Optimal bases for data-driven estimation of forecast operators

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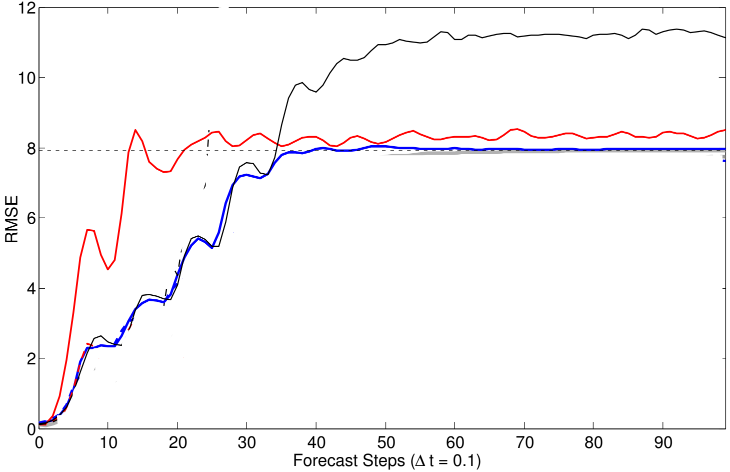
Types of Forecasting: Deterministic

- Deterministic Forecasting, $x_{k+1} = F(x_k)$
- Regression problem: Learn F from data
- Iterative Methods: $x_{k+n} = \tilde{F}^n(x_k)$ where $\tilde{F} \approx F$
- Direct Methods: $x_{k+n} = \tilde{F}_n(x_k)$ where $\tilde{F}_n \approx F^n$

Direct vs. Iterative vs Probabilistic



Regression Comparison

- Local Linear Regression (x_j near x):

$$F(x) \approx F(x_j) + DF(x_j)(x - x_j)$$

- Kernel Regression (h is bump function):

$$F(x) \approx \sum_j c_j h(||x - x_j||_{A_j})$$

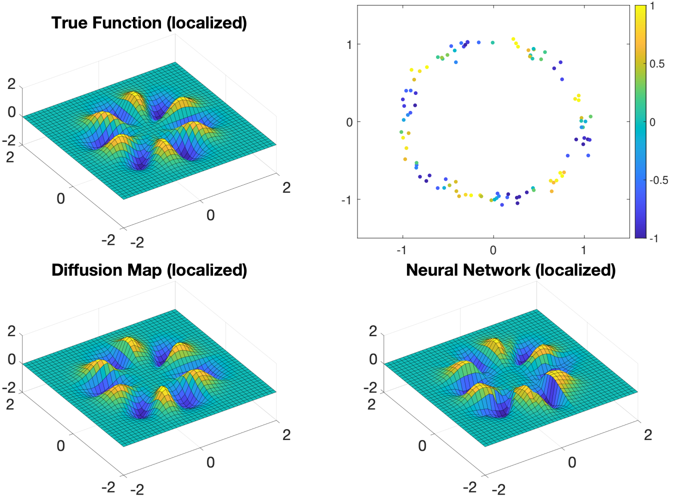
- Neural Network (h is sigmoid):

$$F(x) \approx \sum_j c_j h(a_j^\top (x - \tilde{x}_j))$$

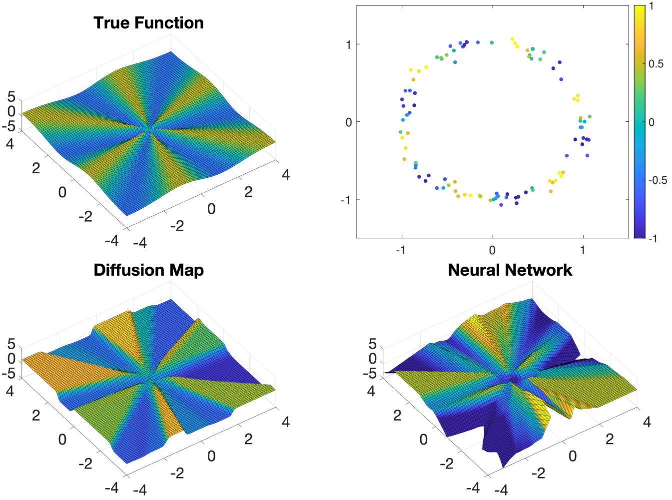
(where we write $b_j = a_j^\top \tilde{x}_j$)

- Deep Network: Composition of Neural Networks
- Reservoir Computer: Fix a_j, b_j, linear regression for c_j

