<u>Context</u>: Smooth Riemannian Geometry.

<u>Motivation</u>: Given data points on an embedded manifold $\{x_i\}_{i=1}^N \subseteq \mathcal{M} \subseteq \mathbb{R}^m$

we can construct graphs (weighted and unweighted) such that the graph Laplacian

$L = D - A$ converges to the Laplace-de Rahm operator $\Delta = \delta d$ spectrally.

(A_{ij} = edge weight between nodes i and j (adjacency matrix) $D_{ii} = \sum_{j=1}^N A_{ij}$ "degree" matrix.)

<u>Note</u>: Only input is the graph, no simplicial complex needed.

$$L = D\Phi\Lambda\Phi^T \quad (L\tilde{\phi}_i = \tilde{\lambda}_i D\tilde{\phi}_i) \quad \text{error} \quad |\tilde{\lambda}_i - \lambda_i| \propto \lambda_i \ \text{ for } \ \lambda_i < \mathcal{O}(N^{\frac{d-2}{d+6}})$$

$$\Delta\phi_i = \lambda_i\phi_i \qquad\qquad |\tilde{\lambda}_i - \lambda_i| \propto \lambda_i^2 \ \text{ for } \ \lambda_i > \mathcal{O}(N^{\frac{d-2}{d+6}})$$

$$\langle f, \Delta\phi_i\rangle = \lambda_i\langle f, \phi_i\rangle \iff \vec{f}^T L\tilde{\phi}_i = \lambda_i \vec{f}^T D\tilde{\phi}_i$$

- So we can learn the Laplacian, but we only trust the smoothest eigenfunctions

ϕ_i orthonormal: $\lambda_i = \langle\phi_i, \Delta\phi_i\rangle_{L^2} = \langle\nabla\phi_i, \nabla\phi_i\rangle_{L^2_X} = \int_{\mathcal{M}} \|\nabla\phi_i\|^2 \, d\text{vol}$ measures roughness.

- Laplacian tells us everything on the manifold. How?

Product Rule: $\Delta(fh) = f\Delta h + h\Delta f - 2\nabla f\cdot\nabla h \qquad \nabla f\cdot\nabla h = g(\nabla f \cdot \nabla h)$

where g is the Riemannian metric.

Given $x \in \mathcal{M}$ we can find functions $f_1, f_2, \dots, f_d$ such that $\nabla f_1(x), \nabla f_2(x), \dots, \nabla f_d(x)$ are an orthonormal basis for $T_x\mathcal{M}$, then

$$g_{ij}(x) = g(\nabla f_i(x), \nabla f_j(x))$$
$$= \tfrac{1}{2}\left(f_i(x)\Delta f_j(x) + f_j(x)\Delta f_i(x) - \Delta(f_i f_j)(x)\right).$$

So Δ tells us g which tells us all of Riemannian geometry. g = matrix valued function "local" or map from pairs of vector fields to functions "global"

- <u>Second Motivation</u>: Local coordinates are annoying. On S^1 θ is a "global" coordinate system: $\frac{\partial}{\partial\theta}$ is a vector field that spans the tangent space $T_\theta\mathcal{M}$ simultanously for all θ.
- On a flat torus, $\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}\}$ simultaneously span all $T_{(\theta,\phi)}\mathcal{M}$.

- On S^2 consider any set of smooth vector fields $\{v_1, v_2\}$

  since v_1 must vanish at a point x so $\{v_1(x), v_2(x)\} = \{0, v_2(x)\}$ cannot span $T_x M$.

- However, there exists an embedding $F: S^2 \to \mathbb{R}^3$ meaning that for all $x \in S^2$ $F = (F_1, F_2, F_3)$ $\{\nabla F_1(x), \nabla F_2(x), \nabla F_3(x)\}$ spans $T_x M$ (but is not a basis.)

- Since every manifold embedds in $\mathbb{R}^n$ for $n \gg d$, there exists a collection of n vector fields that simultanously span all tangent spaces. (but are not a basis.) • overcomplete

- Fact: The eigenfunctions of the Laplacian $(\phi_1, \phi_2, \dots, \phi_J)$ form an embedding for J sufficiently large. Thus $\{\nabla\phi_1, \dots, \nabla\phi_J\}$ span $T_x M$ for all x.

Idea: If we can represent all vector fields in terms of eigenfunctions of the Laplacian we can represent the geometry w/o a simplicial complex!

