pm$ on neighboring cells (see Figure5.1). Because of the even or odd parity of u with respect to the reflections ρ_H and ρ_V, the miniaturized functions are continuous along the boundaries of the cells of order one. Since u satisfies Neumann boundary conditions, it follows that u_+ or u_- satisfy matching conditions along these boundaries, hence they are Neumann eigenfunctions on u_{m+1}, and the eigenvalue is multiplied by λ^{-2} where λ denotes the contraction ratio of the F_i mappings (so $\lambda = \frac{1}{3}$ for SC and $\lambda = \frac{1}{4}$ for the $\frac{12}{16}$ carpet). Note that on the $\frac{12}{16}$ carpet, the miniaturized eigenfunction has the same representation type as u, while on SC, u_+ preserves representation type while u_- maps $1+-$ to $1++$ and $1--$ to $1-+$.

There is also a two-dimensional representation of D_4, that we denote by 2. The representation space is spanned by functions u and v satisfying $v = \rho_H u = -\rho_V u$ and $\rho''_D u = -\rho'_D u = u$, $\rho'_D v = -\rho''_D v = v$. The miniaturized functions u_2 and v_2 are shown in Figure 5.2. Once again we see that u_2 and v_2 are Neumann eigenfunctions on Ω_{m+1} with eigenvalue multiplied by λ^{-2}, and the pair transform according to the representation 2.

What does this tell us about the Neumann spectrum on the corresponding fractal? If we believe (1.4) then there will be an eigenvalue renormalization factor $R = r\lambda^{-2}$. For every eigenvalue λ_n, there will be an eigenvalue equal

u	$-u$	u
$-u$		$-u$
u	$-u$	u

(a)

u	$-u$	u	$-u$
$-u$			u
u			$-u$
$-u$	u	$-u$	u

(b)

Figure 5.1: 1-D Miniaturized Carpet Eigenfunctions

u	$-v$	u
v		v
u	$-v$	u

v	$-u$	v
u		u
v	$-u$	v

(a)

u	$-v$	u	$-v$
v			$-u$
u			$-v$
v	$-u$	v	$-u$

v	$-u$	$-v$	$-u$
u			$-v$
v			$-u$
u	$-v$	$-u$	$-v$

(b)

Figure 5.2: 1-D Miniaturized Carpet Eigenfunctions

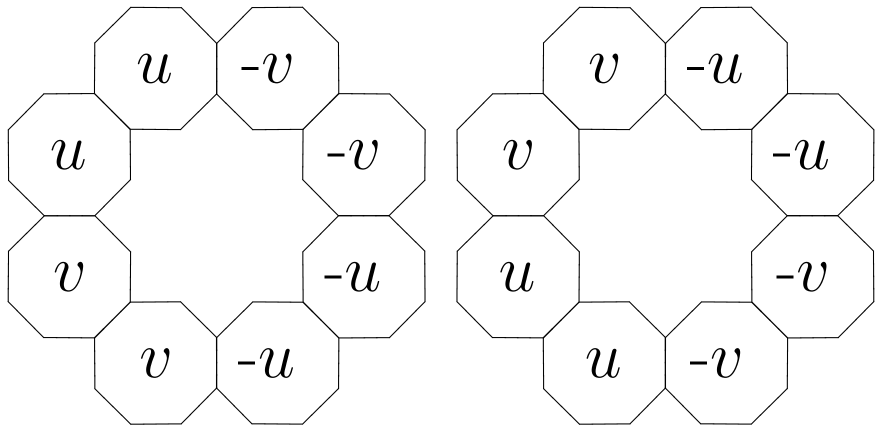


Figure 5.3: The miniaturizations (a) u_2 and (b) v_2 for a 2_1 or 2_3 eigenspace

to $R\lambda_n$ with equal multiplicity, and the corresponding eigenfunctions will be miniaturizations as illustrated.

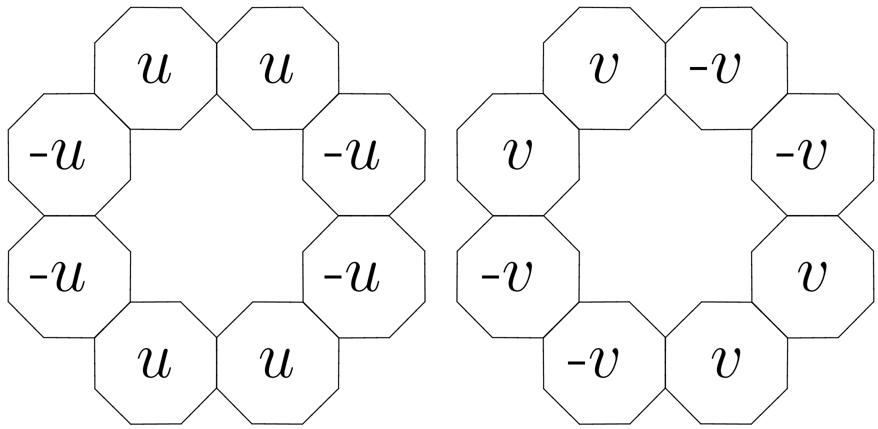
But in fact we can run the same miniaturization argument directly on the fractal. Indeed, in both cases we know that there exists a Laplacian Δ on the fractal satisfying a self-similar identity

$$\Delta(u \circ F_i) = R^{-1}(\Delta u) \circ F_i \tag{5.5}$$

for a certain constant R. Then the miniaturization recipes given above create eigenfunctions with eigenvalue multiplied by R. This is true independent of the validity of the outer approximation method. Incidentally, the miniaturization recipes given above extend easily to any D_4 symmetric carpet type fractal.

In our last example, the octagasket, the symmetry group is D_8. Here we have four one-dimensional representations. Since $D_4 \subset D_8$ we may sort the reflections in D_8 into those that are in D_4 and those that are not. The representation $1{+}{+}$ is described by functions even with respect to all reflections, and $1--$ by all functions odd with respect to all reflections. Similarly, $1+-$ functions are odd with respect to D_4 reflections and even with respect to all other reflections, while for $1-+$ functions the situation is reversed. The miniaturizations u_+ (for $1{+}{+}$ or $1{+}{-}$ eigenfunctions) and u_- (for $1{-}{+}$ or $1{-}{-}$ eigenfunctions) are again given by (5.3) and (5.4), where the $\pm$ signs alternate along the eight small octagons. We note that the representation type is preserved under miniaturization.

In this case there are three two-dimensional representations, denoted $2_1, 2_2, 2_3$. In terms of complex valued functions on the circle, 2_1 is spanned by $e^{\pm 2\pi i\theta/8}$, 2_2 is spanned by $e^{\pm 2\pi i 2\theta/8}$, and 2_3 is spanned by $e^{\pm 2\pi 3 i\theta/8}$. If x, y, z denote any



(a) (b)

Figure 5.4: The miniaturizations (a) u_2' and (b) v_2' for a 2_2 eigenspace

consecutive points on an eight element orbit of D_8, then 2_1 functions satisfy

$$u(y) = \frac{\sqrt{2}}{2}(u(x) + u(z)), \tag{5.6}$$

2_2 functions satisfy

$$u(x) + u(z) = 0, \tag{5.7}$$

and 2_3 functions satisfy

$$u(y) = -\frac{\sqrt{2}}{2}(u(x) + u(z)). \tag{5.8}$$

The 2_1 and 2_3 representations have the property that restricted to D_4 they become the 2 representation. So if u, v are the basis described above, the miniaturization u_2, v_2 are given in Figure 5.3. On the other hand, the restriction of 2_2 to D_4 splits into a direct sum of a $1+-$ and a $1-+$ representation. So we can choose a basis u, v such that $\rho_H u = \rho_V u = u = -\rho_D' u = -\rho_D'' u$ and $-\rho_H v = -\rho_V v = v = \rho_D' v = \rho_D'' v$, and the miniaturization u_2', v_2' is given in Figure 5.4. Again the representation type is preserved under miniaturization.

Some types of miniaturization on the pentagasket are described in [Adams et al. 2003].

6 Random Carpets

For $j \in \mathbb{Z}$, $j > 1$, we partition the unit square into a grid of j by j smaller, equally sized squares of width $1/j$. We then randomly remove k of these smaller squares, where k is a small positive integer, and the result is our level 1 domain Ω_1. To produce Ω_2, we partition each square of width $1/j$ into a grid of j by j equally sized squares of width $1/j^2$, and we then randomly remove m squares of width $1/j^2$ from each square of width $1/j$. Iterating this process yields a sequence of nested compact domains $\{\Omega_m\}_{m=1}^{\infty}$ where Ω_m is a union of squares of side length j^{-m}. Matlab's `rand('state')` function, a modified version of Marsaglia's Subtract-with-Borrow algorithm, makes our random choices. The number generator's state is set according to the exact date and time of the computation, so that the generator's own state is essentially randomly determined. Also, to shorten FEM computation time we triangulate Ω_m with the four sides and two diagonals of each square of side length j^{-m}.

The problem we find with our FEM eigenvalue problem on these domains is connectivity. How can we guarantee that each Ω_m has only one path component? Also, if two squares are disjoint except at a common vertex, with no other squares in a neighborhood of that vertex, how can we avoid the problem we saw in Section 3? Recall that in this case, the spline space of our finite element solver couples these squares at the common vertex. For simplicity we resolve both questions by choosing small k and altering the above algorithm so that this coupling problem is avoided, as follows. When we pass from Ω_m to Ω_{m+1} we partition a square of side length j^{-m} into squares of side length j^{-m-1} and delete k of the smaller squares randomly. We then check if this deletion process has produced the above coupling problem. If it has, then we go back and try again; otherwise, we move on to the next m^{th} level square, and so on. For k small enough, the algorithm terminates. Figure 6.1 shows a typical result of the above algorithm. Notice that we have only one path component.

Now, we study our spectral information with the eigenvalue counting function $N : [0, \infty) \to \mathbb{Z}$, where $N(x)$ is the number of nonnegative eigenvalues less than or equal to x. Then, we examine the Weyl ratio

$$W(x) = \frac{N(x)}{x^{\alpha}} \tag{6.1}$$

where x^{α} is an approximate asymptotic bound for $N(x)$, i.e. we choose $\alpha \in \mathbb{R}$ so that $N(x) \sim x^{\alpha}$ in accordance with the experimental data. So, finding α corresponds to finding the slope of a linear approximation of $N(x)$ on a log-log plot. In fact, since we are dealing with domains in the plane, the Weyl asymptotic law implies that $\alpha = 1$ is the correct value as $x \to \infty$. The point is that we truncate our computations well before we reach the region where this asymptotic behavior is approximated, so we observe values of α considerably smaller than 1.

