Data assimilation with and without a model

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This work is in collaboration with:

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DATA ASSIMILATION

$$
\begin{aligned}
x_k &= f(x_{k-1}) + \eta_k \qquad & \eta_k \in \mathcal{N}(0, Q) \\
y_k &= h(x_k) + \nu_k \qquad & \nu_k \in \mathcal{N}(0, R)
\end{aligned}
$$

Main Problem: Given the model above plus observations y_k,

- **Filtering:** Estimate the current state $p(x_k \,|\, y_1, ..., y_k)$
- **Forecasting:** Estimate a future state $p(x_{k+\ell} \,|\, y_1, ..., y_k)$
- **Smoothing:** Estimate a past state $p(x_{k-\ell} \,|\, y_1, ..., y_k)$
- **Parameter estimation**

Apply ensemble Kalman filter (EnKF) to achieve these goals.

Data assimilation

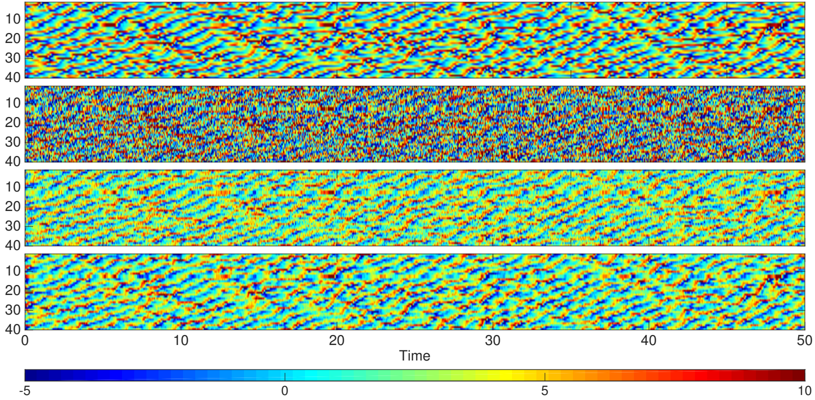
$$\begin{aligned} x_k &= f(x_{k-1}) + \eta_k & \eta_k \in \mathcal{N}(0, Q) \\ y_k &= h(x_k) + \nu_k & \nu_k \in \mathcal{N}(0, R) \end{aligned}$$

Possible obstructions:

- Observations y_k mix system noise η_k with obs noise ν_k
- Observations may be sparse in space or time
- Model error
    - Q and R may be unknown
    - Known model with unknown parameters
    - Wrong model, even with best fit parameters
    - Have model for some, not all of the variables

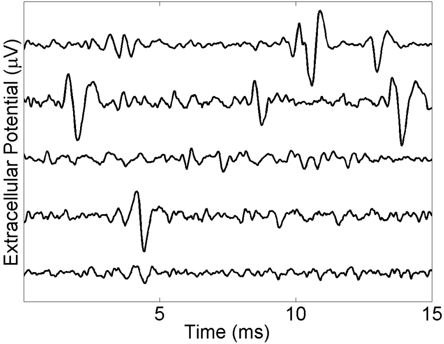
Example 1. Lorenz 96

$$\frac{dx^i}{dt} = -x^{i-2}x^{i-1} + x^{i-1}x^{i+1} - x^i + F$$



EXAMPLE 2. MEA RECORDINGS

Given: Voltages + Model



Want to find:

- State:
  - Sodium
  - Potassium
  - Currents
- Parameters
  - Neuron
  - Network

Two Step Filtering to Find $p(x_k \mid y_1, ..., y_k)$

- Assume we have $p(x_{k-1} \mid y_1, ..., y_{k-1})$
- **Forecast Step:** Find $p(x_k \mid y_1, ..., y_{k-1})$
- **Assimilation Step:** Perform a Bayesian update,

$$p(x_k \mid y_1, ..., y_k) \propto p(x_k \mid y_1, ..., y_{k-1}) p(y_k \mid x_k, y_1, ..., y_{k-1})$$

Posterior $\propto$ **Prior** $\times$ **Likelihood**

- **Alternative:** Variational methods (e.g. 3DVAR, 4DVAR)
  - Minimize a cost functional $\Rightarrow$ Hard optimization problem

BEST POSSIBLE SCENARIO

$$
\begin{aligned}
x_k &= f(x_{k-1}) + \eta_k \qquad & \eta_k \in \mathcal{N}(0, Q) \\
y_k &= h(x_k) + \nu_k \qquad & \nu_k \in \mathcal{N}(0, R)
\end{aligned}
$$

f and h are linear, all parameters known.

$$
\begin{aligned}
x_k &= F_{k-1} x_{k-1} + \eta_k \qquad & \eta_k \in \mathcal{N}(0, Q) \\
y_k &= H_k x_k + \nu_k \qquad & \nu_k \in \mathcal{N}(0, R)
\end{aligned}
$$

KALMAN FILTER

- Assume linear dynamics/obs and additive Gaussian noise

$$
\begin{aligned}
x_k &= F_{k-1}x_{k-1} + \omega_k \qquad & \omega_k \sim \mathcal{N}(0, Q) \\
y_k &= H_k x_k + \nu_k \qquad & \nu_k \sim \mathcal{N}(0, R)
\end{aligned}
$$

- For linear systems, easy observability condition:

$$
\tilde{H}_k^\ell = \begin{pmatrix} H_k \\ H_{k+1}F_k \\ \vdots \\ H_{k+\ell+1}F_{k+\ell}\cdots F_k \end{pmatrix}
$$

Must be full rank for some $\ell \Rightarrow$ KF guaranteed to work!