Let V be a smooth vector field on M, then for each x we can write

$$V_x = c_1(x)\nabla\phi_1(x) + \dots + c_J(x)\nabla\phi_J(x) \quad \text{since } \{\nabla\phi_1(x) \dots \nabla\phi_J(x)\} \text{ span } T_x M.$$
$$= \sum_{j=1}^{J} c_j(x)\nabla\phi_j(x).$$

Since V, M are smooth, we can choose $c_j(x)$ to be a smooth function of M.

Recall that $\{\phi_i\}_{i=1}^{\infty}$ is an orthonormal basis for $L^2(M)$.

So in $L^2(M)$, $c_j(x) = \sum_{i=1}^{\infty} \langle c_j, \phi_i\rangle \phi_i(x) = \sum_{i=1}^{\infty} c_{ij}\phi_i(x)$

So $V_x = \sum c_{ij}\phi_i(x)\nabla\phi_j(x)$ so $V = \sum c_{ij}\phi_i\nabla\phi_j$

In other words $\{\phi_i\nabla\phi_j\}$ spans the set of all smooth vector fields V.

More formally: Let v, w vector fields on M and set $\langle v, w\rangle_{L^2_X} = \int g(v, w)\, dvol$

and $L^2_X = \{ v : \|v\|_{L^2_X} = \sqrt{\langle v, v\rangle_{L^2_X}} < \infty \}$

then $\{\phi_i \nabla\phi_j\}_{\substack{i=1,\dots,\infty \\ j=1,\dots,J}}$ is a frame for L^2_X meaning that there exists

nonzero constants A, B such that for all $v \in L^2_X$ we have the frame condition

$$A\|v\|^2_{L^2_X} \leq \sum_{i=1}^{\infty}\sum_{j=1}^{J} \langle v, \phi_i \nabla\phi_j\rangle^2_{L^2_X} \leq B\|v\|^2_{L^2_X}$$

(Note that for ONB we have $\sum_{i=1}^{n} |\langle v, e_i\rangle|^2 = \|v\|^2$ for $v \in \mathbb{R}^n$)

$A \neq 0 \Rightarrow \phi_i \nabla\phi_j$ spans (something in every direction so can't get zero for all inner products)

$B \neq 0 \Rightarrow$ limited redundancy (can't stack up infinitely many elements in same direction)

For each $v \in L^2_X$ $\exists\, c_{ij}$ s.t. $v = \sum c_{ij}\, \phi_i \nabla\phi_j$ but the c_{ij} are not unique.

Fact: There are unique coefficients $v = \sum v^{ij} \phi_i \nabla\phi_j$ such that $\sum_{i=1}^{\infty}\sum_{j=1}^{J} |v^{ij}|^2$ is minimal.

$\boxed{\text{Set } b_{ij} = \phi_i \nabla\phi_j}$

Now we can compute $g(v, w) = v \cdot w = \sum_{ijkl} v^{ij} w^{kl} (\phi_i \nabla\phi_j \cdot \phi_k \nabla\phi_l)$

$$= \sum_{ijkl} v^{ij} w^{kl}\, \phi_i \phi_k \nabla\phi_j \cdot \nabla\phi_l = \sum_{ijkl} v^{ij} w^{kl}\, \phi_i \phi_k\, \tfrac{1}{2}\big(\phi_j \Delta\phi_l + \phi_l \Delta\phi_j - \Delta(\phi_j \phi_l)\big)$$

Notice $\phi_j \phi_l \in L^2$ so $\phi_j \phi_l = \sum_s c_{jls} \phi_s$ where $\underbrace{c_{jls} = \langle \phi_j \phi_l, \phi_s\rangle}_{\text{Structure constants of the multiplicative algebra on } C^\infty(M)}$

c_{ij}^{k}

$$= \sum_{ijkls} v^{ij} w^{kl}\, \phi_i \phi_k \underbrace{\tfrac{1}{2}(\lambda_l + \lambda_j - \lambda_s)\, c_{jls}}_{g_{sjl} = \langle \nabla\phi_j \cdot \nabla\phi_l, \phi_s\rangle} \phi_s$$

So spectrally, we want $\langle g(v,w), \phi_t \rangle = \sum_{ijkl} v^{ij} w^{kl} \sum_s g_{sjl} \langle \phi_i \phi_k \phi_s, \phi_t \rangle$