In our first example, we let $j = 4$ and $k = 2$ and run our algorithm up to level 4 to get $\{\Omega_i\}_{i=1}^{4}$, where Ω_4 is the upper left carpet in Figure 6.2. From this initial carpet, we can restart our algorithm three separate times, beginning

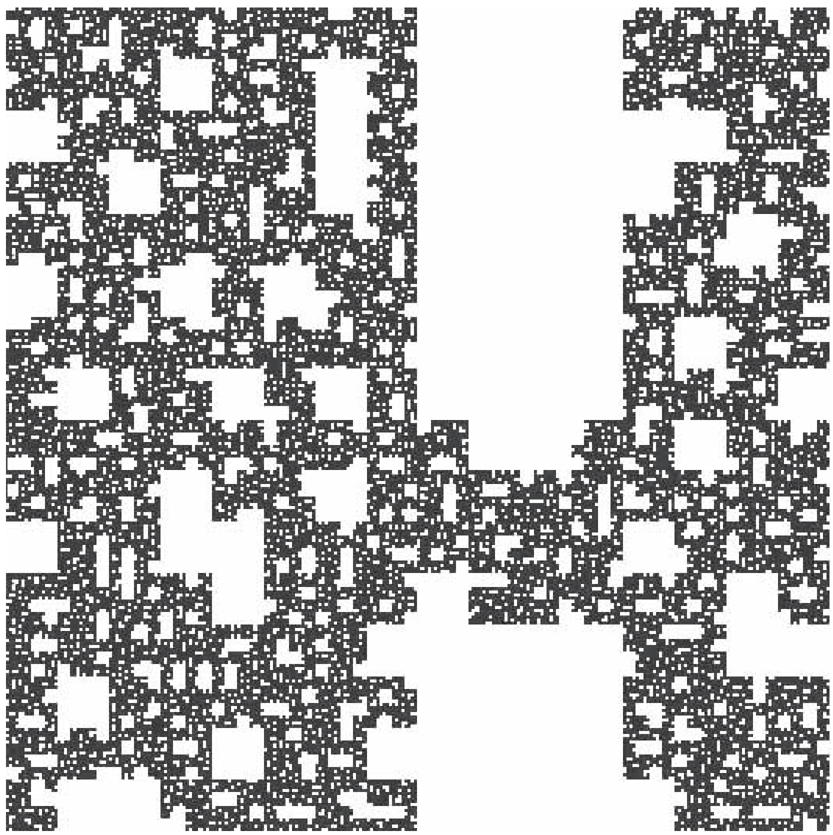


Figure 6.1: Level 4 Domain Ω_4 for $j = 4$, $k = 3$

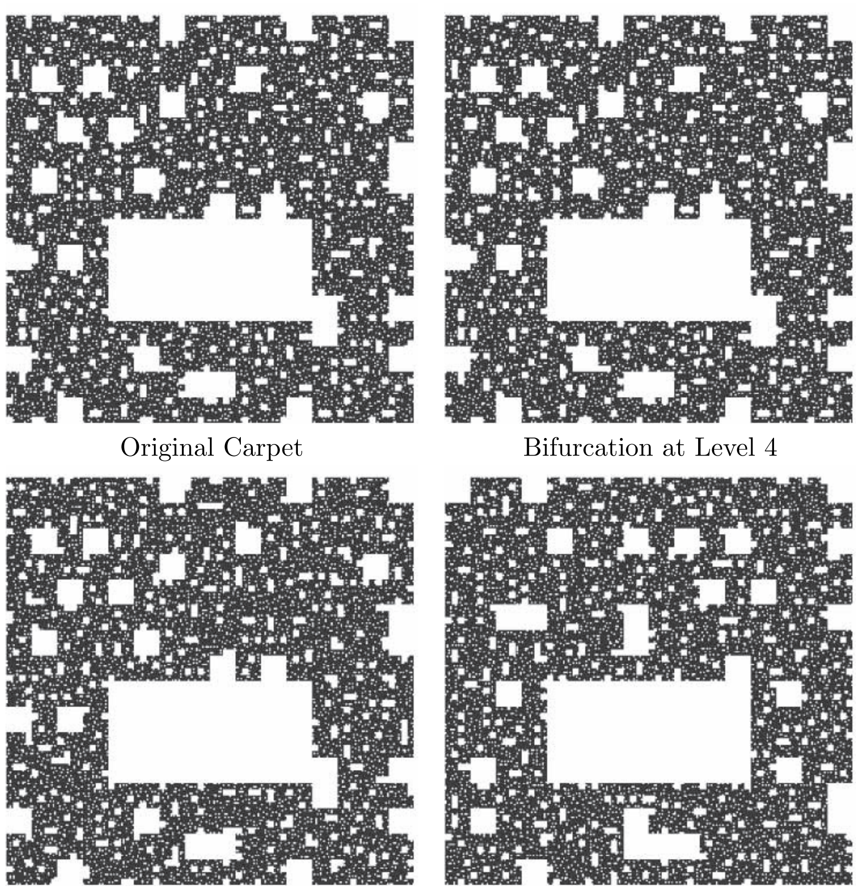
at Ω_i once for each $i = 1, 2, 3$. We then end the algorithm again at level 4 and we call the resulting (level 4) carpet which was started at Ω_i the bifurcation of Ω_4 at level $i + 1$. The carpets are shown in Figure 6.2 and the eigenvalue data in Tables 6.1 and 6.2. Next, we let $j = 4$ and $k = 3$ and do the same bifurcation study. The carpets are shown in Figure 6.4 and the eigenvalue data in Tables 6.3 and 6.4.

Finally, we fix $j = 4$ and vary k on different levels so that at level 1 we set $k = 2$, at level 2 we set $k = 3$, etc. A similar procedure for gaskets rather than carpets is discussed in [Drenning and Strichartz 2008]. Our sequence of k values for the carpet in Figure 6.6 is $k = \{2, 3, 2, 3, 2\}$. The eigenvalue data appears in Tables 6.5 and 6.6. The level-to-level eigenvalue ratios in Table 6.5 appear to roughly alternate between the same ratios in Tables 6.3 and 6.1. This is the strongest evidence that the geometry of the domain at different scales is reflected in the spectrum of the Laplacian. Such a correlation is more striking in [Drenning and Strichartz 2008], but the fractals there have a more coherent structure.

The Weyl ratios of our first example (where $j = 4$ and $k = 2$) appear in Figure 6.3. We now look closely at the agreement of the graph of the original carpet to each individual bifurcation. We see that the original agrees with the bifurcation at Level 4 up to about $x = 300$, the original agrees with that at Level 3 up to around $x = 65$, and it agrees with the Level 2 bifurcation up to about $x = 25$. In our second example (where $j = 4$ and $k = 4$) we find the Weyl ratios in Figure 6.5. We do the same comparison. The original agrees with the the Level 4 bifurcation to around $x = 150$, it agrees with the Level 3 one up to approximately $x = 30$, and it agrees with the Level 2 bifurcation to approximately $x = 10$. In other words, the added detail at finer resolutions has only a minimal effect on some initial segment of the spectrum. This is consistent with results in [Drenning and Strichartz 2008]. Our final example's Weyl ratios (where $j = 4$ and $k = \{2, 3, 2, 3, 2\}$) are found in Figure 6.7.

For further comparison of the Weyl ratios, we show those from another trial with $j = 4$ and $k = 2$, and those from another trial where $j = 4$ and $k = 3$. The carpets for the new $j = 4$, $k = 2$ trial appear in Figure 6.8 with Weyl ratios in Figure 6.9, while the carpets for the new $j = 4$, $k = 3$ trial appear in Figure 6.10 with Weyl ratios in Figure 6.11. It is clear that different random choices in the construction make a big difference in the spectrum. We leave to the future the problem of formulating precise conjectures concerning the spectra of different random carpets.

Acknowledgments: We are grateful to Stacie Goff who contributed to the numerical experiments.



Original Carpet — Bifurcation at Level 4

Bifurcation at Level 3 — Bifurcation at Level 2

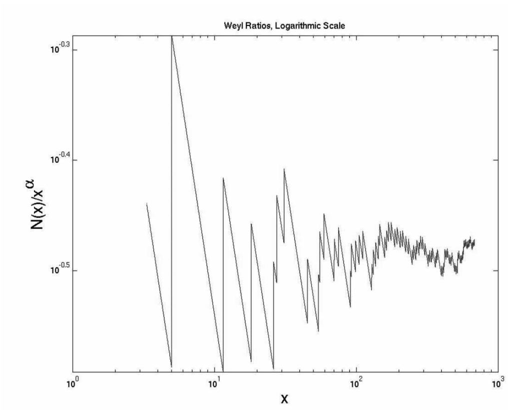
Figure 6.2: Carpet Bifurcations Ω_4 for j=4, k=2