Kalman Filter

- Assume linear dynamics/obs and additive Gaussian noise

$$\begin{aligned} x_k &= F_{k-1}x_{k-1} + \omega_k & \omega_k \sim \mathcal{N}(0, Q) \\ y_k &= H_k x_k + \nu_k & \nu_k \sim \mathcal{N}(0, R) \end{aligned}$$

- Assume current estimate is Gaussian:

$$p(x_{k-1} \,|\, y_1, ..., y_{k-1}) = \mathcal{N}(x^a_{k-1}, P^a_{k-1})$$

- **Forecast:** Linear combinations of Gaussians
  - **Prior:** $p(x_k \,|\, y_1, ..., y_{k-1}) = \mathcal{N}(x^f_k, P^f_k)$
    - $x^f_k = F_{k-1}x^a_{k-1}$
    - $P^f_k = F_{k-1}P_{k-1}F^\top_{k-1} + Q$
  - **Likelihood:** $p(y_k \,|\, x_k, y_1, ..., y_{k-1}) = \mathcal{N}(y^f_k, P^y_k)$
    - $y^f_k = H_k x^f_k$
    - $P^y_k = H_k P^f_k H^\top_k + R$

Kalman Filter

- Forecast: Linear combinations of Gaussians
  - Prior: $p(x_k \,|\, y_1, \ldots, y_{k-1}) = \mathcal{N}(x_k^f, P_k^f)$
    - $x_k^f = F_{k-1} x_{k-1}^a$
    - $P_k^f = F_{k-1} P_{k-1} F_{k-1}^\top + Q$
  - Likelihood: $p(y_k \,|\, x_k, y_1, \ldots, y_{k-1}) = \mathcal{N}(y_k^f, P_k^y)$
    - $y_k^f = H_k x_k^f$
    - $P_k^y = H_k P_k^f H_k^\top + R$
- **Assimilation:** Product of Gaussians (complete the square)

$$p(x_k \,|\, y_1, \ldots, y_k) = \mathcal{N}(x_k^f, P_k^f) \times \mathcal{N}(y_k^f, P_k^y) = \mathcal{N}(x_k^a, P_k^a)$$

- Define the **Kalman gain:** $K_k = P_k^f H_k^T (P_k^y)^{-1}$
  - $x_k^a = x_k^f + K_k(y_k - y_k^f)$
  - $P_k^a = (I - K_k H_k) P_k^f$

Kalman Filter Summary

Forecast

$$x_k^f = F_{k-1} x_{k-1}^a$$
$$y_k^f = H_k x_k^f$$

Covariance update

$$P_k^f = F_{k-1} P_{k-1}^a F_{k-1}^T + Q$$
$$P_k^y = H_k P_k^f H_k^T + R$$

Kalman gain & Innovation

$$K_k = P_k^f H_k^T (P_k^y)^{-1}$$
$$\epsilon_k = y_k - y_k^f$$

Assimilation

$$x_k^a = x_k^f + K_k \epsilon_k$$
$$P_k^a = (I - K_k H_k) P_k^f$$

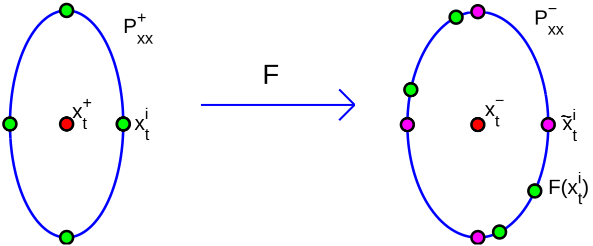
WHAT ABOUT NONLINEAR SYSTEMS?

- Consider a system of the form:

$$\begin{aligned} x_{k+1} &= f(x_k) + \omega_{k+1} & \omega_{k+1} \sim \mathcal{N}(0, Q) \\ y_{k+1} &= h(x_{k+1}) + \nu_{k+1} & \nu_{k+1} \sim \mathcal{N}(0, R) \end{aligned}$$

- More complicated observability condition (Lie derivatives)
- **Extended Kalman Filter (EKF)**:
  - Linearize $F_k = Df(x_k^a)$ and $H_k = Dh(x_k^f)$
- **Problem:** State estimate x_k^a may not be well localized
- **Solution: Ensemble Kalman Filter (EnKF)**

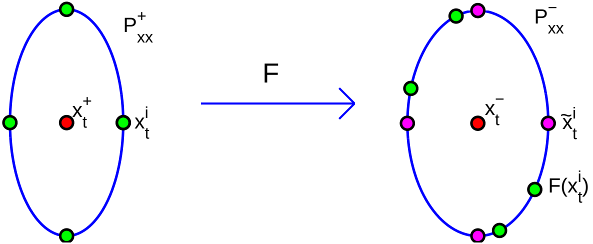
Ensemble Kalman Filter (EnKF)



Generate an ensemble with the current statistics (use matrix square root):

$$
\begin{aligned}
x_t^i &= \text{“sigma points” on semimajor axes} \\
x_t^f &= \frac{1}{2n}\sum F(x_t^i) \\
P_{xx}^f &= \frac{1}{2n-1}\sum (F(x_t^i)-x_t^f)(F(x_t^i)-x_t^f)^T + Q
\end{aligned}
$$

Ensemble Kalman Filter (EnKF)



Calculate $y_t^i = H(F(x_t^i))$. Set $y_t^f = \frac{1}{2n}\sum_i y_t^i$.

$$
\begin{aligned}
P_{yy} &= (2n-1)^{-1}\sum(y_t^i - y_t^f)(y_t^i - y_t^f)^T + R \\
P_{xy} &= (2n-1)^{-1}\sum(F(x_t^i) - x_t^f)(y_t^i - y_t^f)^T \\
K &= P_{xy}P_{yy}^{-1} \text{ and } P_{xx}^a = P_{xx}^f - KP_{yy}K^T \\
x_{t+1}^a &= x_t^f + K(y_t - y_t^f)
\end{aligned}
$$

