Learning manifolds from data

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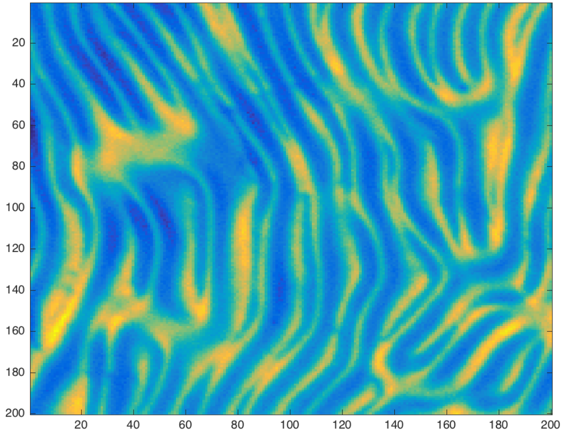
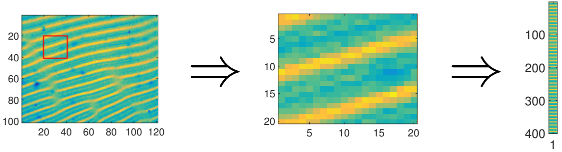
Motivating Example: Nematic Liquid Crystal



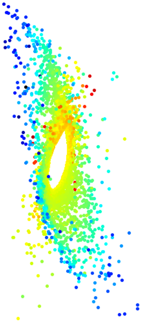
Video provided by Rob Cressman, Krasnow Institute, GMU

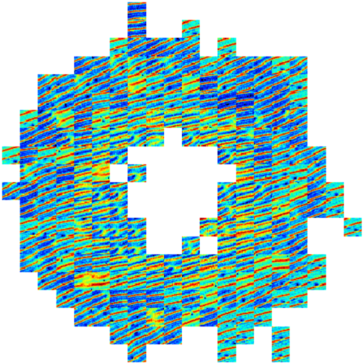
Finding Hidden Structure in Data



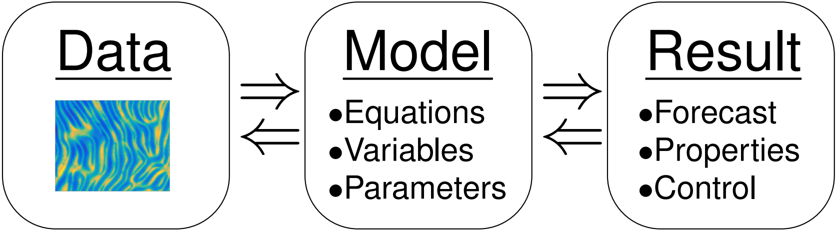
$\in \mathbb{R}^{400}$

The sub-image geometry:



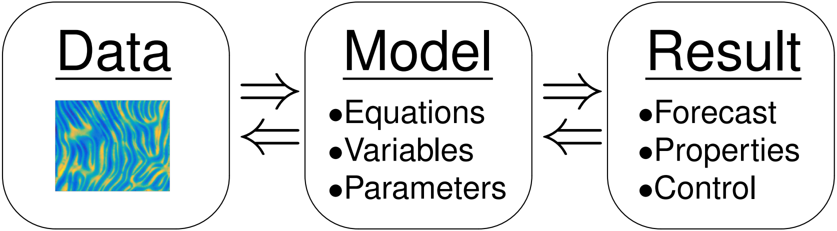


PARAMETRIC MODELING



- **Model error:**
  - Trade off resolution and complexity
  - Stationarity/Homogeneity of parameters
- **Assimilate Data:** Fit Parameters/Variables
  - Lumps together noise and model error

Parametric Modeling



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DATA ASSIMILATION

- Estimate state/parameters from noisy observations
- EnKF requires noise covariances (unknown)
- Adaptive data assimilation
  - Estimates covariances, compensates for model error
  - Extends (Mehra 1970,1972) to nonlinear systems

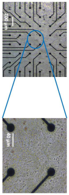
Hamilton, Berry, Peixoto, Sauer (PRE, 2013)

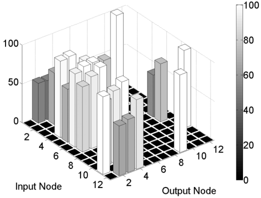
Berry & Sauer (Tellus A, 2013)