NYSTRÖM VS. DEEP NET, $(r, \theta) \mapsto sin(6\theta)$



NYSTRÖM VS. DEEP NET, $(r,\theta) \mapsto \sin(6\theta)$



Types of Forecasting: UQ

- Deterministic Forecasting, $x_{k+1} = F(x_k), x_0 \sim p_0$
- Uncertainty Quantification, $p_{k+1} = \mathcal{F}(p_k) = p_k \circ F$
- *Can be* considered a regression problem
- *Option 1:* Learn F, then apply UQ (MC, PC, etc.)
- *Option 2:* Learn $\mathcal{F}$ directly in a basis

$$A_{ij} = \langle \phi_i, \mathcal{F}\phi_j \rangle = \langle \phi_i, \phi_j \circ F \rangle \approx \frac{1}{N} \sum_{k=1}^{N} \phi_i(x_k)\phi_j(x_{k+1})$$

Types of Forecasting: Stochastic

- Stochastic Forecasting, $x_{k+1} = F(x_k, \omega_k)$
- *Not* a regression problem
- Don't just want $\bar{F}(\cdot) = \mathbb{E}_\omega[F(\cdot,\omega)]$
- We want the forward operator

$$p_{k+1} = \mathcal{F}(p_k) = \int p_k \circ F(\cdot,\omega)\, d\pi(\omega)$$

- Note: $\int p_k \circ F(\cdot,\omega)\, d\pi(\omega) \neq p_k \circ \int F(\cdot,\omega)\, d\pi(\omega)$

STOCHASTIC FORECASTING = OPERATOR ESTIMATION

- Represent $\mathcal{F}$ in a basis

$$A_{ij} = \langle \phi_i, \mathcal{F}\phi_j \rangle = \langle \phi_i, \phi_j \circ F \rangle \approx \frac{1}{N}\sum_{k=1}^{N} \phi_i(x_k)\phi_j(x_{k+1})$$

- Error Sources: Bias, variance, and truncation
- Which basis?
  - Respect the measure $\Rightarrow$ Eliminate bias
  - Leverage smoothness $\Rightarrow$ Minimize variance
  - Capture global structure $\Rightarrow$ Minimize truncation

WHAT IS MANIFOLD LEARNING?

- **Manifold learning** $\Leftrightarrow$ **Estimating Laplace-Beltrami**
- Eigenfunctions $\Delta\varphi_i = \lambda_i\varphi_i$ orthonormal basis for $L^2(\mathcal{M})$
- Smoothest functions: φ_i minimizes the functional

$$\lambda_i = \min_{\substack{f \perp \varphi_k \\ k=1,\ldots,i-1}} \left\{ \frac{\int_{\mathcal{M}} ||\nabla f||^2 \, dV}{\int_{\mathcal{M}} |f|^2 \, dV} \right\}$$

- Eigenfunctions of Δ are custom Fourier basis
  - Smoothest orthonormal basis for $L^2(\mathcal{M})$
  - Can be used to define wavelets
  - Define the Hilbert/Sobolev spaces on $\mathcal{M}$

CONFORMALLY INVARIANT DIFFUSION MAPS (CIDM)

- Data samples $\{x_i\}_{i=1}^N \subset \mathcal{M} \subset \mathbb{R}^n$ of volume $p_{\mathrm{eq}}\, dV$
- Continuous k-Nearest Neighbors (CkNN) dissimilarity:

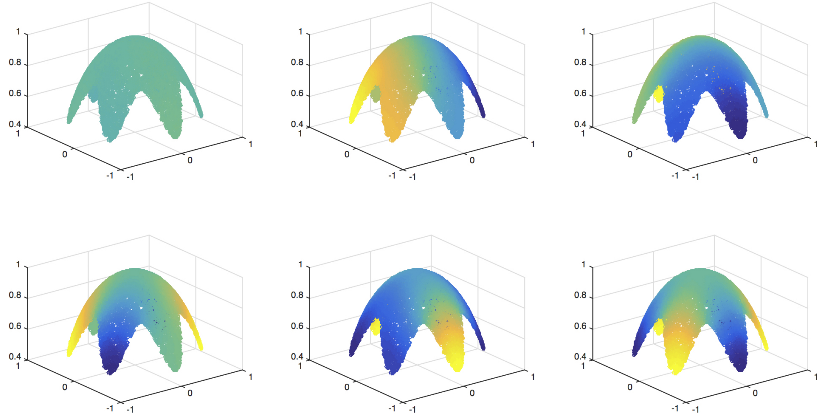
$$d(x_i, x_j) \equiv \frac{||x_i - x_j||}{\sqrt{||x_i - x_{kNN(i)}||\,||x_j - x_{kNN(j)}||}}$$

- Variable bandwidth kernel, $K_{ij} = \exp\left(\frac{-d(x_i,x_j)^2}{\delta^2}\right)$
- Degree matrix $D_{ii} = \sum_j K_{ij}$ (diagonal)
- Graph Laplacian, $L = \frac{D-K}{\delta^{d+2}}$
- **Theorem:** $L\vec{f} = \Delta_{\hat{g}} f + \mathcal{O}\left(\delta^2, N^{-1/2}\delta^{-1-d/2}\right),\ \hat{g} = p_{\mathrm{eq}}^{2/d} g$
- **Solve:** $(I - D^{-1/2}KD^{-1/2})\vec{v} = \lambda\vec{v}$, set $\vec{\varphi} = D^{-1/2}\vec{v}$