PARAMETER ESTIMATION (STATE AUGMENTATION)

- When the model has parameters θ,

$$x_{k+1} = f(x_k, \theta) + \omega_{k+1}$$

- *Augment* the state $\tilde{x}_k = [x_k, \theta_k]$, $\tilde{Q} = \begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0} & Q^\theta \end{bmatrix}$
- Introduce trivial dynamics $d\theta = Q^\theta d\omega^\theta$

$$\begin{aligned} x_{k+1} &= f(x_k, \theta_k) + \omega_{k+1} \\ \theta_{k+1} &= \theta_k + \omega_{k+1}^\theta \end{aligned}$$

- Need to tune the covariance Q^θ of ω^θ
- Can track slowly varying parameters, $Q^\theta \approx \text{var}(\theta)$

EXAMPLE OF PARAMETER ESTIMATION

Consider a network of n Hodgkin-Huxley neurons

$$\begin{aligned}
\dot{V}_i &= -g_{Na}m^3h(V_i - E_{Na}) - g_K n^4(V_i - E_K) - g_L(V_i - E_L) \\
&\quad + I + \sum_{j\neq i}^{n} \Gamma_{\mathrm{HH}}(V_j)V_j \\
\dot{m}_i &= a_m(V_i)(1 - m_i) - b_m(V_i)m_i \\
\dot{h}_i &= a_h(V_i)(1 - h_i) - b_h(V_i)h_i \\
\dot{n}_i &= a_n(V_i)(1 - n_i) - b_n(V_i)n_i \\
\Gamma_{\mathrm{HH}}(V_j) &= \beta_{ij}/(1 + e^{-10(V_j+40)})
\end{aligned}$$

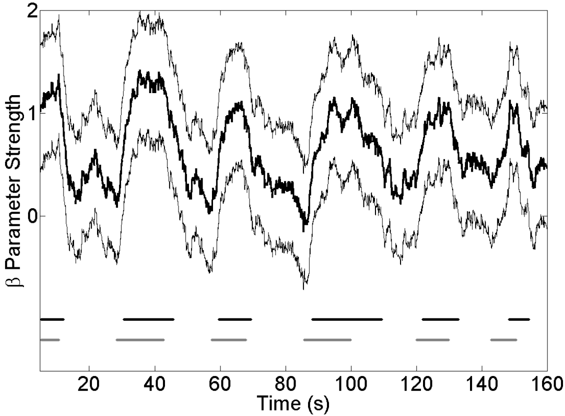
Only observe the voltages V_i

Recover all variables and the connection parameters β

Example of Parameter Estimation

Can even turn connections on and off (grey dashes)

Variance estimate ⇒ statistical test (black dashes)



Robustness to model error?

Fit a generic spiking model (Hindmarsh-Rose)

$$\begin{aligned}
\dot{V}_i &= a_i V_i^2 - V_i^3 - y_i - z_i + I_i + \sum_{j \neq i}^{n} \Gamma_{\mathrm{HH}}(V_j) V_j \\
\dot{y}_i &= (a_i + \alpha_i) V_i^2 - y_i \\
\dot{z}_i &= \mu_i (b_i V_i + c_i - z_i) \\
\Gamma_{\mathrm{HH}}(V_j) &= \beta_{ij} / (1 + e^{-10(V_j + 40)})
\end{aligned}$$

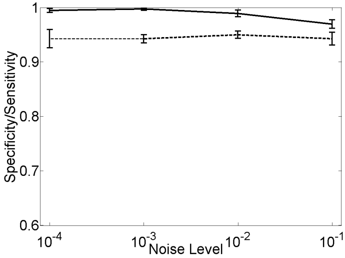
Observe voltages V_i from Hodgkin-Huxley!

Fit parameters to match neuron characteristics

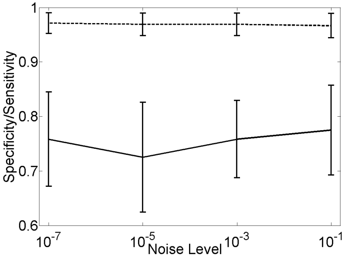
Recover the connection parameters β

Link detection from networks of model neurons

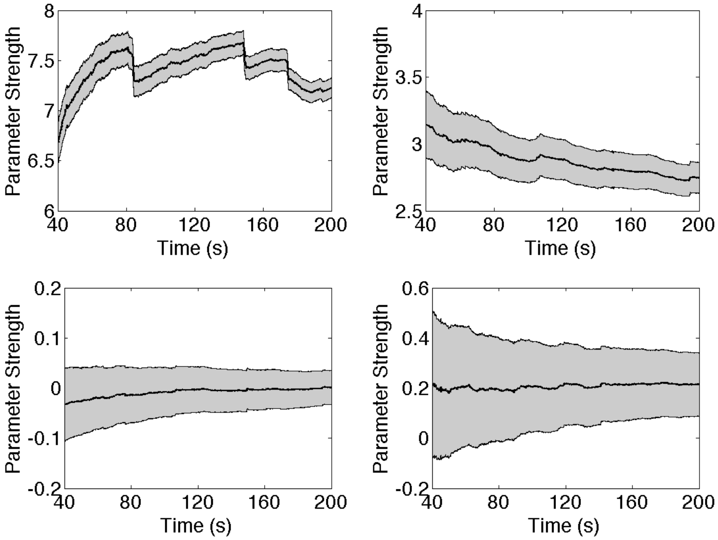
Network of Hindmarsh-Rose neurons, modeled by Hindmarsh-Rose



