Data-Driven Correction of Model Error for Forecasting

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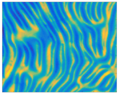
Joint work with John Harlim, PSU and Dimitris Giannakis, NYU

PARAMETRIC MODELING

Data



⇒ ⇐

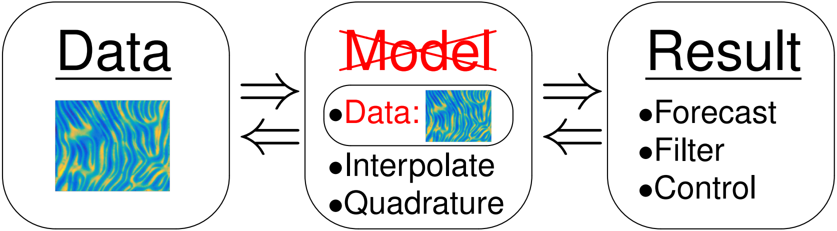
Model
- Equations
- Variables
- Parameters

⇒ ⇐

Result
- Forecast
- Filter
- Control

- **Design Model:** Limited resolution and complexity
- **Assimilate Data:** Fit Parameters/Variables
  - Kalman Filter, EKF, EnKF, Variational methods
  - Observability and noise
  - Model error
- **Study/Apply:** Ensemble Forecast

NONPARAMETRIC MODELING



- **Data IS the model:**
  - Assume a model exists
    - Data lies on/near an unknown sub-manifold
    - Data obeys an unknown dynamical system
  - Represent the model using training data

ROADMAP: CORRECTING MODEL ERROR

- What is manifold learning? $\Rightarrow$ Custom Fourier Basis
- Nonparametric methods (no model)
  - Diffusion Forecast
  - Diffusion Filter
- Semiparametric methods (model error)

MANIFOLD LEARNING

- **Manifold learning $\Leftrightarrow$ Estimating Laplace-Beltrami**
- Laplacian Eigenmaps, Diffusion Maps,
  Variable Bandwidth Diffusion Kernels, Local Kernels
- Provably estimate L-B and eigenfunctions from data
- Eigenfunctions $\Delta\varphi_i = \lambda_i\varphi_i$ orthonormal basis for $L^2(\mathcal{M}, g)$
- Smoothest functions: φ_i minimizes the functional

$$\lambda_i = \min_{f \perp \varphi_k, k<i} \left\{ \frac{\int_{\mathcal{M}} ||\nabla f||^2 \, dV}{\int_{\mathcal{M}} |f|^2 \, dV} \right\}$$