Harmonic Analysis on Manifolds/Data Sets

- Manifolds with boundary, (R. Vaughn)

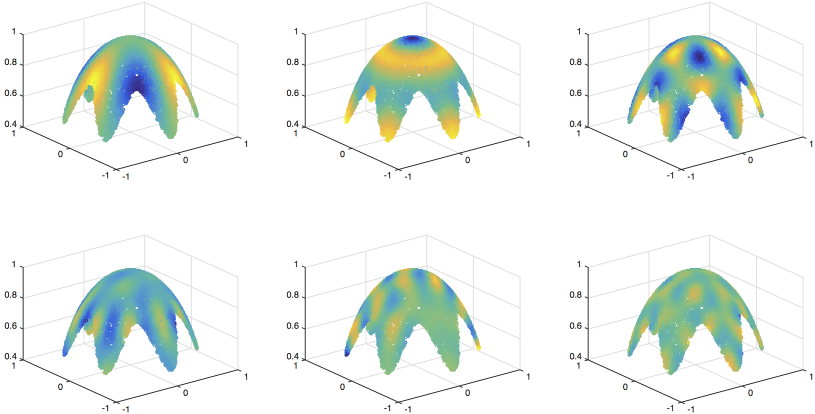
$$\vec{h}^\top L\vec{f} \to \int (\nabla h \cdot \nabla f)\, p_{\mathrm{eq}} dV$$



HARMONIC ANALYSIS ON MANIFOLDS/DATA SETS

- Manifolds with boundary, (R. Vaughn)

$$\vec{h}^\top L\vec{f} \to \langle\langle \nabla_{\hat{g}} h, \nabla_{\hat{g}} f \rangle\rangle_{\hat{g}} = \int \hat{g}(\nabla_{\hat{g}} h, \nabla_{\hat{g}} f)\, dV_{\hat{g}}$$



FORECASTING THE FOKKER-PLANK PDE

- Dynamical system: $dx = a(x)\,dt + b(x)\,dW_t$
- Uncertain initial state $x(0)$ with density $p(x, 0)$
- Density solves Fokker-Planck PDE, $p_t = \mathcal{L}^* p$ where

$$\mathcal{L}^* p = -\nabla \circ (pa) + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(p \sum_k b_{ik} b_{jk} \right)$$

- Semigroup solution, $p(x, t) = e^{t\mathcal{L}^*} p(x, 0)$

THE SHIFT MAP

- Given data samples $x_i = x(t_i)$ with $\tau = t_{i+1} - t_i$
- Define the *shift map* of a function by $Sf(x_i) = f(x_{i+1})$
- Using the Itô lemma we can show:

$$Sf(x_i) = f(x_{i+1}) = e^{\tau\mathcal{L}} f(x_i) + \int_{t_i}^{t_{i+1}} \nabla f^\top b \, dW_s + \int_{t_i}^{t_{i+1}} Bf \, ds$$

- Notice: $\mathbb{E}[S(f)] = e^{\tau\mathcal{L}} f$
- Need to minimize the stochastic integrand $\nabla f^\top b$

Forecasting with the Shift Map

$$
\begin{array}{ccc}
p(x,t) & \xrightarrow{\text{Diffusion Forecast}} & p(x,t+\tau) \\
\Big\downarrow \langle p,\varphi_j\rangle & & \Big\uparrow \sum_j c_j\varphi_j p_{\mathrm{eq}} \\
\vec{c}(t) & \xrightarrow{A_{lj}\equiv\mathbb{E}[\langle\varphi_j,S\varphi_l\rangle_{p_{\mathrm{eq}}}]} & \vec{c}(t+\tau)=A\vec{c}(t).
\end{array}
$$

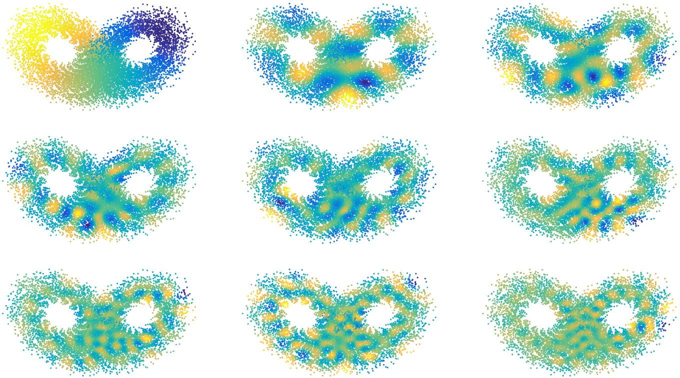
- Estimate A_{lj} with $\hat{A}_{lj}=\frac{1}{N}\sum_{i=1}^{N}\varphi_j(x_i)\varphi_l(x_{i+1})$
- $\mathbb{E}[\hat{A}_{lj}]=A_{lj}$ with error $\mathcal{O}(||\nabla\varphi_l||_{p_{\mathrm{eq}}}\sqrt{\tau/N})$