	Original Carpet				Bifurc. at Level 4	Bifurcation at Level 3		Bifurcation at Level 2			Original Carpet Ratios $\lambda_n^{j+1}/\lambda_n^j$		
Level:	1	2	3	4	4	3	4	2	3	4			
Refinement:	2	1	0	0	0	0	0	1	0	0			
n													
1	5.580	4.524	3.961	3.331	3.349	3.885	3.248	5.011	4.393	3.689	0.811	0.875	0.841
2	7.666	6.734	5.914	5.008	5.009	5.963	4.990	6.384	5.528	4.639	0.878	0.878	0.847
3	18.031	15.575	13.671	11.556	11.543	13.514	11.311	15.597	13.664	11.478	0.864	0.878	0.845
4	31.079	24.699	21.630	18.234	18.269	21.733	18.259	27.549	24.099	20.091	0.795	0.876	0.843
5	40.933	35.373	31.097	26.152	25.983	31.230	26.123	35.484	31.262	26.392	0.864	0.879	0.841
6	46.442	37.463	32.776	27.549	27.640	32.326	27.255	38.315	33.722	28.249	0.807	0.875	0.841
7	49.757	41.840	36.519	30.975	30.850	36.427	30.614	45.424	39.840	33.521	0.841	0.873	0.848
8	72.354	62.389	54.211	45.450	45.767	54.977	45.924	56.607	49.171	41.245	0.862	0.869	0.838
9	88.309	74.938	65.259	54.360	54.880	65.444	54.860	70.649	61.489	51.576	0.849	0.871	0.833
10	96.790	77.384	65.694	55.376	55.582	66.288	54.919	74.244	63.358	53.123	0.799	0.849	0.843
11	103.355	80.835	70.391	59.301	59.323	70.224	59.343	83.948	72.797	60.697	0.782	0.871	0.842
12	106.680	96.799	83.310	70.018	69.713	84.547	70.628	91.564	78.693	65.923	0.907	0.861	0.840
13	138.947	114.330	90.751	74.898	75.720	93.994	78.280	100.291	88.203	73.751	0.823	0.794	0.825
14	162.163	123.498	107.533	91.169	90.934	105.799	88.143	118.054	102.261	85.535	0.762	0.871	0.848
15	162.163	124.726	109.598	92.432	91.856	109.995	92.539	133.591	114.904	95.879	0.769	0.879	0.843
16	168.598	138.139	117.904	99.056	99.110	118.118	99.034	140.919	121.513	101.298	0.819	0.854	0.840
17	170.390	144.519	125.422	104.987	105.280	128.532	108.234	149.625	134.221	112.025	0.848	0.868	0.837
18	183.444	155.084	131.723	111.310	111.070	133.795	111.329	159.175	141.125	117.324	0.845	0.849	0.845
19	198.436	172.548	152.591	128.765	129.400	150.123	125.782	164.504	143.437	119.346	0.870	0.884	0.844
20	206.000	179.213	156.432	132.112	132.276	158.088	133.030	177.161	154.685	129.358	0.870	0.873	0.845
21	214.522	188.011	162.499	137.582	137.272	164.643	138.022	183.438	158.717	132.738	0.876	0.864	0.847
22	240.719	196.395	170.702	143.091	143.624	169.629	141.827	201.824	173.072	144.509	0.816	0.869	0.838
23	250.543	206.038	173.663	146.420	146.806	173.727	146.243	209.485	180.972	151.168	0.822	0.843	0.843
24	257.351	218.075	188.898	159.001	159.718	186.572	154.557	218.375	189.522	158.937	0.847	0.866	0.842
25	266.899	224.684	195.801	164.282	164.779	195.558	163.810	230.264	195.063	162.503	0.842	0.871	0.839
26	276.814	231.084	200.575	168.720	168.320	201.320	170.096	235.294	204.404	171.042	0.835	0.868	0.841
27	279.112	253.950	209.948	176.419	174.688	220.595	183.338	241.413	207.898	173.358	0.910	0.827	0.840
28	302.034	265.227	223.157	187.434	187.788	233.403	193.925	256.896	223.343	186.535	0.878	0.841	0.840
29	331.424	277.522	232.146	194.488	193.716	238.930	199.095	278.109	245.174	203.467	0.837	0.836	0.838
30	339.761	284.208	242.094	204.379	203.672	246.169	205.403	292.874	257.954	215.964	0.836	0.852	0.844
31	372.054	299.978	251.492	212.246	211.069	257.519	215.248	317.191	271.234	226.783	0.806	0.838	0.844
32	385.306	316.612	265.554	223.592	224.233	266.844	222.676	331.605	287.096	238.995	0.822	0.839	0.842
33	393.257	324.027	278.512	233.507	232.679	278.277	231.720	342.980	301.702	254.300	0.824	0.860	0.838
34	395.854	339.584	294.971	248.618	248.375	286.407	241.061	355.614	311.708	261.206	0.858	0.869	0.843
35	405.837	361.323	303.434	254.390	253.273	301.936	253.741	371.636	324.228	269.291	0.890	0.840	0.838
36	420.811	364.392	315.821	263.606	264.886	313.507	260.755	380.539	327.137	273.775	0.866	0.867	0.835
37	424.847	375.125	320.413	268.424	268.127	317.266	261.999	386.150	333.468	277.348	0.883	0.854	0.838
38	451.376	386.611	330.029	278.419	277.801	325.821	270.028	397.166	346.083	287.709	0.857	0.854	0.844
39	462.812	405.467	338.324	282.917	283.186	339.347	282.063	409.025	359.088	301.704	0.876	0.834	0.836
40	502.632	408.831	347.163	292.032	290.267	361.005	302.953	429.651	366.719	306.863	0.813	0.849	0.841
41	528.577	447.618	366.670	305.564	307.685	387.191	316.739	448.759	389.395	325.524	0.847	0.819	0.833
42	545.770	452.183	388.507	323.312	324.962	388.234	323.430	456.622	395.283	329.216	0.829	0.859	0.832
43	551.007	464.111	394.833	329.113	330.967	389.063	324.844	462.266	399.880	333.236	0.842	0.851	0.834
44	552.842	477.614	405.344	338.488	338.553	405.026	335.237	486.916	414.255	346.218	0.864	0.849	0.835
45	564.216	494.116	413.127	345.354	345.961	416.434	344.196	495.591	429.603	356.793	0.876	0.836	0.836
46	578.850	501.300	431.951	362.010	361.376	432.872	359.480	500.567	436.894	364.233	0.866	0.862	0.838
47	598.075	523.765	440.593	367.808	366.624	444.307	363.054	512.076	438.755	367.034	0.876	0.841	0.835
48	613.847	551.828	449.590	374.306	377.290	449.181	369.326	547.600	457.011	378.966	0.899	0.815	0.833
49	631.941	558.302	477.459	399.598	402.493	476.826	396.391	554.463	465.816	386.917	0.883	0.855	0.837
50	715.621	567.952	488.904	409.795	410.350	480.033	397.220	576.594	477.564	397.180	0.794	0.861	0.838
51	725.020	592.871	499.609	416.054	418.770	488.846	401.513	587.455	492.776	410.999	0.818	0.843	0.833
52	727.913	600.091	503.672	419.499	422.080	493.842	415.384	609.850	512.419	423.682	0.824	0.839	0.833
53	731.705	616.830	508.602	421.296	423.380	505.115	419.245	612.367	521.155	428.834	0.843	0.825	0.828
54	735.932	631.051	524.965	431.091	444.056	522.488	436.796	631.828	538.115	449.802	0.857	0.832	0.821
55	736.330	637.768	532.002	438.897	447.894	538.642	448.956	640.834	543.298	455.736	0.866	0.834	0.825
56	737.370	656.446	548.081	455.651	457.588	555.001	459.264	648.905	549.244	458.336	0.890	0.835	0.831
57	763.993	671.049	563.190	467.554	468.467	567.954	471.301	668.772	568.342	472.118	0.878	0.839	0.830
58	770.748	681.736	573.285	472.499	493.796	576.961	480.894	688.591	576.277	480.247	0.885	0.841	0.824
59	771.833	688.547	591.230	495.228	497.748	581.500	488.252	705.282	604.405	502.457	0.892	0.859	0.838
60	772.741	699.743	601.577	508.440	502.614	596.141	494.604	712.415	609.218	509.436	0.906	0.860	0.845

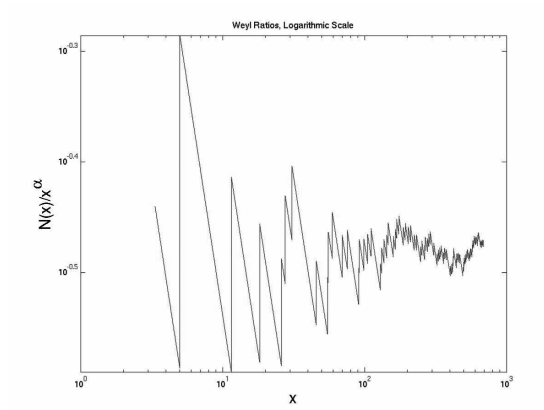
Table 6.1: Carpet Bifurcation Unnormalized Eigenvalues for $j = 4$, $k = 2$