- Eigenfunctions of Δ define Sobolev and wavelets on $\mathcal{M}$

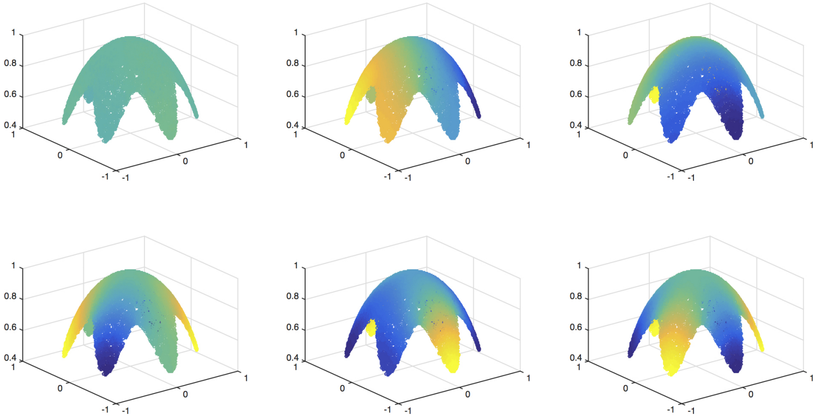
DISCRETE ANALOGS OF CONTINUOUS OBJECTS

Continuous, $\mathcal{M}$	Discrete, $\{x_i\}_{i=1}^N$
$L^2(\mathcal{M}, q)$	$\mathbb{R}^N$
Functions, $f : \mathcal{M} \to \mathbb{R}$	Vectors, $\vec{f}_i = f(x_i)$
'Basis', δ_x	Basis, $\vec{e}_i = \delta_{x_i}$
Laplace-Beltrami, Δ	Normalized Graph Laplacian, **L**
Eigenfunctions, $\Delta\varphi_j = \lambda_j\varphi_j$	Eigenvectors, $\mathbf{L}\vec{\varphi}_j = \lambda_j\vec{\varphi}_j$
Inner product, $\langle f, h\rangle_{L^2}$	Dot Product, $\frac{1}{N}\vec{f} \cdot \vec{h}$

$$\frac{1}{N}\vec{f} \cdot \vec{h} = \frac{1}{N}\sum_{i=1}^{N} f(x_i)h(x_i) \to_{N\to\infty} \int_{\mathcal{M}} f(x)h(x)\, dV(x)$$

Harmonic Analysis on Manifolds/Data Sets

- Unit circle: $\Delta = \frac{d^2}{d\theta^2}$ eigenfunctions are Fourier basis
- General manifold or data set $\Rightarrow$ Custom Fourier basis



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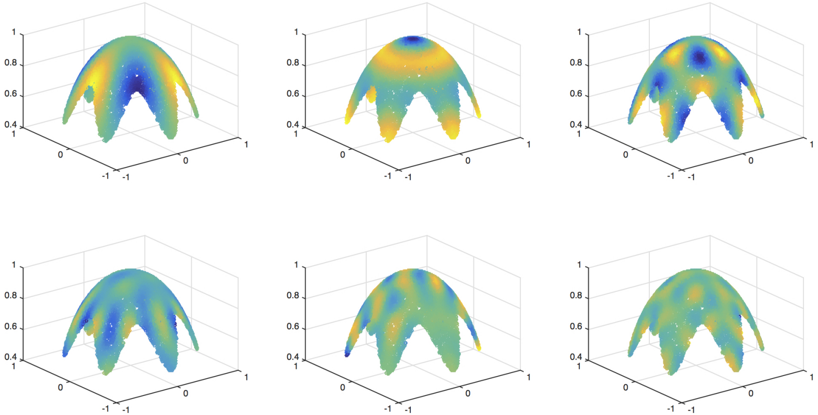


DIFFUSION FORECAST

- Autonomous SDE: $dx = a(x)\,dt + b(x)\,dW_t$
- Density solves Fokker-Planck PDE: $\frac{\partial}{\partial t}p = \mathcal{L}^* p$
- Shift map: $S(p)(x_i) = p(x_{i+1})$
- Estimates: $\mathbb{E}[S(p)] = e^{\tau\mathcal{L}}p$
- Project onto custom Fourier basis (spectral method)

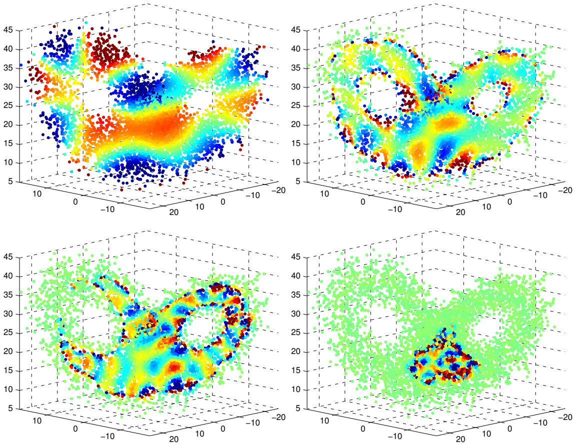
$$p(x,t) \xrightarrow{\text{Diffusion Forecast}} p(x,t+\tau) = e^{\tau\mathcal{L}^*}p(x,t)$$

$$\downarrow \langle p, \varphi_j \rangle \qquad\qquad \uparrow \textstyle\sum_j c_j \varphi_j q$$

$$\vec{c}(t) \xrightarrow{A_{lj} \equiv \mathbb{E}[\langle \varphi_j, S\varphi_l \rangle_q]} \vec{c}(t+\tau) = A\vec{c}(t).$$