CHOOSING A BASIS

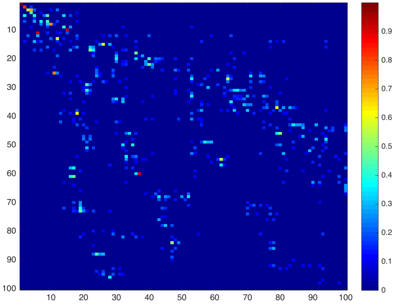
- Need to minimize the error term $\mathcal{O}(||\nabla\varphi_l||_{p_{\mathrm{eq}}}\sqrt{\tau/N})$
- The eigenfunctions $\Delta_{\hat{g}}\varphi_j = \lambda_j\varphi_j$ minimize $||\nabla\varphi_j||_{p_{\mathrm{eq}}} = \lambda_j$
- Find φ_j with Manifold Learning: CIDM

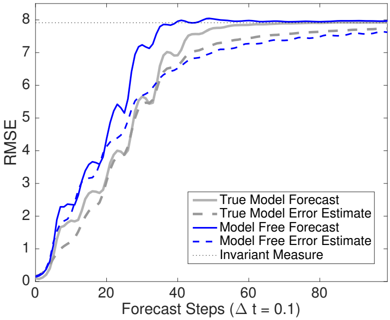
MANIFOLD LEARNING ⇒ CUSTOM ‘FOURIER’ BASIS

- **Optimal basis:** Minimum variance $A_{lj} \equiv \mathbb{E}[\langle \varphi_j, S\varphi_l \rangle_{p_{\mathrm{eq}}}]$



Shift map ⇒ Markov matrix





DIFFUSION FORECAST EXAMPLE

(Loading Video...)

RELATIONSHIP TO CLASSICAL METHODS

- For partial observations, use Takens' reconstruction
- Local linear representations
  - Nearest neighbor interpolation
  - Diffusion forecast extends the map to distributions
- Partition state space $\Rightarrow$ Markov matrix
  - Also uses the shift map, just a different basis
  - Diffusion forecast is optimal basis for estimation

RELATIONSHIP TO RESERVOIR COMPUTERS

- Create a random (recurrent) network $v_k \in \mathbb{R}^N$

$$v_{k+1} = f(Av_k + Bx_k)$$

- Continuously feed in the time series x_k

$$\begin{aligned} v_{k+1} &= f(Af(Av_{k-1} + Bx_{k-1}) + Bx_k) = \cdots \\ &= f(Af(A \cdots f(Av_{k-\tau} + Bx_{k-\tau}) + \cdots) + Bx_k) \\ &= g(x_k, x_{k-1}, ..., x_{k-\tau}) \end{aligned}$$

- Predict: $x_{k+1} = Wv_k = Wg(x_k, ..., x_{k-\tau})$
- Since $\lambda_{\max}(A) < 1$ network forgets distant past
- Effectively a random diffeomorphism of a delay embedding
- Effectively uses a linear combination W of random basis!

PROBLEM: CURSE OF DIMENSIONALITY

- Nonparametric methods $\rightarrow$ Data required grows like a^{dim}

Projections of High Dimensional Dynamics

- Consider the 40-dimensional Lorenz-96 system:

$$\dot{x}_i = x_{i-1}x_{i+1} - x_{i-1}x_{i-2} - x_i + 8$$

- Assume we only observe a projection of this system

$$y = h(x_1, ..., x_{40})$$

- Example: Spatial Fourier mode $y = \hat{x}_\omega = \sum_{k=1}^{40} x_i e^{-k\omega}$
- Evolution of y is not closed, sometimes modeled by SDEs

Attractor Reconstruction

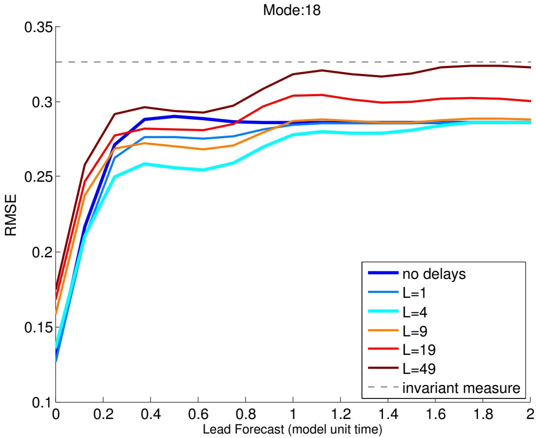
- Evolution of $y = h(x)$ is not closed (missing information)
- Idea: Use delay-embedding to recover the missing info
- Problem 1: Delay embeddings are biased towards stable directions

$$\tilde{y}_t \equiv (y_t, y_{t-\tau}, ..., y_{t-L\tau}) = (h(x_t), h(F_{-\tau}(x_t), ..., h(F_{-L\tau}(x_t))$$