Network of Hodgkin-Huxley neurons, modeled by Hindmarsh-Rose

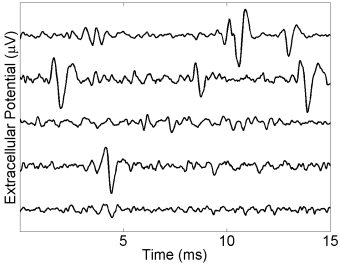


LINK DETECTION FROM MEA RECORDINGS

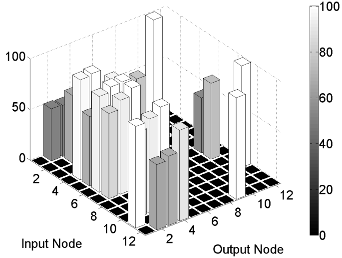


LINK DETECTION FROM MEA RECORDINGS

MEA Recording



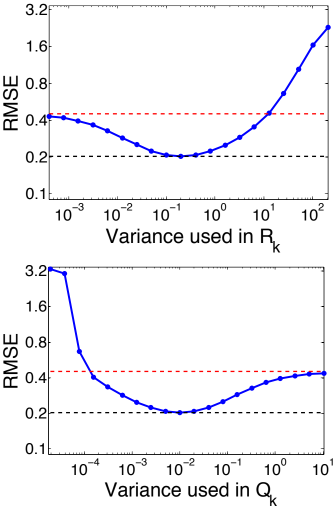
Recovered Network



% of 160 sec each connection was statistically significant

EnKF: Influence of Q and R

- Simple example with full observation and diagonal noise covariances
- Red indicates RMSE of unfiltered observations
- Black is RMSE of 'optimal' filter (true covariances known)



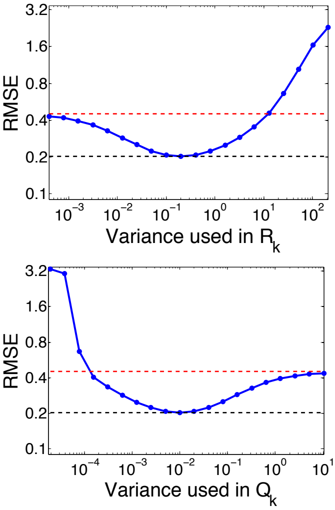
EnKF: Influence of Q and R

Standard Kalman Update:

$$P_k^f = F_{k-1} P_{k-1}^a F_{k-1}^T + Q_{k-1}$$
$$P_k^y = H_k P_k^f H_k^T + R_{k-1}$$

$$K_k = P_k^f H_k^T (P_k^y)^{-1}$$
$$P_k^a = (I - K_k H_k) P_k^f$$

$$\epsilon_k = y_k - y_k^f = y_k - H_k x_k^f$$
$$x_k^a = x_k^f + K_k \epsilon_k$$



ADAPTIVE FILTER: ESTIMATING Q AND R

- Innovations contain information about Q and R

$$\begin{aligned}\epsilon_k &= y_k - y_k^f \\ &= h(x_k) + \nu_k - h(x_k^f) \\ &= h(f(x_{k-1}) + \omega_k) - h(f(x_{k-1}^a)) + \nu_k \\ &\approx H_k F_{k-1}(x_{k-1} - x_{k-1}^a) + H_k \omega_k + \nu_k\end{aligned}$$

- IDEA: Use innovations to produce samples of Q and R :

$$\begin{aligned}\mathbb{E}[\epsilon_k \epsilon_k^T] &\approx HP^f H^T + R \\ \mathbb{E}[\epsilon_{k+1} \epsilon_k^T] &\approx HFP^e H^T - HFK\mathbb{E}[\epsilon_k \epsilon_k^T] \\ P^e &\approx FP^a F^T + Q\end{aligned}$$

- In the linear case this is rigorous and was first solved by Mehra in 1970

ADAPTIVE FILTER: ESTIMATING Q AND R

- To find Q and R we estimate H_k and F_{k-1} from the ensemble and invert the equations:

$$\begin{aligned}
\mathbb{E}[\epsilon_k \epsilon_k^T] &\approx HP^f H^T + R \\
\mathbb{E}[\epsilon_{k+1} \epsilon_k^T] &\approx HFP^e H^T - HFK\mathbb{E}[\epsilon_k \epsilon_k^T]
\end{aligned}$$

- This gives the following *empirical* estimates of Q_k and R_k:

$$\begin{aligned}
P_k^e &= (H_{k+1}F_k)^{-1}(\epsilon_{k+1}\epsilon_k^T + H_{k+1}F_k K_k \epsilon_k \epsilon_k^T)H_k^{-T} \\
Q_k^e &= P_k^e - F_{k-1}P_{k-1}^a F_{k-1}^T \\
R_k^e &= \epsilon_k \epsilon_k^T - H_k P_k^f H_k^T
\end{aligned}$$

- Note: P_k^e is an empirical estimate of the background covariance

Adaptive EnKF

We combine the estimates of Q and R with a moving average

Original Kalman Eqs.

$$
\begin{aligned}
P_k^f &= F_{k-1} P_{k-1}^a F_{k-1}^T + Q_{k-1} \\
P_k^y &= H_k P_k^f H_k^T + R_{k-1} \\
K_k &= P_k^f H_k^T (P_k^y)^{-1} \\
P_k^a &= (I - K_k H_k) P_k^f \\
\epsilon_k &= y_k - y_k^f \\
x_k^a &= x_k^f + K_k \epsilon_k
\end{aligned}
$$