	Original Carpet				Bifurc. at Level 4	Bifurcation at Level 3		Bifurcation at Level 2		
Level:	1	2	3	4	4	3	4	2	3	4
Refinement:	2	1	0	0	0	0	0	1	0	0
n										
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	1.374	1.489	1.493	1.503	1.496	1.535	1.536	1.274	1.258	1.257
3	3.231	3.443	3.452	3.469	3.447	3.479	3.482	3.113	3.110	3.111
4	5.570	5.460	5.461	5.473	5.455	5.595	5.621	5.498	5.485	5.445
5	7.336	7.819	7.852	7.850	7.758	8.039	8.042	7.082	7.116	7.153
6	8.323	8.281	8.276	8.270	8.253	8.322	8.391	7.647	7.676	7.657
7	8.917	9.249	9.221	9.298	9.211	9.377	9.425	9.066	9.068	9.086
8	12.967	13.791	13.688	13.643	13.665	14.152	14.138	11.297	11.192	11.179
9	15.826	16.565	16.477	16.318	16.387	16.847	16.889	14.100	13.996	13.979
10	17.346	17.106	16.587	16.623	16.596	17.064	16.907	14.817	14.421	14.399
11	18.522	17.868	17.773	17.801	17.713	18.077	18.269	16.754	16.570	16.451
12	19.118	21.397	21.035	21.018	20.816	21.764	21.743	18.274	17.912	17.868
13	24.901	25.272	22.914	22.483	22.609	24.196	24.099	20.016	20.076	19.989
14	29.062	27.299	27.151	27.367	27.152	27.235	27.135	23.561	23.276	23.183
15	29.062	27.570	27.673	27.746	27.427	28.315	28.489	26.662	26.154	25.987
16	30.215	30.535	29.770	29.734	29.593	30.406	30.488	28.124	27.658	27.456
17	30.536	31.946	31.668	31.515	31.435	33.087	33.320	29.862	30.551	30.363
18	32.875	34.281	33.259	33.413	33.164	34.442	34.273	31.768	32.122	31.800
19	35.562	38.141	38.528	38.652	38.637	38.645	38.723	32.831	32.648	32.348
20	36.918	39.615	39.498	39.657	39.496	40.696	40.954	35.357	35.208	35.061
21	38.445	41.559	41.029	41.299	40.988	42.383	42.491	36.610	36.126	35.977
22	43.140	43.413	43.101	42.952	42.885	43.666	43.662	40.279	39.394	39.168
23	44.900	45.544	43.848	43.952	43.835	44.721	45.022	41.808	41.192	40.973
24	46.120	48.205	47.695	47.728	47.690	48.028	47.581	43.583	43.138	43.078
25	47.831	49.666	49.438	49.313	49.201	50.341	50.430	45.955	44.399	44.045
26	49.608	51.081	50.643	50.646	50.259	51.825	52.365	46.959	46.525	46.359
27	50.020	56.135	53.010	52.957	52.160	56.786	56.442	48.180	47.321	46.987
28	54.128	58.628	56.345	56.263	56.071	60.083	59.701	51.271	50.836	50.559
29	59.395	61.346	58.615	58.381	57.841	61.506	61.293	55.504	55.805	55.148
30	60.889	62.824	61.126	61.350	60.814	63.370	63.235	58.451	58.714	58.535
31	66.676	66.310	63.499	63.711	63.023	66.291	66.265	63.304	61.737	61.467
32	69.051	69.986	67.050	67.117	66.954	68.692	68.552	66.181	65.347	64.777
33	70.476	71.626	70.322	70.093	69.475	71.635	71.337	68.451	68.672	68.926
34	70.942	75.064	74.477	74.629	74.162	73.728	74.212	70.972	70.949	70.797
35	72.731	79.870	76.614	76.362	75.624	77.725	78.116	74.170	73.799	72.989
36	75.414	80.548	79.742	79.128	79.092	80.704	80.275	75.947	74.461	74.204
37	76.138	82.921	80.901	80.574	80.060	81.672	80.658	77.067	75.902	75.173
38	80.892	85.460	83.329	83.574	82.948	83.874	83.130	79.265	78.773	77.981
39	82.941	89.628	85.424	84.925	84.556	87.356	86.835	81.632	81.734	81.774
40	90.078	90.371	87.655	87.661	86.670	92.931	93.266	85.748	83.471	83.172
41	94.727	98.945	92.581	91.723	91.871	99.672	97.510	89.562	88.632	88.230
42	97.808	99.954	98.094	97.050	97.030	99.941	99.570	91.131	89.972	89.231
43	98.747	102.591	99.691	98.792	98.823	100.154	100.005	92.258	91.019	90.320
44	99.076	105.576	102.345	101.606	101.088	104.263	103.205	97.177	94.290	93.839
45	101.114	109.223	104.311	103.667	103.300	107.200	105.963	98.908	97.784	96.705
46	103.737	110.811	109.063	108.667	107.903	111.431	110.668	99.902	99.444	98.722
47	107.182	115.777	111.246	110.407	109.470	114.375	111.768	102.198	99.867	99.481
48	110.008	121.981	113.517	112.357	112.654	115.630	113.699	109.288	104.022	102.715
49	113.251	123.412	120.554	119.949	120.180	122.746	122.031	110.658	106.026	104.870
50	128.248	125.545	123.443	123.010	122.526	123.572	122.287	115.075	108.701	107.652
51	129.932	131.053	126.147	124.889	125.040	125.840	123.608	117.242	112.163	111.397
52	130.451	132.649	127.172	125.923	126.028	127.126	127.878	121.712	116.634	114.835
53	131.130	136.349	128.417	126.463	126.416	130.029	129.067	122.214	118.622	116.231
54	131.888	139.493	132.549	129.403	132.590	134.501	134.470	126.098	122.483	121.915
55	131.959	140.977	134.325	131.746	133.736	138.659	138.214	127.896	123.663	123.523
56	132.145	145.106	138.385	136.775	136.630	142.870	141.387	129.506	125.016	124.228
57	136.916	148.334	142.200	140.348	139.879	146.205	145.093	133.471	129.363	127.963
58	138.127	150.696	144.749	141.833	147.442	148.523	148.046	137.427	131.169	130.166
59	138.322	152.202	149.280	148.655	148.622	149.692	150.311	140.758	137.571	136.186
60	138.484	154.677	151.893	152.621	150.075	153.461	152.267	142.182	138.667	138.078

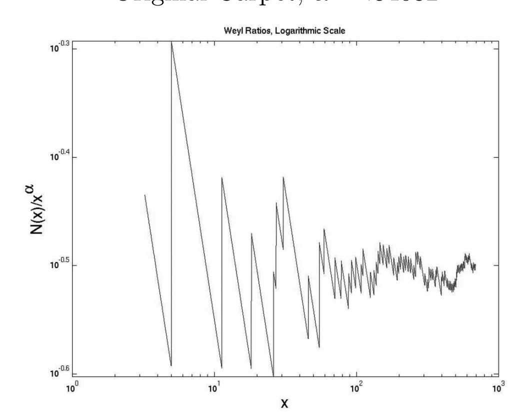
Table 6.2: Carpet Bifurcation Normalized Eigenvalues for $j = 4$, $k = 2$



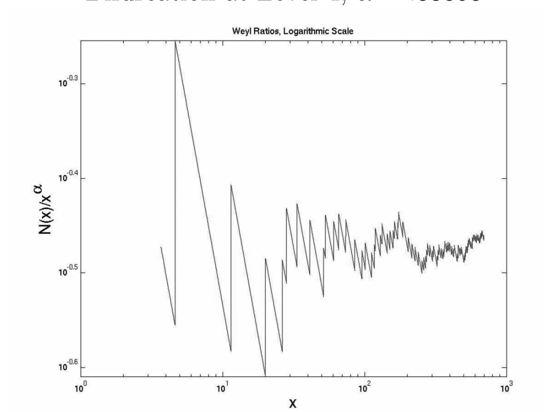
Original Carpet, $\alpha = .84032$



Bifurcation at Level 4, $\alpha = .83853$

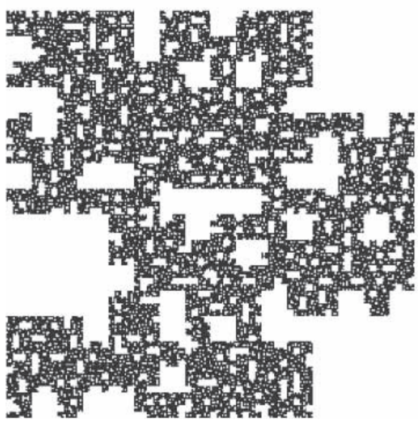


Bifurcation at Level 3, $\alpha = .85007$

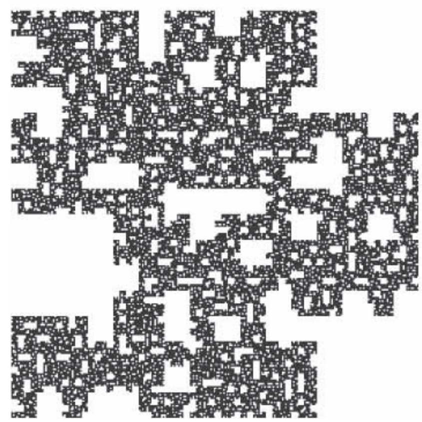


Bifurcation at Level 2, $\alpha = .83383$

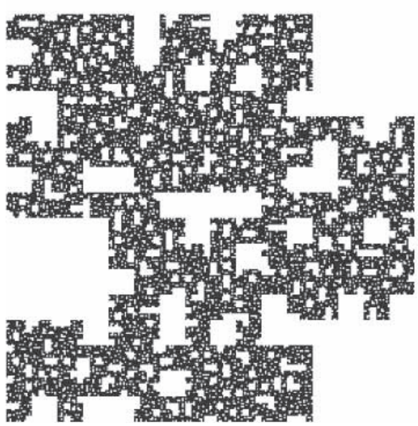
Figure 6.3: Weyl Ratios for $j = 4$, $k = 2$



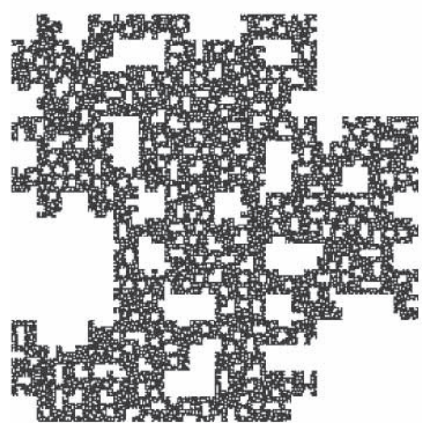
Original Carpet



Bifurcation at Level 4



Bifurcation at Level 3



Bifurcation at Level 2

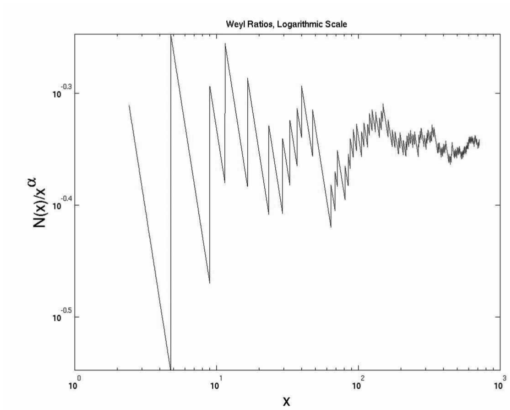
Figure 6.4: Carpet Bifurcations Ω_4 for j=4, k=3