MANIFOLD LEARNING ⇒ CUSTOM 'FOURIER' BASIS

- **Optimal basis:** Minimum variance $A_{lj} \equiv \mathbb{E}[\langle \varphi_j, S\varphi_l \rangle_q]$



Diffusion Forecast Example

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For details: MS138, John Harlim, 4:15pm, Room: Wasatch A

Filtering with the Shift Map

Introduce an observable $y = h(x) + \nu$ with $\nu = y - h(x) \sim q$

- Likelihood is $p(y \mid x) = q(y - h(x))$
- Bayesian Posterior: $p^a(x_i) \propto p^f(x_i) q(y - h(x_i))$
- Psuedo-spectral method

$$
\begin{array}{ccccc}
p^a(x, t-\tau) & \xrightarrow{\text{Diffusion Forecast}} & p^f(x,t) & \xrightarrow{p^f(x)p(y \mid x)} & p^a(x,t) \\
\Big\downarrow \langle p^a, \varphi_j \rangle & & \Big\uparrow \sum_j c_j^f \varphi_j p_{\text{eq}} & & \langle p^a, \varphi_j \rangle \Big\downarrow \\
\vec{c}^{\,a}(t-\tau) & \xrightarrow{A_{lj} c^{\,a}(t-\tau)} & \vec{c}^{\,f}(t) & \xrightarrow{\text{Bayesian Update}} & \vec{c}^{\,a}(t)
\end{array}
$$

RECOVERING THE KALMAN FILTER FOR LINEAR SYSTEMS

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- **Linear Dynamics:** $dx = ax\,dt + b\,dW_t$
- **Linear Observation:** $dy = x\,dt + R\,dW_t$

NON-OBSERVABILITY

(Loading Video...)

- **Double Well Potential:** $dx = x(1 - x^2)\,dt + b\,dW_t$
- **Absolute Value Observation:** $dy = |x|\,dt + R\,dW_t$

Restricted Observability

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- **Double Well Potential:** $dx = x(1-x^2)\,dt + b\,dW_t$
- **Tough Observation:** $dy = (x-0.05)^2\,dt + R\,dW_t$

PROBLEM: CURSE OF DIMENSIONALITY

- Learning the basis → Data exponential in manifold dim
- Monte-Carlo type estimates $\mathcal{O}(N^{-1/2})$:
  - Coefficients:

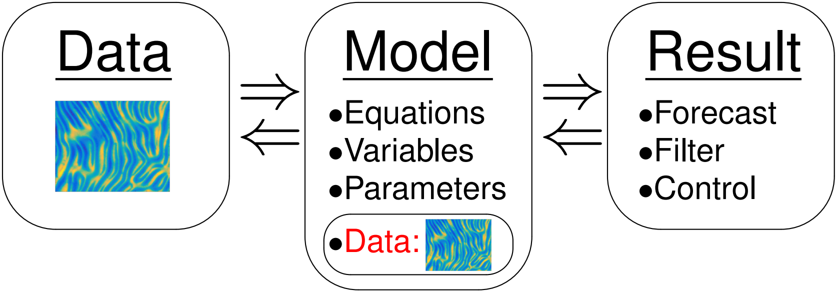
$$c_l(t) = \langle p(x,t), \varphi_l \rangle \approx \frac{1}{N} \sum_{i=1}^{N} \varphi_l(x_i) p(x_i,t)/p_{\mathrm{eq}}(x_i)$$

  - Markov Matrix:

$$A_{lj} = \left\langle \varphi_j, e^{\tau \mathcal{L}} \varphi_l \right\rangle_{p_{\mathrm{eq}}} \approx \frac{1}{N} \sum_{i=1}^{N} \varphi_j(x_i) \varphi_l(x_{i+1})$$

- Maybe we shouldn't throw out the model...
- Use diffusion forecast to fix model error!