- Problem 2: Curse-of-dimensionality prevents learning the full attractor
- Adding some delays helps, but adding too many hurts

Attractor Reconstruction

- Evolution of $y = h(x)$ is not closed
- Adding some delays helps, but adding too many hurts



Next steps: Mori-Zwanzig formalism

- Evolution of $y = h(x)$ is not closed
- Delay-embedding, $\tilde{y}_t$ only yeilds partial reconstruction
- Projections of dynamical systems can be closed as

$$\text{Mori-Zwanzig formalism:} \qquad \frac{d}{dt}\tilde{y} = V + K + R$$

- Diffusion Forecast includes: V (Markovian), R (stochastic)
- Missing the memory term: $K = \int_{-\infty}^{t} K(s, \tilde{y}_t, \tilde{y}_s)\tilde{y}_s \, ds$

What about the other Riemannian structure?

- A Riemannian manifold has an exterior calculus:
  - Calculus of tensors and differential forms
  - Built entirely from the Riemannian metric $g \Leftrightarrow \Delta$
  - Formulates the generalization of the FTC (Stokes' Thm)
  - Can construct Laplacians on k-forms, Δ_k
  - Eigenforms of Δ_k are smoothest basis for k-forms
- **Question:** Given only the eigenfunctions of the Laplacian how can we construct the rest of the exterior calculus?

What about the other Riemannian structure?

- **Good News:** Laplacian $\Leftrightarrow$ Riemannian metric

$$g(\nabla f, \nabla h) = \nabla f \cdot \nabla h = \frac{1}{2}(f\Delta h + h\Delta f - \Delta(fh))$$

- Let $v, w \in T_x\mathcal{M}$, there exists $f_1, ..., f_d$ such that $\nabla f_1, ..., \nabla f_d$ span $T_x\mathcal{M}$ and

$$g(v, w) = v \cdot w = \sum_{ij} v_i w_j \nabla f_i \cdot \nabla f_j$$

- **Bad News:** There may be no $f_1, ..., f_d$ that work for all x

- Hairy Ball Thm: Every smooth vector field on S^2 must vanish: at these points the gradients do not span $T_x\mathcal{M}$.

How can we use the Laplacian eigenfunctions?

- Cannot find $\nabla f_1, ..., \nabla f_d$ basis for all $T_x\mathcal{M}$
- **Whitney:** We can find $\nabla f_1, ..., \nabla f_{2d}$ span all $T_x\mathcal{M}$
- **Thm[1]:** $\exists J$ such that $\nabla\varphi_1, ..., \nabla\varphi_J$ span all $T_x\mathcal{M}$
- Representing vector fields in a frame (overcomplete set)
  - Let $v(x) \in T_x\mathcal{M}$ be a smooth vector field
  - Then $v(x) = \sum_{j=1}^{J} c_j(x)\nabla\varphi_j(x)$ where $c_j(x)$ are smooth
  - So $c_j(x) = \sum_{i=1}^{\infty} c_{ij}\varphi_i(x)$
  - Finally $v = \sum_{i,j} c_{ij}\varphi_i\nabla\varphi_j$ (not uniquely)

[1] J. Portegies, Embeddings of Riemannian Manifolds with Heat Kernels and Eigenfunctions. (2014).

How can we use the Laplacian eigenfunctions?

- **Thm (Berry & Giannakis)** Let φ_i be the eigenfunctions of the Laplacian then $\{\varphi_i \nabla \varphi_j \,:\, j = 1, ..., J, i = 1, ..., \infty\}$ is a **frame** for the L^2 space of vector fields on $\mathcal{M}$.
- A **frame** is an overcomplete spanning set commonly used in Harmonic analysis, must satisfy the frame inequalities:

$$A||v||^2 \leq \sum_{i,j} \left\langle v, \varphi_i \nabla \varphi_j \right\rangle^2 \leq B||v||^2$$

where $A, B > 0$ and $|| \cdot ||^2 = \langle \cdot, \cdot \rangle$ is the Hodge inner prod.

The Spectral Exterior Calculus (SEC)

- We extend Thm to frames for Sobolev spaces of tensors
- SEC formulates the entire exterior calculus in these frames
- Key accomplishment: Representation of the 1-Laplacian

$$\Delta_1 = d\delta + \delta d$$

- Key challenge: Frame representations are not unique, requires Sobolev regularizations for numerical stability

A CALCULUS NEEDS FORMULAS!