	Original Carpet				Bifurc. at Level 4	Bifurcation at Level 3		Bifurcation at Level 2			Original Carpet Ratios $\lambda_n^{j+1}/\lambda_n^j$		
Level:	1	2	3	4	4	3	4	2	3	4			
Refinement:	2	1	0	0	0	0	0	1	0	0			
n													
1	7.092	4.504	3.375	2.426	2.445	3.493	2.631	6.127	4.897	3.695	0.635	0.749	0.719
2	11.728	8.197	6.565	4.765	4.809	6.560	4.906	9.482	7.315	5.436	0.699	0.801	0.726
3	24.546	15.759	12.132	9.007	8.969	12.351	9.397	21.222	16.397	12.210	0.642	0.770	0.742
4	30.185	18.800	15.634	11.511	11.431	15.081	11.247	23.081	17.654	13.228	0.623	0.832	0.736
5	42.518	27.342	22.736	16.597	17.156	21.355	15.651	35.771	28.705	21.604	0.643	0.832	0.730
6	58.544	39.332	31.633	23.474	23.415	29.450	21.692	42.024	34.860	25.822	0.672	0.804	0.742
7	61.533	48.008	39.592	29.343	28.460	37.253	26.997	51.343	38.677	28.823	0.780	0.825	0.741
8	77.637	55.316	44.954	33.046	33.484	43.576	33.038	62.867	47.085	35.453	0.712	0.813	0.735
9	83.257	65.486	50.292	37.106	37.063	50.448	38.068	69.055	53.226	39.577	0.787	0.768	0.738
10	104.768	73.075	55.186	39.936	40.286	56.936	42.856	82.013	62.297	45.968	0.697	0.755	0.724
11	113.834	80.749	63.592	47.756	46.867	61.053	45.682	89.184	71.478	53.499	0.709	0.788	0.751
12	150.330	107.208	87.607	64.387	64.396	83.453	61.747	104.399	78.918	58.162	0.713	0.817	0.735
13	162.163	114.553	93.828	68.725	68.441	87.807	65.440	121.120	97.702	73.003	0.706	0.819	0.732
14	162.163	123.236	97.285	71.388	72.414	96.939	73.436	128.037	102.533	76.207	0.760	0.789	0.734
15	172.077	134.430	108.788	80.883	81.781	105.258	78.237	141.405	111.031	82.675	0.781	0.809	0.743
16	175.214	142.039	112.803	85.010	83.188	110.001	79.855	148.384	120.540	89.074	0.811	0.794	0.754
17	188.670	151.413	120.069	88.113	88.222	116.098	84.337	157.156	125.459	92.636	0.803	0.793	0.734
18	195.969	155.785	124.187	92.302	93.366	125.759	91.453	166.793	129.710	96.150	0.795	0.797	0.743
19	202.684	172.478	133.879	97.392	98.402	139.613	103.979	175.989	137.547	102.029	0.851	0.776	0.727
20	240.609	181.954	145.005	105.777	106.864	147.803	108.886	181.217	144.039	106.966	0.756	0.797	0.729
21	244.457	187.959	146.514	109.927	109.130	153.475	113.767	189.592	153.756	113.071	0.769	0.780	0.750
22	274.309	195.733	161.327	116.845	117.350	158.670	118.341	215.292	165.384	119.406	0.714	0.824	0.724
23	280.954	213.619	164.799	120.048	120.767	168.550	127.045	218.276	174.245	129.845	0.760	0.771	0.728
24	310.772	221.652	171.028	125.428	126.050	176.933	132.774	240.206	186.285	134.818	0.713	0.772	0.733
25	315.838	234.301	178.988	132.647	130.963	181.365	135.580	264.516	204.123	154.817	0.742	0.764	0.741
26	325.052	249.165	191.045	141.451	139.979	191.300	140.370	271.605	208.368	156.500	0.767	0.767	0.740
27	331.424	256.361	197.974	145.517	144.028	202.974	148.117	277.958	218.970	162.376	0.774	0.772	0.735
28	360.177	265.361	201.315	149.385	146.794	211.209	159.046	285.512	224.683	167.545	0.737	0.759	0.742
29	385.417	283.819	227.801	162.546	167.689	217.654	162.927	309.537	242.878	180.433	0.736	0.803	0.714
30	389.957	302.005	238.307	174.063	174.823	234.133	173.773	315.867	251.600	187.139	0.774	0.789	0.730
31	404.260	313.891	253.256	181.953	186.426	234.618	175.749	322.848	265.116	192.433	0.776	0.807	0.718
32	415.828	319.907	261.191	192.491	192.163	254.854	184.953	340.518	267.937	197.979	0.769	0.816	0.737
33	438.883	339.911	275.572	197.205	202.084	263.163	194.662	357.545	271.540	198.556	0.774	0.811	0.716
34	446.211	351.282	280.260	208.609	207.833	268.417	200.150	360.353	289.120	213.750	0.787	0.798	0.744
35	459.422	358.047	294.617	217.058	218.615	289.697	212.752	375.481	300.539	223.974	0.779	0.823	0.737
36	496.933	372.237	303.439	221.374	219.487	294.416	216.619	409.773	314.193	228.308	0.749	0.815	0.730
37	497.815	399.070	310.002	227.464	231.842	305.155	225.425	420.361	322.296	239.171	0.802	0.777	0.734
38	508.950	411.214	318.971	233.374	233.036	312.163	229.676	430.050	326.045	240.535	0.808	0.776	0.732
39	527.137	420.241	330.335	243.651	240.021	323.612	237.958	442.270	336.285	246.673	0.797	0.786	0.738
40	551.265	426.786	333.384	249.500	243.044	332.791	248.954	446.720	343.015	251.129	0.774	0.781	0.748
41	567.597	450.642	355.913	265.783	258.013	345.979	254.543	457.895	357.701	264.114	0.794	0.790	0.747
42	583.940	472.465	361.077	267.320	261.429	352.508	260.712	464.320	367.546	271.991	0.809	0.764	0.740
43	597.055	484.208	368.120	272.725	269.302	356.974	266.468	480.125	380.877	280.646	0.811	0.760	0.741
44	604.989	488.128	377.891	278.169	275.595	379.520	279.242	519.333	390.951	285.829	0.807	0.774	0.736
45	666.885	501.510	396.592	291.488	284.793	390.723	287.972	534.943	417.099	303.615	0.752	0.791	0.735
46	696.767	528.048	410.599	301.706	303.345	402.895	293.498	542.283	422.342	309.099	0.758	0.778	0.735
47	724.838	544.995	416.860	305.621	305.132	411.603	304.224	559.085	432.436	320.626	0.752	0.765	0.733
48	725.020	560.630	430.081	311.371	318.362	418.850	306.840	571.554	444.148	325.198	0.773	0.767	0.724
49	731.705	564.489	442.746	321.252	325.115	437.407	323.790	593.800	457.423	335.551	0.771	0.784	0.726
50	733.652	579.186	445.729	328.599	329.796	442.425	324.682	619.347	472.066	348.351	0.789	0.770	0.737
51	740.114	612.913	452.752	331.363	332.623	460.158	336.138	624.425	485.492	356.737	0.828	0.739	0.732
52	742.828	630.679	475.019	341.589	343.604	468.450	344.312	653.206	500.482	372.797	0.849	0.753	0.719
53	760.150	646.953	497.418	364.433	360.065	479.135	353.972	658.561	506.439	378.011	0.851	0.769	0.733
54	770.961	655.027	523.642	372.127	371.947	500.853	366.851	663.004	520.723	380.865	0.850	0.799	0.711
55	772.604	686.275	532.901	385.695	377.962	513.718	377.732	689.445	521.847	388.528	0.888	0.777	0.724
56	809.840	696.798	544.755	393.847	396.853	517.891	381.072	717.226	545.963	404.472	0.860	0.782	0.723
57	815.252	711.641	553.161	399.410	403.815	544.312	395.388	723.980	575.366	428.062	0.873	0.777	0.722
58	840.638	722.733	572.093	408.348	412.322	549.102	400.724	751.200	582.018	429.608	0.860	0.792	0.714
59	884.506	746.735	586.715	413.175	425.816	566.350	409.882	771.766	595.182	434.700	0.844	0.786	0.704
60	925.356	769.117	598.816	436.475	434.588	576.353	421.201	801.163	615.474	448.107	0.831	0.779	0.729

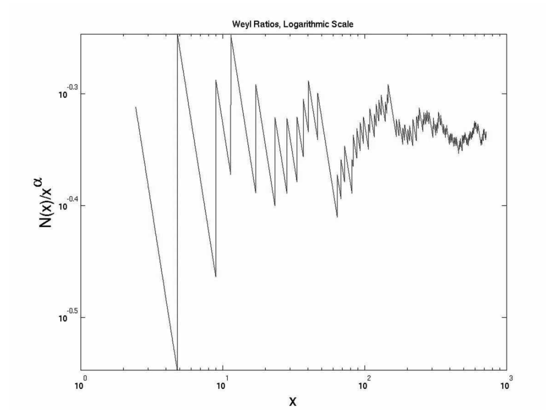
Table 6.3: Carpet Bifurcation Unnormalized Eigenvalues for $j = 4$, $k = 3$