Our Additional Update

$$
\begin{aligned}
P_{k-1}^e &= F_{k-1}^{-1} H_k^{-1} \epsilon_k \epsilon_{k-1}^T H_{k-1}^{-T} \\
&+ K_{k-1} \epsilon_{k-1} \epsilon_{k-1}^T H_{k-1}^{-T} \\
Q_{k-1}^e &= P_{k-1}^e - F_{k-2} P_{k-2}^a F_{k-2}^T \\
R_{k-1}^e &= \epsilon_{k-1} \epsilon_{k-1}^T - H_{k-1} P_{k-1}^f H_{k-1}^T \\
Q_k &= Q_{k-1} + (Q_{k-1}^e - Q_{k-1})/\tau \\
R_k &= R_{k-1} + (R_{k-1}^e - R_{k-1})/\tau
\end{aligned}
$$

ADAPTIVE FILTER: APPLICATION TO LORENZ-96

- We will apply the adaptive EnKF to the 40-dimensional Lorenz96 model integrated over a time step $\Delta t = 0.05$

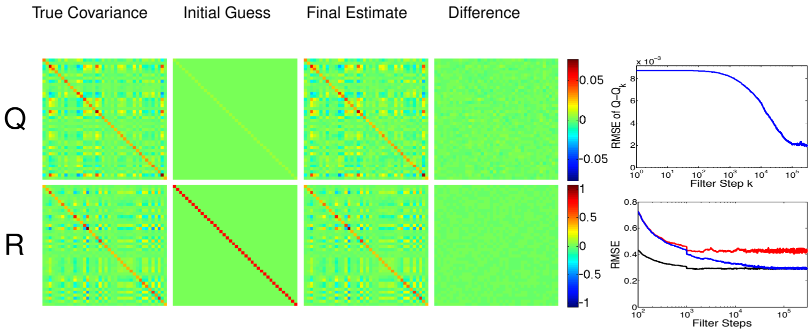
$$\frac{dx^i}{dt} = -x^{i-2}x^{i-1} + x^{i-1}x^{i+1} - x^i + F$$

- We augment the model with Gaussian white noise

$$\begin{aligned} x_k &= f(x_{k-1}) + \omega_k \qquad & \omega_k = \mathcal{N}(0, Q) \\ y_k &= h(x_k) + \nu_k \qquad & \nu_k = \mathcal{N}(0, R) \end{aligned}$$

- The Adaptive EnKF uses $F = 8$
- We will consider model error where the true $F^i = \mathcal{N}(8, 16)$

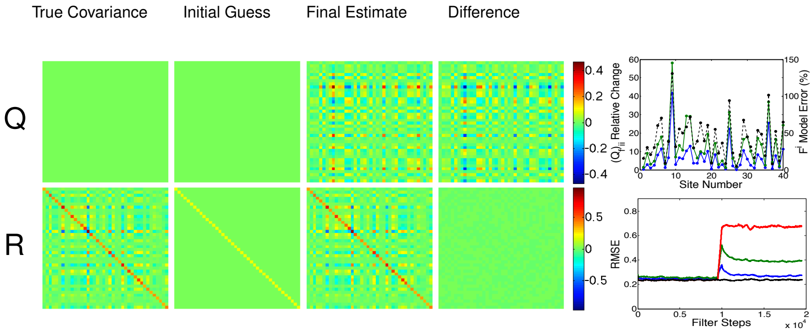
RECOVERING Q AND R, PERFECT MODEL



RMSE (bottom right) for the initial guess covariances (red) the true Q and R (black) and the adaptive filter (blue)

Compensating for Model Error

The adaptive filter compensates for errors in the forcing F^i



RMSE (bottom right) for the initial guess covariances (red) the perfect model (black) and the adaptive filter (blue)

Open problem

- Determining Q^θ for parameters fails: $d\theta = Q^\theta d\omega^\theta$
- Often Q^θ increases unrealistically or diverges
- Brownian motion $\Rightarrow$ Non-identifiability?
- Ornstein-Uhlenbeck works: $d\theta = \alpha(\overline{\theta} - \theta)d\theta + Q^\theta d\omega^\theta$
- But now need to estimate $\alpha, \overline{\theta}$!

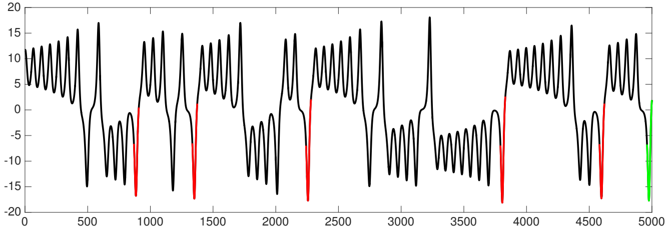
KALMAN-TAKENS FILTER: THROWING OUT THE MODEL...