Object	Symbolic	Spectral
Function	f	$\hat{f}_k = \langle \phi_k, f \rangle_{L^2}$
Laplacian	Δf	$\langle \phi_k, \Delta f \rangle_{L^2} = \lambda_k \hat{f}_k$
L^2 Inner Product	$\langle f, h \rangle_{L^2}$	$\sum_i \hat{f}_i^* \hat{h}_i$
Dirichlet Energy	$\langle f, \Delta f \rangle_{L^2}$	$\sum_i \lambda_i \|\hat{f}_i\|^2$
Multiplication	$\phi_i \phi_j$	$c_{ijk} = \left\langle \phi_i \phi_j, \phi_k \right\rangle_{L^2}$
Function Product	fh	$\sum_{ij} c_{kij} \hat{f}_i \hat{h}_j$
Riemannian Metric	$\nabla \phi_i \cdot \nabla \phi_j$	$g_{kij} \equiv \left\langle \nabla \phi_i \cdot \nabla \phi_j, \phi_k \right\rangle_{L^2}$ $= \frac{1}{2}(\lambda_i + \lambda_j - \lambda_k) c_{kij}$
Gradient Field	$\nabla f(h) = \nabla f^* \cdot \nabla h$	$\langle \phi_k, \nabla f(h) \rangle_{L^2} = \sum_{ij} g_{kij} \hat{f}_i \hat{h}_j$
Exterior Derivative	$df(\nabla h) = df^* \cdot dh$	$\sum_{ij} g_{kij} \hat{f}_i \hat{h}_j$
Vector Field (basis)	$v(f) = v^* \cdot \nabla f$	$\sum_j v_{ij} \hat{f}_j$
Divergence	$\text{div} v$	$\langle \phi_i, \text{div} v \rangle_{L^2} = -v_{0i}$
Frame Elements	$b_{ij}(\phi_l) = \phi_i \nabla \phi_j(\phi_l)$	$G_{ijkl} \equiv \left\langle b_{ij}(\phi_l), \phi_k \right\rangle_{L^2} = \sum_m c_{mik} g_{mjl}$
Vector Field (frame)	$v(f) = \sum_{ij} v^{ij} b_{ij}(f)$	$\langle \phi_k, v(f) \rangle_{L^2} = \sum_{ijl} G_{ijkl} v^{ij} \hat{f}_l$
Frame Elements	$b^{ij}(v) = b^i \, db^j(v)$	$\left\langle \phi_k, b^{ij}(v) \right\rangle_{L^2} = \sum_{nlm} c_{kmi} G_{nlmj} v^{nl}$
1-Forms (frame)	$\omega = \sum_{ij} \omega_{ij} b^{ij}$	$\langle \phi_k, \omega(v) \rangle_{L^2} = \sum_{ij} \omega_{ij} \left\langle \phi_k, b^{ij}(v) \right\rangle_{L^2}$

Operator	Tensor	Symmetries
Quadruple Product	$c^0_{ijkl} = \left\langle \phi_i \phi_j, \phi_k \phi_l \right\rangle_{L^2} = \sum_s c_{ijs} c_{skl}$	Fully symmetric
Product Energy	$c^p_{ijkl} = \left\langle \Delta^p(\phi_i \phi_j), \phi_k \phi_l \right\rangle_{L^2} = \sum_s \lambda^p_s c_{ijs} c_{skl}$	(1,2), (3,4), (1,3), (2,4)
Hodge Grammian	$G_{ijkl} = \left\langle b^{ij}, b^{kl} \right\rangle_{L^2_1} = \frac{1}{2}\left[(\lambda_j + \lambda_l) c^0_{ijkl} - c^1_{ijkl}\right]$	(1,3), (2,4)
Antisymmetric	$\hat{G}_{ijkl} = \langle \hat{b}^{ij}, \hat{b}^{kl} \rangle_{L^2_1} = G_{ijkl} + G_{jilk} - G_{jikl} - G_{ijlk}$	(1,3), (2,4)
Dirichlet Energy $E_{ijkl} = \left\langle b^{ij}, \Delta_1 (b^{kl}) \right\rangle_{L^2_1}$	$E_{ijkl} = \frac{1}{4}\left[(\lambda_i + \lambda_j + \lambda_k + \lambda_l)(c^1_{iljk} - c^1_{ikjl})\right.$ $\left. + (\lambda_j + \lambda_l - \lambda_i - \lambda_k) c^1_{ijkl} + (c^2_{ijkl} + c^2_{ikjl} - c^2_{iljk})\right]$	(1,3), (2,4)
Antisymmetric	$\hat{E}_{ijkl} = \langle \hat{b}^{ij}, \Delta_1 \hat{b}^{kl} \rangle_{L^2_1}$ $= (\lambda_i + \lambda_j + \lambda_k + \lambda_l)(c^1_{iljk} - c^1_{ikjl}) + (c^2_{ikjl} - c^2_{iljk})$	(1,3), (2,4)
Sobolev H^1 Grammian	$G^1_{ijkl} = E_{ijkl} + G_{ijkl}, \quad \hat{G}^1_{ijkl} = \hat{E}_{ijkl} + \hat{G}_{ijkl}$	(1,3), (2,4)