	Original Carpet				Bifurc. at Level 4	Bifurcation at Level 3		Bifurcation at Level 2		
Level:	1	2	3	4	4	3	4	2	3	4
Refinement:	2	1	0	0	0	0	0	1	0	0
n										
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	1.654	1.820	1.945	1.964	1.967	1.878	1.865	1.548	1.494	1.471
3	3.461	3.499	3.595	3.712	3.668	3.536	3.572	3.464	3.349	3.305
4	4.256	4.174	4.633	4.744	4.675	4.317	4.275	3.767	3.605	3.580
5	5.995	6.071	6.737	6.840	7.016	6.113	5.950	5.839	5.862	5.847
6	8.255	8.732	9.374	9.674	9.576	8.431	8.246	6.859	7.119	6.989
7	8.676	10.659	11.732	12.093	11.640	10.665	10.263	8.380	7.899	7.801
8	10.947	12.281	13.321	13.619	13.694	12.475	12.559	10.262	9.616	9.595
9	11.740	14.539	14.903	15.292	15.158	14.442	14.471	11.272	10.870	10.712
10	14.773	16.224	16.353	16.459	16.476	16.300	16.291	13.387	12.723	12.441
11	16.051	17.928	18.844	19.681	19.167	17.478	17.366	14.557	14.597	14.479
12	21.197	23.802	25.960	26.536	26.336	23.891	23.473	17.040	16.117	15.742
13	22.866	25.433	27.804	28.324	27.991	25.137	24.876	19.770	19.953	19.758
14	22.866	27.361	28.828	29.421	29.616	27.752	27.916	20.899	20.940	20.625
15	24.264	29.846	32.237	33.334	33.446	30.133	29.741	23.081	22.675	22.376
16	24.706	31.535	33.427	35.035	34.022	31.491	30.356	24.220	24.617	24.108
17	26.604	33.617	35.580	36.314	36.081	33.236	32.060	25.652	25.622	25.072
18	27.633	34.587	36.800	38.040	38.184	36.002	34.765	27.225	26.490	26.023
19	28.580	38.294	39.672	40.138	40.244	39.968	39.527	28.726	28.090	27.614
20	33.927	40.398	42.969	43.593	43.705	42.313	41.392	29.579	29.416	28.950
21	34.470	41.731	43.416	45.304	44.632	43.937	43.248	30.946	31.401	30.603
22	38.679	43.457	47.805	48.155	47.993	45.424	44.987	35.141	33.775	32.317
23	39.616	47.428	48.834	49.475	49.391	48.252	48.295	35.628	35.585	35.143
24	43.821	49.211	50.680	51.692	51.551	50.652	50.473	39.208	38.044	36.488
25	44.535	52.020	53.039	54.667	53.561	51.921	51.540	43.176	41.687	41.901
26	45.834	55.320	56.612	58.296	57.248	54.765	53.361	44.333	42.554	42.357
27	46.733	56.917	58.665	59.972	58.904	58.107	56.305	45.370	44.719	43.947
28	50.787	58.916	59.655	61.566	60.035	60.465	60.460	46.603	45.886	45.346
29	54.346	63.013	67.503	66.990	68.581	62.310	61.936	50.524	49.602	48.834
30	54.986	67.051	70.616	71.736	71.499	67.027	66.059	51.557	51.383	50.649
31	57.003	69.690	75.046	74.988	76.244	67.166	66.810	52.697	54.143	52.082
32	58.634	71.026	77.398	79.331	78.590	72.959	70.309	55.581	54.719	53.583
33	61.885	75.467	81.659	81.273	82.647	75.338	73.999	58.360	55.455	53.739
34	62.918	77.992	83.048	85.974	84.999	76.842	76.086	58.819	59.045	57.852
35	64.781	79.494	87.303	89.455	89.409	82.934	80.876	61.288	61.377	60.619
36	70.071	82.644	89.917	91.234	89.765	84.285	82.346	66.885	64.166	61.792
37	70.195	88.602	91.862	93.744	94.818	87.360	85.694	68.614	65.821	64.732
38	71.765	91.298	94.519	96.180	95.306	89.366	87.310	70.195	66.586	65.101
39	74.329	93.302	97.887	100.415	98.163	92.643	90.458	72.190	68.678	66.762
40	77.732	94.755	98.790	102.826	99.399	95.271	94.638	72.916	70.052	67.968
41	80.035	100.052	105.466	109.536	105.521	99.047	96.763	74.740	73.051	71.483
42	82.339	104.897	106.996	110.170	106.918	100.916	99.108	75.789	75.062	73.614
43	84.188	107.504	109.084	112.397	110.138	102.194	101.304	78.369	77.784	75.957
44	85.307	108.374	111.979	114.641	112.712	108.649	106.152	84.768	79.842	77.360
45	94.035	111.345	117.521	120.130	116.474	111.856	109.471	87.316	85.182	82.174
46	98.248	117.237	121.671	124.341	124.061	115.340	111.571	88.514	86.252	83.658
47	102.206	121.000	123.526	125.955	124.792	117.833	115.648	91.257	88.314	86.778
48	102.232	124.471	127.444	128.325	130.202	119.908	116.643	93.292	90.706	88.015
49	103.175	125.328	131.197	132.397	132.964	125.221	123.087	96.923	93.417	90.817
50	103.449	128.591	132.081	135.425	134.879	126.657	123.426	101.093	96.407	94.281
51	104.360	136.079	134.162	136.564	136.035	131.734	127.781	101.922	99.149	96.551
52	104.743	140.023	140.760	140.778	140.526	134.108	130.888	106.620	102.210	100.898
53	107.186	143.637	147.398	150.193	147.258	137.166	134.560	107.494	103.427	102.309
54	108.710	145.429	155.169	153.363	152.117	143.384	139.456	108.219	106.344	103.081
55	108.942	152.367	157.912	158.955	154.578	147.067	143.592	112.535	106.574	105.155
56	114.192	154.703	161.425	162.315	162.303	148.262	144.862	117.069	111.499	109.471
57	114.955	157.999	163.916	164.608	165.151	155.825	150.304	118.172	117.504	115.855
58	118.535	160.461	169.526	168.291	168.630	157.197	152.332	122.615	118.862	116.274
59	124.721	165.790	173.859	170.281	174.149	162.134	155.814	125.972	121.551	117.652
60	130.481	170.760	177.445	179.883	177.736	164.998	160.117	130.770	125.695	121.280

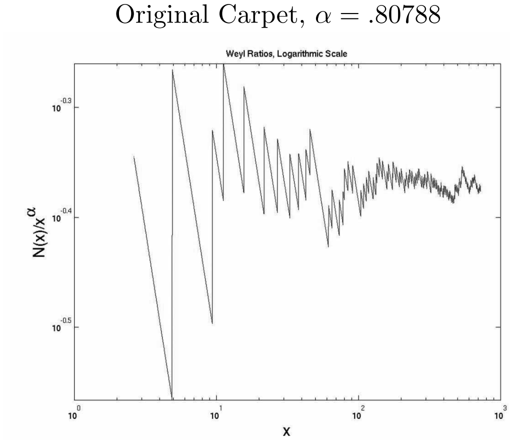
Table 6.4: Carpet Bifurcation Normalized Eigenvalues for $j = 4$, $k = 3$