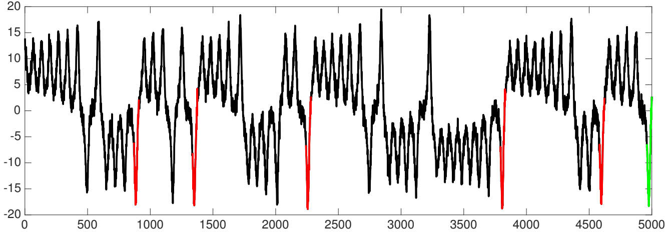
- Starting with historical observations $\{y_0, ..., y_n\}$
- Form Takens delay-embedding state vectors

$$x_i = (y_i, y_{i-1}, ..., y_{i-d})^\top$$

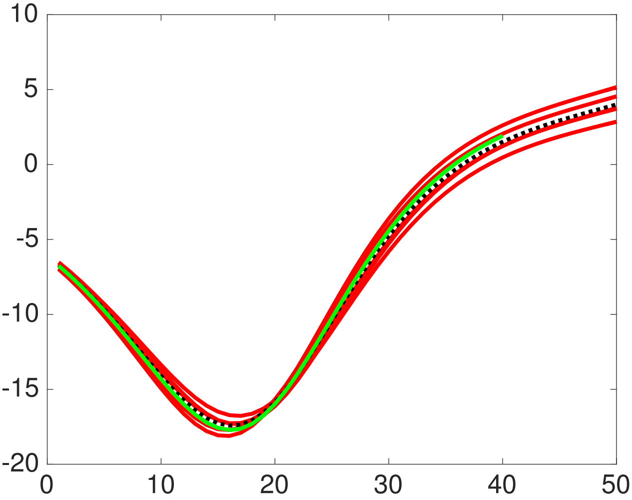
- Build an EnKF:
  - Apply analog forecast to each ensemble member
  - Use the observation function $Hx_i = y_i$
  - Crucial to estimate Q and R

Kalman Reconstruction

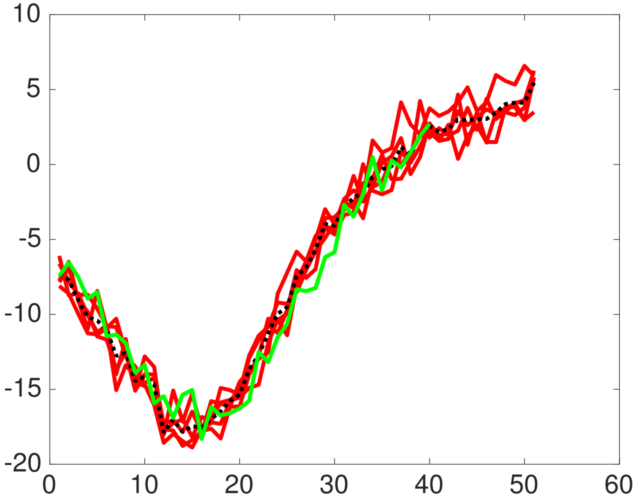




Analog Forecast

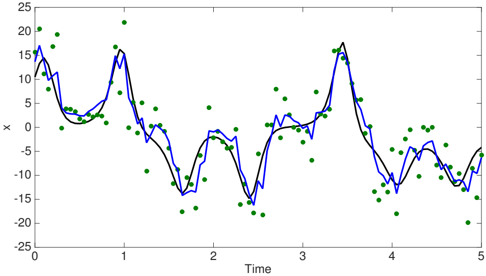


Analog Forecast

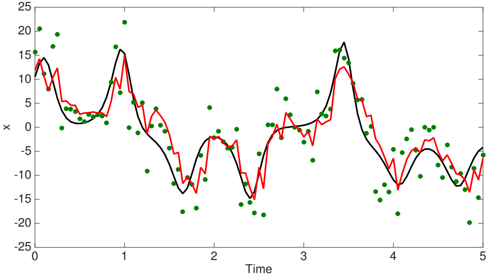


Kalman-Takens filter: Lorenz-63

EnKF w/ model

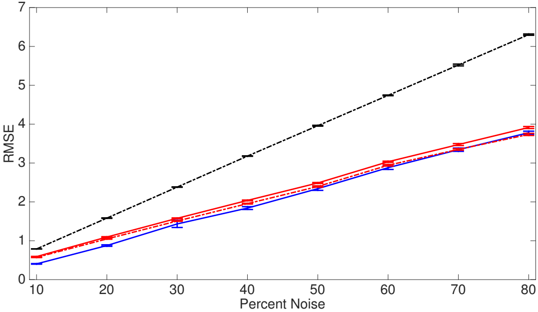


Kalman-Takens

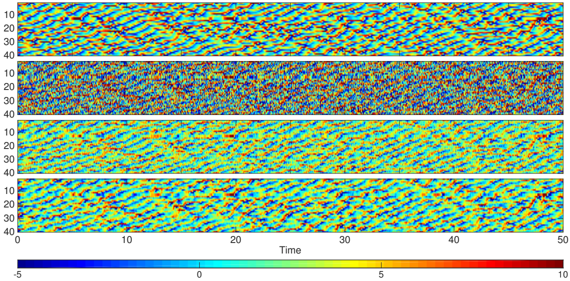


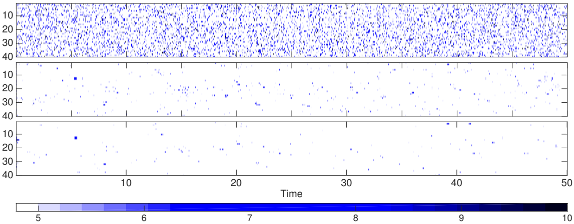
Kalman-Takens filter

Comparing K-T (red) with full model (blue)