Object	Symbolic	Spectral
Multiple Product	$c^0_I = \left\langle b^{i_0} \cdots b^{i_k}, 1 \right\rangle_H$	$c^0_I = \sum_s c_{i_0 i_1 s} c^0_{s i_2 \cdots i_k}$
Tensor Evaluation	$H^{\tilde{I}\tilde{J}} = (db^{i_1} \cdot db^{j_1}) \cdots (db^{i_k} \cdot db^{j_k})$ $= \nabla b^{i_1} \otimes \cdots \otimes \nabla b^{i_k} (b^{j_1}, \ldots, b^{j_k})$	$\hat{H}^{\tilde{I}\tilde{J}}_l \equiv \langle H^{\tilde{I}\tilde{J}}, b^l \rangle_H$ $= \sum_{n=1}^{k^2} \prod_{s,r=1}^{k} g_{i_s j_r m_n} c_{l m_1 \cdots m_{k^2}}$
Tensor Product	$b_J = b^{j_0} \nabla b^{j_1} \otimes \cdots \otimes \nabla b^{j_k}$	$\left\langle b_J(b^{i_1}, \ldots, b^{i_k}), b^l \right\rangle = \sum_s \hat{H}^{\tilde{J}\tilde{I}}_s c_{s j_0 l}$
Frame Elements	$b^I = b^{i_0} db^{i_1} \wedge \cdots \wedge db^{i_k}$	$\left\langle b^I(b_J), b^l \right\rangle_H = \left\langle b^I \cdot b^J, b^l \right\rangle_H$
Riemannian Metric	$b^I \cdot b^J = b^{i_0} b^{j_0} \det([db^{i_q} \cdot db^{j_n}])$	$\left\langle b^I \cdot b^J, b_l \right\rangle_H = \sum_s \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) c_{s i_0 j_0 l} \hat{H}^{\tilde{I}\sigma(\tilde{J})}_s$
Hodge Grammian	$G_{IJ} = \left\langle b^I, b^J \right\rangle_{H_k} = \left\langle b^I \cdot b^J, 1 \right\rangle_H$	$\sum_s \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) c_{s i_0 j_0} \hat{H}^{\tilde{I}\sigma(\tilde{J})}_s$
d-Energy	$E^d_{IJ} = \left\langle db^I, db^J \right\rangle_{H_{k+1}}$	$\left\langle db^I \cdot db^J, 1 \right\rangle_{H_{k+1}} = \hat{H}^{IJ}_0$

Back to Basis

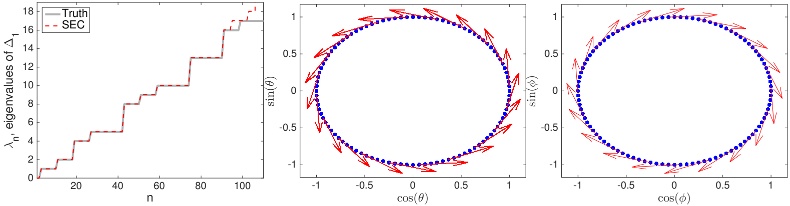
- We need the frame representation to build the 1-Laplacian

$$\Delta_1 = d\delta + \delta d$$

- Once we have Δ_1, the eigenfields form the smoothest possible basis for vector fields
- Can use to smooth vector fields and represent operators

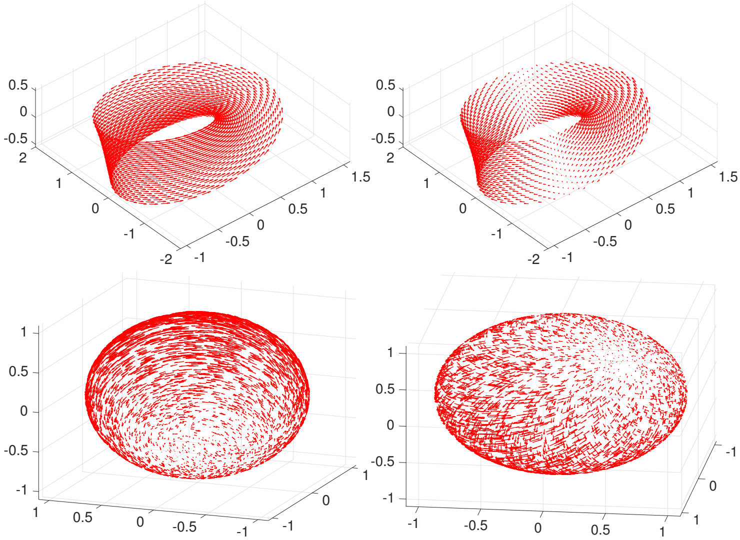
Numerical Verification on Flat Torus

Captures the true spectrum of the Hodge Laplacian.