We also compute the Hodge inner product

$$\langle v, w \rangle_{L^2_X} = \int g(v,w) \, dvol = \sum_{ijkl} v^{ij} w^{kl} \int \phi_i \nabla\phi_j \cdot \phi_k \nabla\phi_l \, dvol$$

- Where $G_{ijkl} = \langle b_{ij}, b_{kl} \rangle = \langle \phi_i \phi_k, \nabla\phi_j \cdot \nabla\phi_l \rangle = \sum_s c_{iks} g_{sjl}$

$$= \boxed{\frac{1}{2} \sum_s c_{iks} c_{sjl} (\lambda_j + \lambda_l - \lambda_s)}$$

Notice: Only c's and λ's

- Is the Hodge Grammian. Structure of L^2_X in terms of c and λ.

[By re-indexing b_{ij} $j = 1, \ldots, J$, $i = 1, \ldots, \infty$ we can ~~then~~ turn $G_{ij,kl}$ into a symmetric Grammian matrix.

Longer Derivation: $\langle \phi_i d\phi_j, \Delta_1(\phi_k d\phi_l) \rangle = E_{ijkl}$ is the Dirichlet Energy for the 1-Laplacian.

Set $c^p_{ijkl} = \sum_s \lambda^p_s c_{ijs} c_{skl}$

Then $\hat{E}_{ijkl} = \langle b^{ij} - b^{ji}, \Delta_1 (b^{kl} - b^{lk}) \rangle = (\lambda_i + \lambda_j + \lambda_k + \lambda_l)(c^1_{il,jk} - c^1_{ik,jl}) + (c^2_{ik,jl} - c^2_{il,jk})$

Harder Theorem: Set $\langle v, w \rangle_{H^1} = \langle v, w \rangle_{L^2} + \langle v, \Delta_1(w) \rangle_{L^2}$

and $H^1 = \{ v : \|v\|_{H^1} = \sqrt{\langle v, v \rangle_{H^1}} < \infty \}$ the $b_{ij} = \phi_i \nabla\phi_j$ is a frame for H^1.

Vector Fields

- **Derivations:** Consider $C^\infty(\mathcal{M})$ $\quad v : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$

  Liebniz rule: $v(fg) = f v(g) + g v(f)$.

  Interpretation: $v(f) = v \cdot \nabla f = \overbrace{g(\underset{\substack{\uparrow \\ \text{vector field}}}{v}, \underset{\substack{\uparrow \\ \text{vector field}}}{\nabla f})}^{\text{function}}$

- Arrows: Let $F : M \hookrightarrow \mathbb{R}^n$ embedding then $v(F) \equiv v \cdot \nabla F = (v(F_1), \ldots, v(F_n))$

  defines the "arrows" $v(F)(x) \in DF(T_x \mathcal{M})$

Derivations $\Leftrightarrow$ operators $\Leftrightarrow$ Represent in basis as a "matrix"

$$v(f) = v\left(\sum_j \hat{f}_j \phi_j\right) = \sum_j \hat{f}_j v(\phi_j)$$

$$\langle v(f), \phi_i \rangle = \sum_j \hat{f}_j \underbrace{\langle v(\phi_j), \phi_i \rangle}_{V_{ij}} = \sum_j v_{ij} \hat{f}_j$$

So $v_{ij} = \langle \phi_i, v(\phi_j) \rangle$ encodes vector field v as a matrix.

Vector Fields as linear combinations: $v = \sum_{ij} \underset{\substack{\uparrow \\ \text{scalar coeff.}}}{v^{ij}} \overset{\substack{\text{simple vector fields} \\ \swarrow}}{b_{ij}}$

$$v_{lk} = \sum_{ij} v^{ij} (b_{ij})_{lk}$$

$$= \sum_{ij} v^{ij} \langle \phi_l, b_{ij}(\phi_k) \rangle = \sum_{ij} v^{ij} \langle \phi_l, \phi_i \nabla\phi_j (\phi_k) \rangle$$

$$= \sum_{ij} v^{ij} \langle \phi_l \phi_i, \nabla\phi_j \cdot \nabla\phi_k \rangle_{L^2} = \sum v^{ij} \underbrace{\langle \phi_i \nabla\phi_j, \phi_l \nabla\phi_k \rangle_{L^2_X}}_{G_{ijlk}}$$