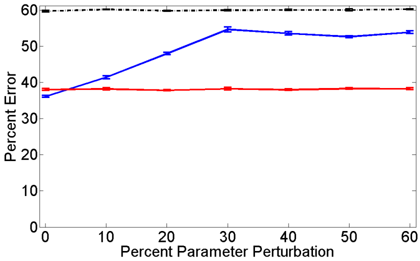
Kalman-Takens filter: Lorenz-96



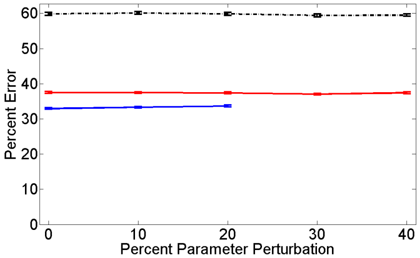


Model Error

Lorenz-63

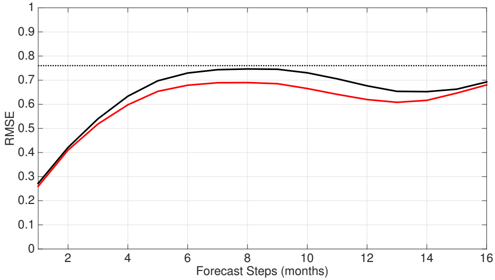


Lorenz-96



KALMAN-TAKENS FILTER

Forecast error: El Nino index



Limits of parameter estimation

- Depends on model complexity and observability
- Typically estimate ≈ 4 parameters per observation
- New work: (F. Hamilton et al., Hybrid modeling and prediction of dynamical systems)
  - Problem: Too many parameters, model is useless
  - Solution: To fit params, replace other equations with K-T
  - Extends the boundaries of parameter estimation

Estimating unmodeled dynamics (Franz)

Want to track un-modeled variable S

$$
\begin{aligned}
\dot{w} &= F(w) + \omega_t \\
\begin{bmatrix} y \\ \vdots \\ y \\ S \end{bmatrix} &= H(w) + \nu_t
\end{aligned}
$$

Run m model with different parameters p_i

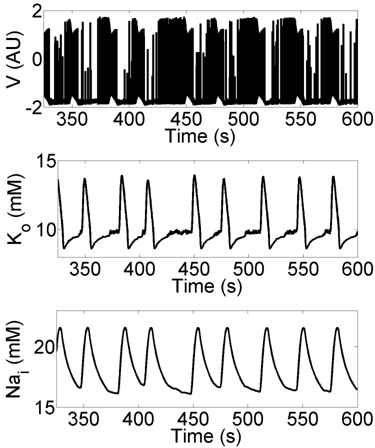
$$
w = \begin{bmatrix} x^1 \\ \vdots \\ x^m \\ c^1 \\ \vdots \\ c^m \\ d \end{bmatrix}, \quad F = \begin{bmatrix} f(x, p_1) \\ \vdots \\ f(x, p_m) \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad H(w) = \begin{bmatrix} h(x^1) \\ \vdots \\ h(x^m) \\ \sum_{i,j} c^i x^i + d \end{bmatrix}
$$

Fit regression params c^i_j and d from training data S

RECONSTRUCTING UNMODELED IONIC DYNAMICS

Observing seizure voltage, reconstruct unmodeled potassium and sodium dynamics (assimilation model is Hindmarsh-Rose)

J. Cressman, G. Ullah, J. Ziburkus, S. Schiff, and E. Barreto, Journal of Comp. Neuroscience **26**, 159 (2009).

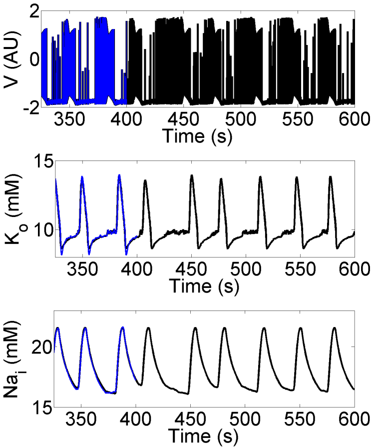


Actual

Reconstructing Unmodeled Ionic Dynamics

Observing seizure voltage, reconstruct unmodeled potassium and sodium dynamics (assimilation model is Hindmarsh-Rose)

J. Cressman, G. Ullah, J. Ziburkus, S. Schiff, and E. Barreto, Journal of Comp. Neuroscience **26**, 159 (2009).



Actual **Observed**

Reconstructing Unmodeled Ionic Dynamics

Observing seizure voltage, reconstruct unmodeled potassium and sodium dynamics (assimilation model is Hindmarsh-Rose)

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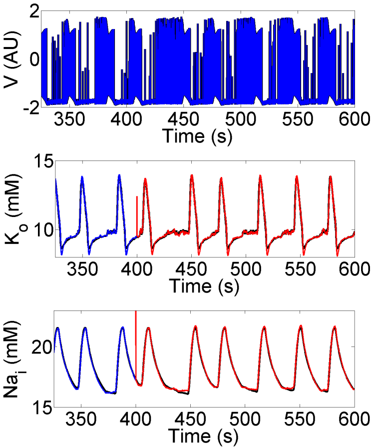
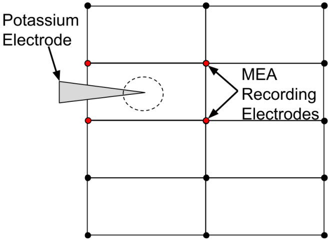


Actual **Observed** **Predicted**

RECONSTRUCTING EXTRACELLULAR POTASSIUM FROM AN *In Vitro* NETWORK

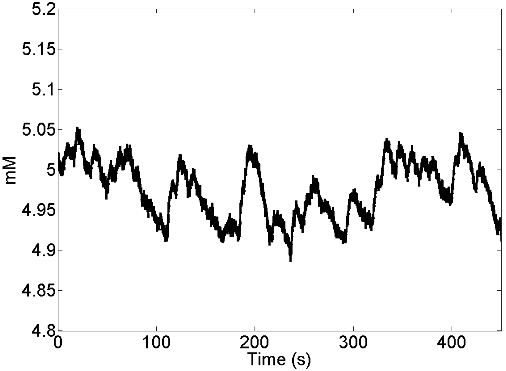
We want to track extracellular potassium dynamics in a network but measurements are difficult and spatially limited