SEMIPARAMETRIC MODELING



- **Data becomes part of the model:**
  - Start with imperfect parametric model
  - Fit training data with time-varying parameters
  - Query data as part of running model
- **Compensate for model error:**
  - Truncated resolution and complexity
  - Non-analytic expressions
  - Non-stationarity/Inhomogeneity

Semiparametric Forecast Model

- Partially known model $\dot{x} = f(x, \theta)$
- Unknown: $d\theta = a(\theta)\, dt + b(\theta)\, dW_t$
- Apply the Diffusion Forecast to $p(\theta, t)$
- Sample $\theta^k(t) \sim p(\theta, t)$ and pair with ensemble $x^k(t)$

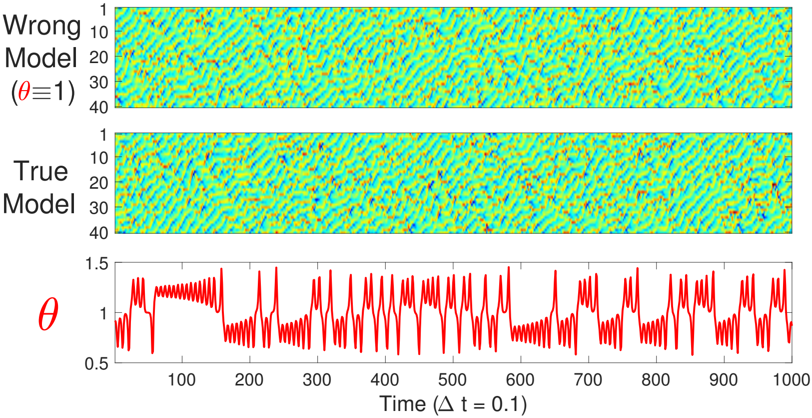
$$(x^k(t), \theta^k(t)) \xrightarrow{\dot{x}=f(x,\theta)} (x^k(t+\tau), \theta^k(t+\tau))$$

$$\uparrow \theta^k(t) \qquad\qquad \uparrow \theta^k(t+\tau)$$

$$p(\theta, t) \quad \dashrightarrow^{\text{Diffusion Forecast}} \quad p(\theta, t+\tau)$$

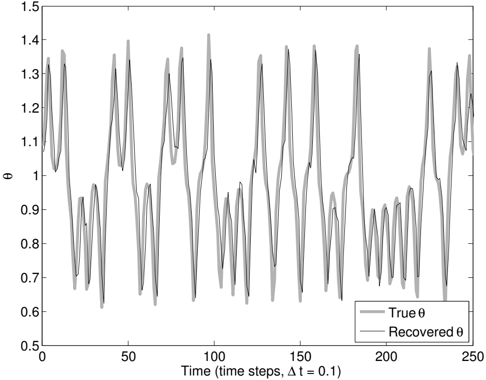
Example: 40-dimensional Lorenz-96 system

$$\dot{x}_i = \theta x_{i-1} x_{i+1} - x_{i-1} x_{i-2} - x_i + 8$$



EXAMPLE: 40-DIMENSIONAL LORENZ-96 SYSTEM

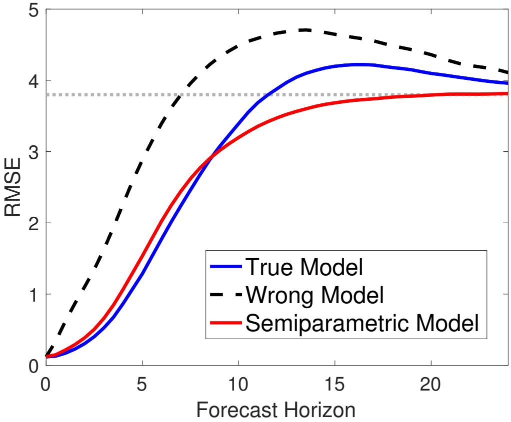
Kalman filter $\Rightarrow$ Estimate time series of θ (training period)