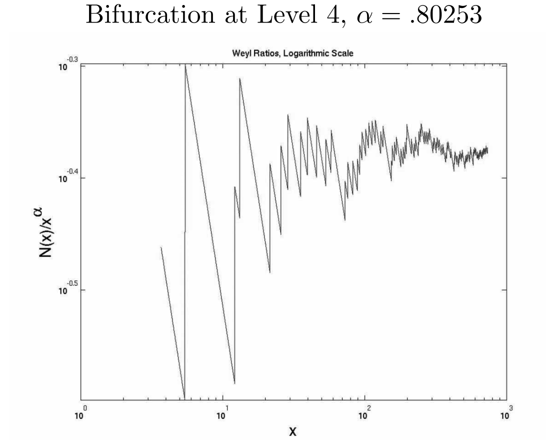
Original Carpet, $\alpha = .80788$



Bifurcation at Level 4, $\alpha = .80253$



Bifurcation at Level 3, $\alpha = .82004$



Bifurcation at Level 2, $\alpha = .81408$

Figure 6.5: Weyl Ratios for $j = 4$, $k = 3$

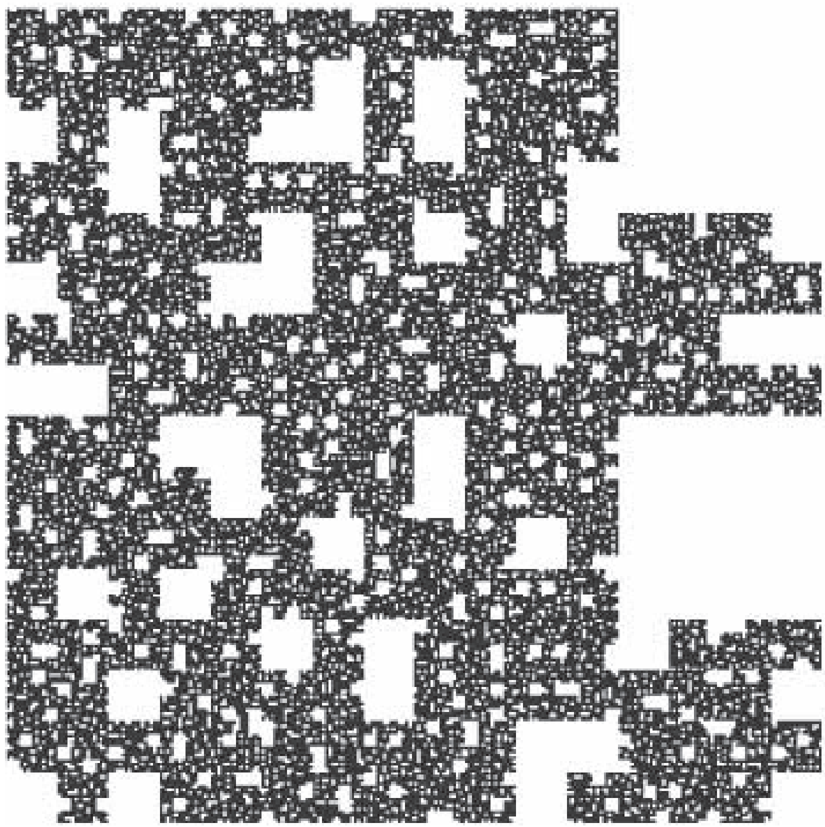


Figure 6.6: Level 4 Domain Ω_4 for $j = 4$, D:2,3,2,3,2

	Eigenvalue Data					Ratios $\lambda_n^{j+1}/\lambda_n^j$			
Level:	1	2	3	4	5	$j=1$	$j=2$	$j=3$	$j=4$
Refinement:	2	1	0	0	0				
n									
1	7.812	5.846	5.041	3.855	3.160	0.748	0.862	0.765	0.820
2	11.846	8.843	7.739	5.918	4.865	0.746	0.875	0.765	0.822
3	16.892	11.188	9.621	7.327	6.019	0.662	0.860	0.762	0.821
4	31.783	21.564	18.685	14.339	11.813	0.678	0.867	0.767	0.824
5	38.849	27.674	24.037	18.351	15.056	0.712	0.869	0.763	0.820
6	44.579	33.739	28.815	21.870	17.953	0.757	0.854	0.759	0.821
7	66.179	53.267	45.651	34.817	28.614	0.805	0.857	0.763	0.822
8	79.132	57.301	50.164	38.328	31.538	0.724	0.875	0.764	0.823
9	91.235	64.613	55.715	42.305	34.727	0.708	0.862	0.759	0.821
10	93.671	71.615	60.579	45.932	37.773	0.765	0.846	0.758	0.822
11	115.299	72.824	65.016	48.923	40.398	0.632	0.893	0.752	0.826
12	118.662	89.806	79.457	60.983	50.215	0.757	0.885	0.768	0.823
13	162.163	95.052	83.209	63.326	51.851	0.586	0.875	0.761	0.819
14	162.163	106.641	90.290	63.836	52.366	0.658	0.847	0.707	0.820
15	163.646	118.545	98.672	74.716	61.516	0.724	0.832	0.757	0.823
16	170.709	125.049	102.625	78.056	63.851	0.733	0.821	0.761	0.818
17	174.715	135.278	113.855	86.167	70.686	0.774	0.842	0.757	0.820
18	188.946	150.973	120.531	91.075	74.673	0.799	0.798	0.756	0.820
19	201.351	161.952	138.235	105.178	86.257	0.804	0.854	0.761	0.820
20	204.687	170.606	144.171	108.697	89.387	0.833	0.845	0.754	0.822
21	244.350	182.087	153.859	110.986	90.735	0.745	0.845	0.721	0.818
22	247.462	182.669	155.701	115.922	95.112	0.738	0.852	0.745	0.820
23	264.337	202.217	175.672	132.411	108.679	0.765	0.869	0.754	0.821
24	274.203	210.504	181.596	137.161	112.420	0.768	0.863	0.755	0.820
25	281.799	218.830	186.415	141.338	115.948	0.777	0.852	0.758	0.820
26	290.260	227.945	196.993	150.327	123.167	0.785	0.864	0.763	0.819
27	325.832	234.838	202.526	153.692	126.141	0.721	0.862	0.759	0.821
28	331.424	266.942	224.635	172.469	141.217	0.805	0.842	0.768	0.819
29	336.355	289.198	236.851	179.569	147.123	0.860	0.819	0.758	0.819
30	361.265	308.120	250.806	187.350	153.399	0.853	0.814	0.747	0.819
31	384.370	312.120	257.020	194.251	160.344	0.812	0.823	0.756	0.825
32	394.115	316.748	264.292	200.424	164.390	0.804	0.834	0.758	0.820
33	409.812	335.456	274.591	205.929	168.413	0.819	0.819	0.750	0.818
34	411.970	343.218	289.720	216.514	177.282	0.833	0.844	0.747	0.819
35	439.508	359.087	304.720	232.617	190.896	0.817	0.849	0.763	0.821
36	440.834	363.915	310.972	233.960	191.294	0.826	0.855	0.752	0.818
37	460.863	375.298	321.912	241.177	196.161	0.814	0.858	0.749	0.813
38	498.602	386.461	336.603	245.761	201.687	0.775	0.871	0.730	0.821
39	510.810	405.711	342.497	255.459	209.461	0.794	0.844	0.746	0.820
40	523.667	413.699	345.709	264.311	217.396	0.790	0.836	0.765	0.822
41	531.231	430.848	352.572	265.834	218.537	0.811	0.818	0.754	0.822
42	546.687	437.950	369.987	281.332	230.437	0.801	0.845	0.760	0.819
43	559.577	440.856	377.907	286.941	235.441	0.788	0.857	0.759	0.821
44	577.182	471.699	383.805	291.534	239.439	0.817	0.814	0.760	0.821
45	581.677	479.787	394.602	298.709	245.557	0.825	0.822	0.757	0.822
46	598.344	498.649	409.685	313.466	255.821	0.833	0.822	0.765	0.816
47	626.841	504.051	421.465	321.144	263.599	0.804	0.836	0.762	0.821
48	633.797	518.365	430.774	326.844	268.188	0.818	0.831	0.759	0.821
49	703.078	527.983	439.453	333.442	272.346	0.751	0.832	0.759	0.817
50	717.798	547.394	449.694	334.683	273.914	0.763	0.822	0.744	0.818
51	725.020	563.709	453.597	344.470	283.262	0.778	0.805	0.759	0.822
52	731.705	579.899	471.292	358.560	294.274	0.793	0.813	0.761	0.821
53	737.699	592.267	491.538	371.267	303.822	0.803	0.830	0.755	0.818
54	741.151	605.924	508.950	382.992	314.146	0.818	0.840	0.753	0.820
55	745.273	622.510	512.829	387.189	317.207	0.835	0.824	0.755	0.819
56	765.924	648.652	543.566	410.317	336.800	0.847	0.838	0.755	0.821
57	769.461	671.205	547.092	413.608	339.283	0.872	0.815	0.756	0.820
58	771.967	672.947	553.461	418.171	343.727	0.872	0.822	0.756	0.822
59	784.362	697.036	579.292	434.757	356.559	0.889	0.831	0.750	0.820
60	819.314	722.128	598.492	448.196	367.600	0.881	0.829	0.749	0.820

Table 6.5: Carpet Mixed Unnormalized Eigenvalues and Ratios, 23232

blah blah

Level:	1	2	3	4	5
Refinement:	2	1	0	0	0
n					
1	1.000	1.000	1.000	1.000	1.000
2	1.516	1.513	1.535	1.535	1.540
3	2.162	1.914	1.908	1.901	1.905
4	4.068	3.689	3.707	3.720	3.739
5	4.973	4.734	4.768	4.761	4.765
6	5.706	5.771	5.716	5.673	5.681
7	8.471	9.112	9.056	9.032	9.055
8	10.129	9.802	9.951	9.943	9.981
9	11.679	11.052	11.052	10.975	10.990
10	11.991	12.250	12.017	11.916	11.954
11	14.759	12.457	12.897	12.692	12.785
12	15.190	15.362	15.762	15.820	15.892
13	20.758	16.259	16.506	16.428	16.409
14	20.758	18.242	17.911	16.561	16.572
15	20.948	20.278	19.574	19.383	19.468
16	21.852	21.390	20.358	20.249	20.207
17	22.365	23.140	22.585	22.354	22.370
18	24.186	25.825	23.910	23.627	23.632
19	25.774	27.703	27.422	27.286	27.298
20	26.201	29.183	28.599	28.198	28.288
21	31.278	31.147	30.521	28.792	28.715
22	31.677	31.247	30.887	30.073	30.100
23	33.837	34.591	34.848	34.351	34.394
24	35.100	36.008	36.023	35.583	35.577
25	36.072	37.432	36.979	36.666	36.694
26	37.155	38.991	39.078	38.998	38.979
27	41.709	40.171	40.175	39.871	39.920
28	42.425	45.662	44.561	44.743	44.691
29	43.056	49.469	46.984	46.584	46.560
30	46.244	52.706	49.752	48.603	48.546
31	49.202	53.390	50.985	50.393	50.744
32	50.449	54.182	52.428	51.995	52.024
33	52.459	57.382	54.471	53.423	53.298
34	52.735	58.710	57.472	56.169	56.104
35	56.260	61.424	60.447	60.346	60.413
36	56.430	62.250	61.688	60.695	60.539
37	58.994	64.197	63.858	62.567	62.079
38	63.824	66.107	66.772	63.756	63.828
39	65.387	69.399	67.941	66.272	66.288
40	67.033	70.766	68.578	68.569	68.799
41	68.001	73.699	69.940	68.964	69.160
42	69.980	74.914	73.394	72.984	72.926
43	71.630	75.411	74.966	74.439	74.510
44	73.883	80.687	76.136	75.631	75.775
45	74.459	82.071	78.277	77.492	77.711
46	76.592	85.297	81.269	81.320	80.960
47	80.240	86.221	83.606	83.312	83.421
48	81.130	88.670	85.453	84.791	84.873
49	89.999	90.315	87.174	86.503	86.189
50	91.883	93.635	89.206	86.825	86.686
51	92.807	96.426	89.980	89.364	89.644
52	93.663	99.195	93.490	93.019	93.129
53	94.431	101.311	97.507	96.315	96.150
54	94.872	103.647	100.961	99.357	99.418
55	95.400	106.484	101.730	100.446	100.387
56	98.044	110.956	107.827	106.446	106.587
57	98.496	114.814	108.527	107.300	107.373
58	98.817	115.112	109.790	108.483	108.779
59	100.404	119.232	114.914	112.786	112.840
60	104.878	123.525	118.723	116.273	116.334

Table 6.6: Carpet Mixed Normalized Eigenvalues 23232

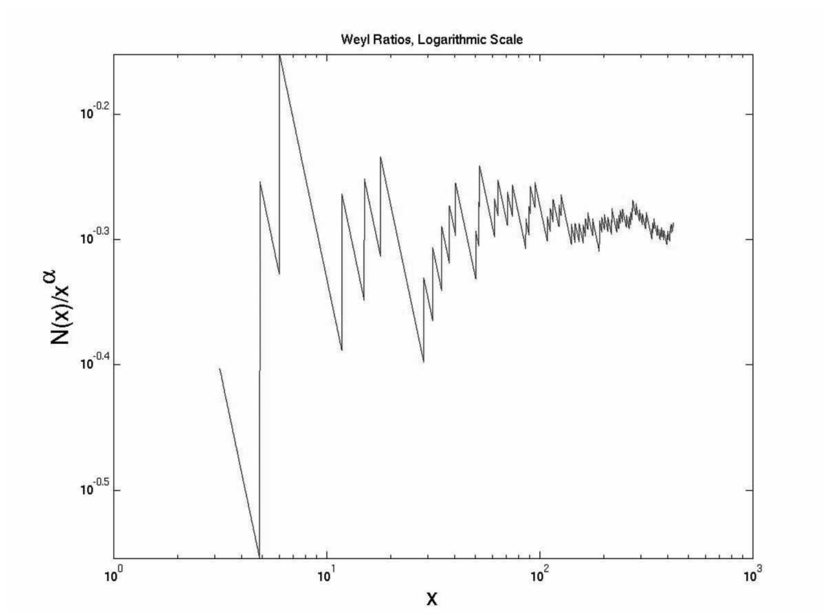
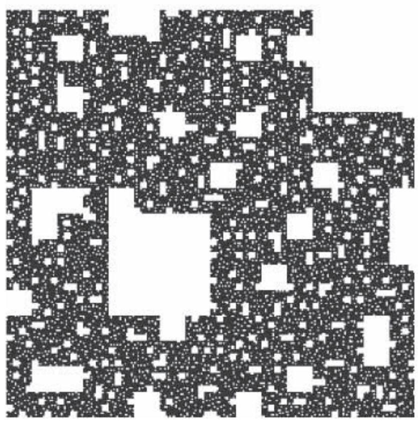
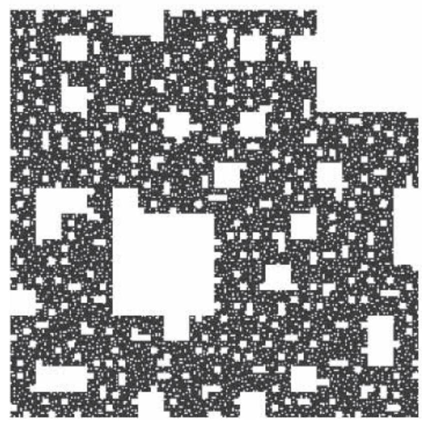


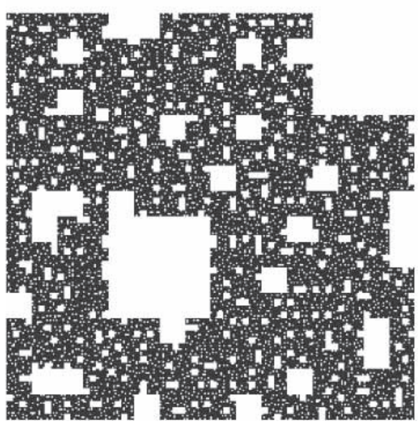
Figure 6.7: Weyl Ratios for $j = 4$, $k = \{2, 3, 2, 3, 2\}$, Level 5 Carpet, $\alpha = .8071$