Extracellular potassium is an **unmodeled variable**

RECONSTRUCTING EXTRACELLULAR POTASSIUM FROM AN *In Vitro* NETWORK

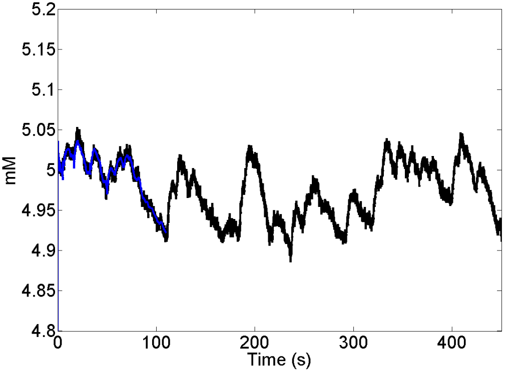
The local extracellular potassium in an MEA network can be reconstructed and predicted using our approach (assimilation model is Hindmarsh-Rose)



Actual

RECONSTRUCTING EXTRACELLULAR POTASSIUM FROM AN *In Vitro* NETWORK

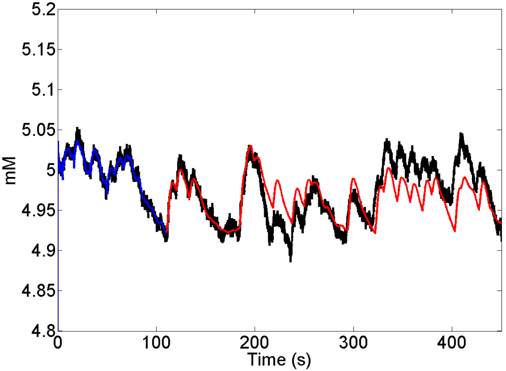
The local extracellular potassium in an MEA network can be reconstructed and predicted using our approach (assimilation model is Hindmarsh-Rose)



Actual **Observed**

Reconstructing Extracellular Potassium from an *In Vitro* Network

The local extracellular potassium in an MEA network can be reconstructed and predicted using our approach (assimilation model is Hindmarsh-Rose)



Actual **Observed** **Predicted**

SUMMARY

- EnKF is a useful data assimilation technique for neurodynamics and other types of data
- Parameter estimation
- Adaptive QR is helpful when Q and R are unknown
- Difficulties
  - Model error
  - Unmodeled variables
  - No model
- Kalman-Takens filter
- Multimodel data assimilation

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Kalman Filter: Forecast Step

- At time $k-1$ we have mean x^a_{k-1} and covariance P^a_{k-1}

$$x_k = F_{k-1}x_{k-1} + \omega_k$$

- Linear combinations of Gaussians are still Gaussian so:
  - $p(F_{k-1}x_{k-1} \mid y_1, ..., y_{k-1}) = \mathcal{N}(F_{k-1}x^a_{k-1}, F_{k-1}P_{k-1}F^\top_{k-1})$
  - $p(x_k \mid y_1, ..., y_{k-1}) = \mathcal{N}(F_{k-1}x^a_{k-1}, F_{k-1}P_{k-1}F^\top_{k-1} + Q)$
- Define the *Forecast mean:* $x^f_k \equiv F_{k-1}x^a_{k-1}$
- Define the *Forecast covariance:* $P^f_k \equiv F_{k-1}P^a_{k-1}F^\top_{k-1} + Q$

KALMAN FILTER: DEFINING THE LIKELIHOOD FUNCTION

- Recall that $y_k = H_k x_k + \nu_k$ where $\nu_k \sim \mathcal{N}(0, R)$ is Gaussian
- The forecast distribution: $p(x_k \,|\, y_1, ..., y_{k-1}) = \mathcal{N}(x_k^f, P_k^f)$
- **Likelihood:** $p(y_k \,|\, x_k, y_1, ..., y_{k-1}) = \mathcal{N}(H_k x_k^f, H_k P_k^f H_k^\top + R)$
- Define the *Observation mean:* $y_k^f = H_k x_k^f$
- Define the *Observation covariance:* $P_k^y = H_k P_k^f H_k^\top + R$

KALMAN FILTER: ASSIMILATION STEP

- Gaussian prior $\times$ Gaussian likelihood $\Rightarrow$ Gaussian posterior

$$
\begin{aligned}
p(y|x)p(x) \propto \exp\Big\{ & -\frac{1}{2}(y - Hx)^\top (P^y)^{-1}(y - Hx) \\
& - \frac{1}{2}(x - x^f)^\top (P^f)^{-1}(x - x^f)\Big\} \\
\propto \exp\Big\{ & -\frac{1}{2}x^\top((P^y)^{-1} + H(P^f)^{-1}H^\top)x \\
& + x^\top(H^\top(P^y)^{-1}y - (P^f)^{-1}x^f)\Big\}
\end{aligned}
$$

- Posterior Covariance: $P^a = \left((P^f)^{-1} + H^\top (P^y)^{-1} H\right)^{-1}$
- Posterior Mean: $x^a = P^a \left(H^\top (P^y)^{-1} y - (P^f)^{-1} x^f\right)$

KALMAN FILTER: ASSIMILATION STEP

- **Kalman Equations:** (after some linear algebra...)
  - Kalman Gain: $K_k = P_k^f H_k^\top (P_k^y)^{-1}$
  - Innovation: $\epsilon_k = y_k - y_k^f$
  - Posterior Mean: $x_k^a = x_k^f + K_k \epsilon_k$
  - Posterior Covariance: $P_k^a = (I - K_k H_k) P_k^f$
- x_k^a is the least squares/minimum variance estimator of x_k