Using this data, build a diffusion forecast model for θ

Example: 40-dimensional Lorenz-96 system

$$\dot{x}_i = \theta x_{i-1}x_{i+1} - x_{i-1}x_{i-2} - x_i + 8$$



SEMIPARAMETRIC FILTER: PUT IT ALL TOGETHER...

$$
\begin{array}{ccccc}
\begin{pmatrix} x^{k,a}(t-\tau) \\ \theta^{k,a}(t-\tau) \end{pmatrix} & \xrightarrow{\dot{x}=f(x,\theta)} & \begin{pmatrix} x^{k,f}(t) \\ \theta^{k,f}(t) \end{pmatrix} & \xrightarrow{\text{EnKF } y^o(t)} & \begin{pmatrix} x^{k,a}(t) \\ \theta^{k,a}(t) \end{pmatrix} \\
\Big\downarrow \theta^a & & \Big\uparrow \theta^{k,f}(t) & & p(\theta^a(t)\,|\,\theta(t)) \Big\downarrow \\
p^a(\theta, t-\tau) & \overset{\text{Diffusion Forecast}}{- - - - \rightarrow} & p^f(\theta, t) & \xrightarrow{p^f(\theta)p(y\,|\,\theta)} & p^a(\theta, t) \\
\Big\downarrow \langle p^a, \varphi_j \rangle & & \Big\uparrow \sum_j c_j^f \varphi_j p_{\mathrm{eq}} & & \langle p^a, \varphi_j \rangle \Big\downarrow \\
\vec{c}^{\,a}(t-\tau) & \xrightarrow{A_{lj} c^a(t-\tau)} & \vec{c}^{\,f}(t) & \overset{\text{Bayesian Update}}{- - - - \rightarrow} & \vec{c}^{\,a}(t)
\end{array}
$$

For more information: http://math.gmu.edu/˜berry/

Building the basis

- Coifman and Lafon, *Diffusion maps.*
- B. and Harlim, *Variable Bandwidth Diffusion Kernels.*
- B. and Sauer, *Local Kernels and Geometric Structure of Data.*

Nonparametric forecast

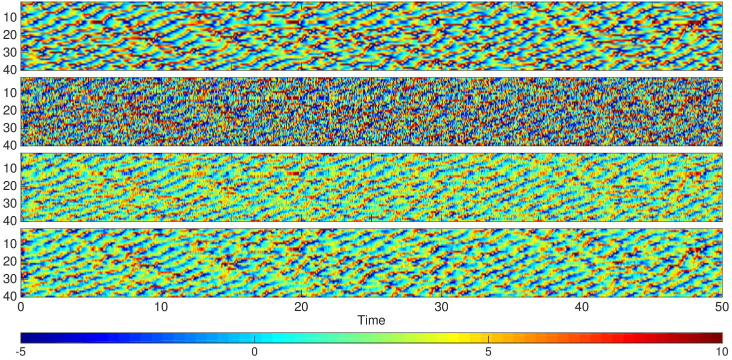
- B., Giannakis, and Harlim, *Nonparametric forecasting of low-dimensional dynamical systems.*
- B. and Harlim, *Forecasting Turbulent Modes with Nonparametric Diffusion Models.*

Semiparametric forecast

- B. and Harlim, *Semiparametric forecasting and filtering: correcting low-dimensional model error in parametric models.*

ADVERTISEMENT: KALMAN-TAKENS FILTER

Filtering Lorenz-96 with no model:



MS140 - New Developments in Attractor Reconstruction

3:45, Franz Hamilton, Room: White Pine