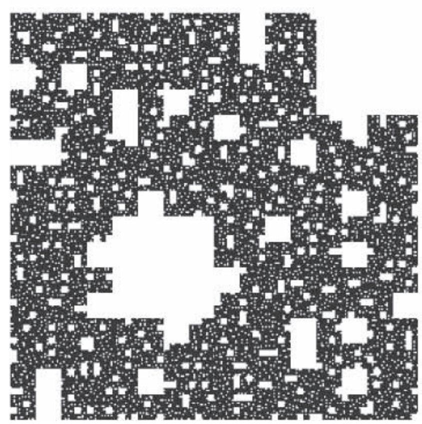
Original Carpet



Bifurcation at Level 4

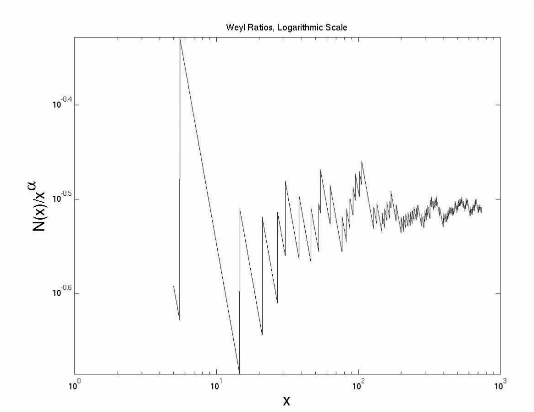


Bifurcation at Level 3

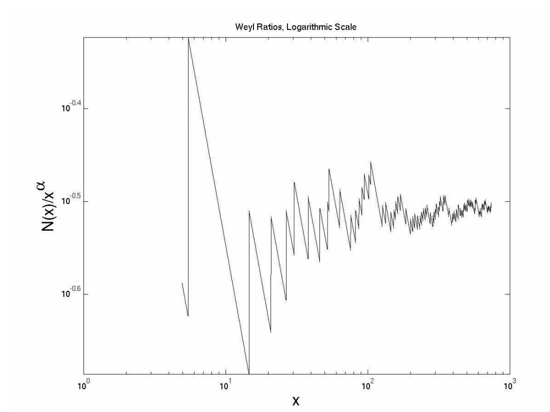


Bifurcation at Level 2

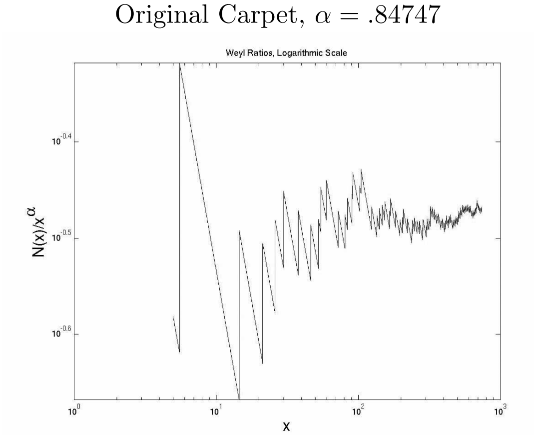
Figure 6.8: Carpet Bifurcations Ω_4 for j=4, k=2



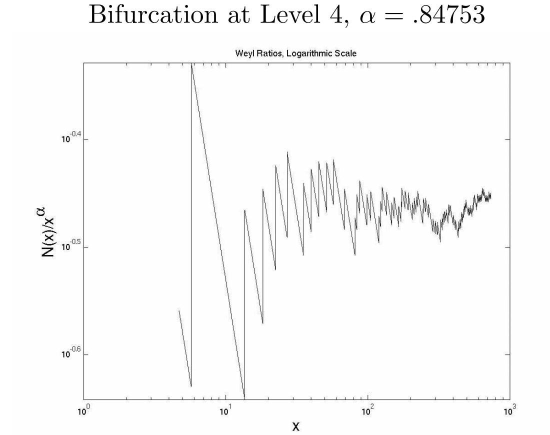
Original Carpet, $\alpha = .84747$



Bifurcation at Level 4, $\alpha = .84753$

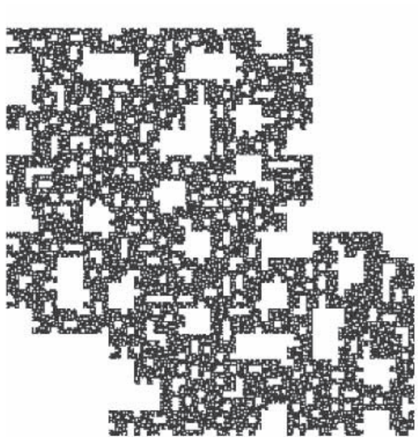


Bifurcation at Level 3, $\alpha = .83368$

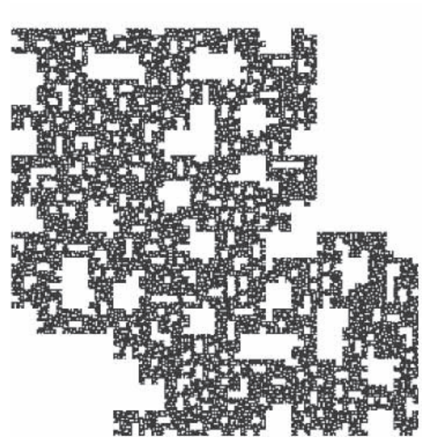


Bifurcation at Level 2, $\alpha = .83019$

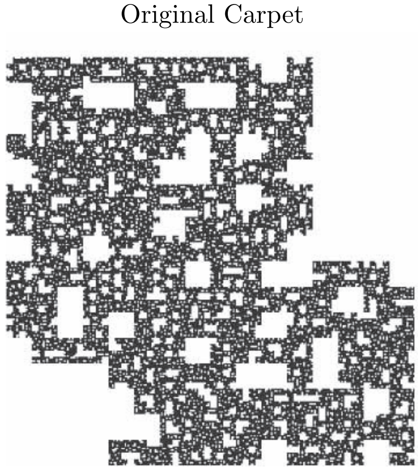
Figure 6.9: Weyl Ratios for $j = 4$, $k = 2$



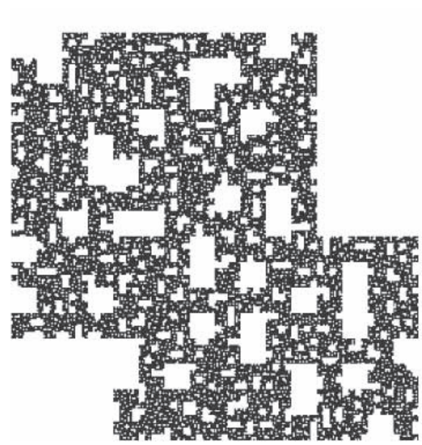
Original Carpet



Bifurcation at Level 4

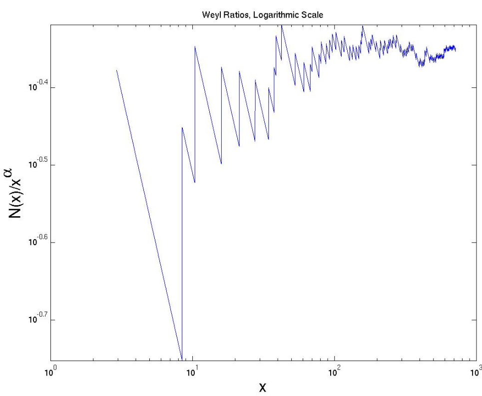


Bifurcation at Level 3

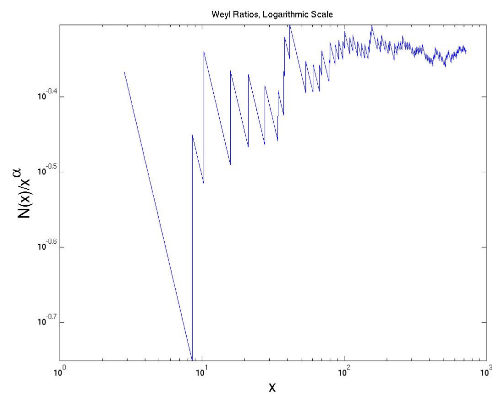


Bifurcation at Level 2

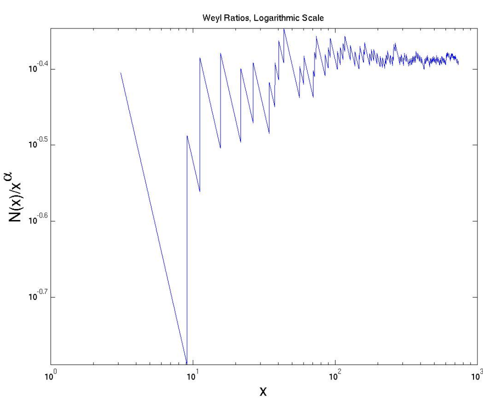
Figure 6.10: Carpet Bifurcations Ω_4 for j=4, k=3



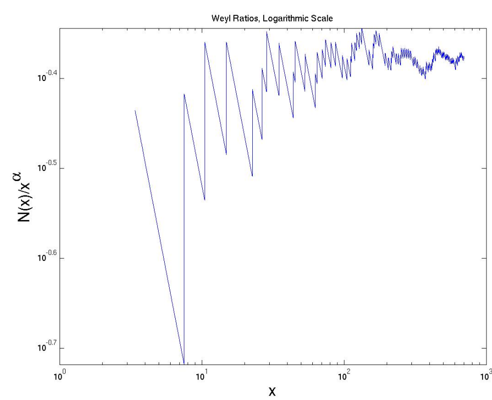
Original Carpet, $\alpha = .81013$



Bifurcation at Level 4, $\alpha = .80544$



Bifurcation at Level 3, $\alpha = .82086$



Bifurcation at Level 2, $\alpha = .81975$

Figure 6.11: Weyl Ratios for $j = 4$, $k = 3$

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Department of Mathematical Sciences, George Mason University, Fairfax, VA 22030
E-mail address: `tberry@gmu.edu`
Department of Mathematics, Cornell University, Ithaca, NY 14850-4201
E-mail address: `smh82@cornell.edu`
Department of Mathematics, Cornell University, Ithaca, NY 14850-4201
E-mail address: `str@math.cornell.edu`