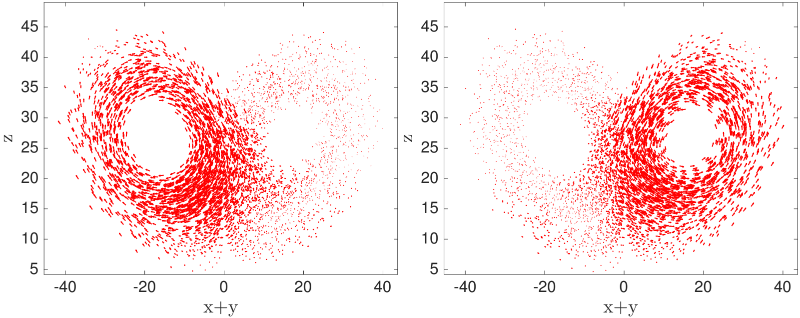


Harmonic forms correspond to unique homology classes.

Smoothest vector fields on the manifold



SEC IS APPLICABLE TO ANY DATA SET



Matlab Code: http://math.gmu.edu/˜berry/

Applying the SEC to Dynamical Systems

- Smooth/Denoise vector fields using SEC basis
- Compute Lyapunov vector fields in the SEC basis
- Next Step: Hodge decomposition

$$v = \nabla U + \delta A + v^{\perp}$$

- U is a potential, A is a tensor field, and $\Delta_1 v^{\perp} = 0$

DECOMPOSING SDE COMPONENTS

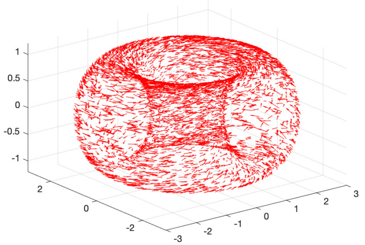
- Given a realization of an SDE on a manifold:

$$dx = f(x)\,dt + B(x)\,dW_t$$

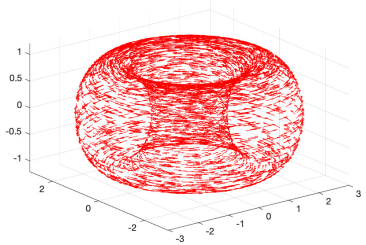
- Want to extract the deterministic component, $f(x)$
- Finite differences $x(t+\tau) - x(t) \approx f(x(t))$ but noisy
- Can smooth component functions using DM basis
- Better to smooth with SEC eigenvectorfields

Decomposing SDE Components

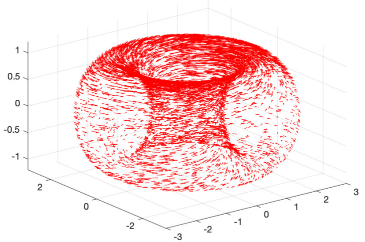
Finite Difference Est.



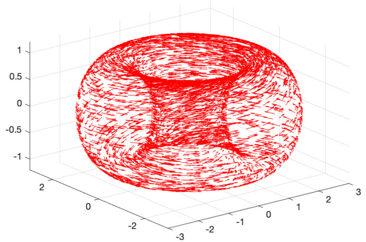
True Vector Field



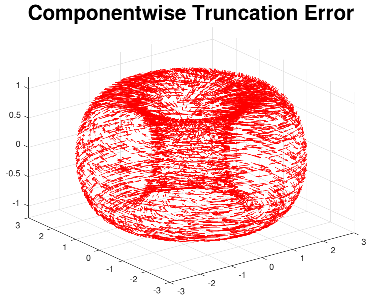
Componentwise Truncation

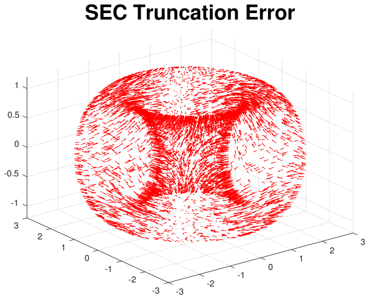


SEC Truncation



DECOMPOSING SDE COMPONENTS





Code and papers available at:

http://math.gmu.edu/˜berry/

Building the basis

- B. and Giannakis, *Spectral Exterior Calclulus.*
- B. and Sauer, *Consistent Manifold Representation for Topological Data Analysis.*
- Coifman and Lafon, *Diffusion maps.*
- B. and Harlim, *Variable Bandwidth Diffusion Kernels.*
- B. and Sauer, *Local Kernels and Geometric Structure of Data.*

Diffusion forecast

- B., Giannakis, and Harlim, *Nonparametric forecasting of low-dimensional dynamical systems.*
- B. and Harlim, *Forecasting Turbulent Modes with Nonparametric Diffusion Models.*