Berry & Harlim (Proc. Royal Society A, 2014)

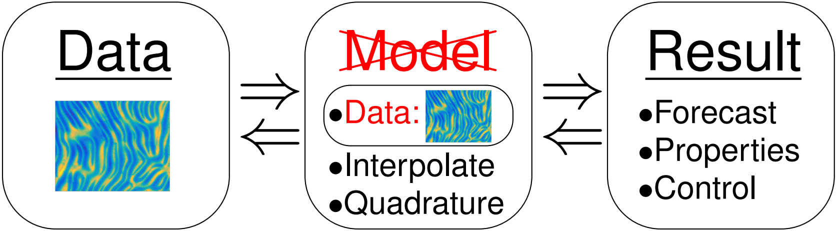
Hamilton, Berry, & Sauer (Physical Review X, 2016)

Harlim & Berry (Monthly Weather Review, in review)





NONPARAMETRIC MODELING



- **Tools:** For functions $f \in \mathcal{H}$ determined by values $\vec{f}_i = f(x_i)$
  - Interpolate: $f(x) = \sum_j \langle f, \varphi_j \rangle \varphi_j(x)$
  - Quadrature: $\langle f, \varphi_i \rangle \approx \sum_i f(x_i)\varphi(x_i)$
  - Operator Representation: $\mathbf{A}_{jk} = \langle \varphi_j, \mathcal{A}\varphi_k \rangle$
- All require a basis $\{\varphi_j\}$!

ROADMAP

- What is manifold learning? $\Rightarrow$ Estimate Laplacian, Δ
- How to find the Laplacian? $\Rightarrow$ Graph Laplacian, $\mathbf{L}$
- Convergence $\mathbf{L} \to \Delta$ and overcoming limitations
- **Key result:** Extension to non-compact manifolds
- New graph construction based on key result (TDA)
- Applications and future directions

WHAT IS MANIFOLD LEARNING?

- Geometric prior: Data on Riemannian manifold $\mathcal{M} \subset \mathbb{R}^m$
- **Goal**: Represent all the information about a manifold
- A smooth embedded manifold $\mathcal{M} \subset \mathbb{R}^m$ inherits:
  - A metric tensor $g_x : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$ (inner product)
  - g completely determines the geometry of $\mathcal{M}$
  - A volume form $dV(x) = \sqrt{\det(g_x)}\, dx^1 \wedge \cdots \wedge dx^d$
- Laplace-Beltrami operator, Δ, is equivalent to g
  - $\Delta f = \text{div} \circ \nabla = \frac{1}{\sqrt{|g|}} \partial_i g^{ij} \sqrt{|g|} \partial_j f$
  - $g(\nabla f, \nabla h) = \frac{1}{2}(f\Delta h + h\Delta f - \Delta(fh))$

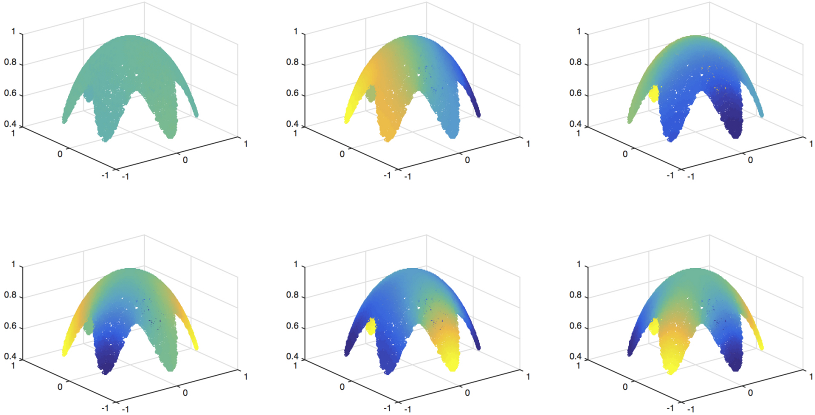
What is Manifold Learning?

- **Manifold learning ⇔ Estimating Laplace-Beltrami**
- **Hodge theorem:**
  Eigenfunctions $\Delta\varphi_i = \lambda_i\varphi_i$ orthonormal basis for $L^2(\mathcal{M}, g)$
- Smoothest functions: φ_i minimizes the functional

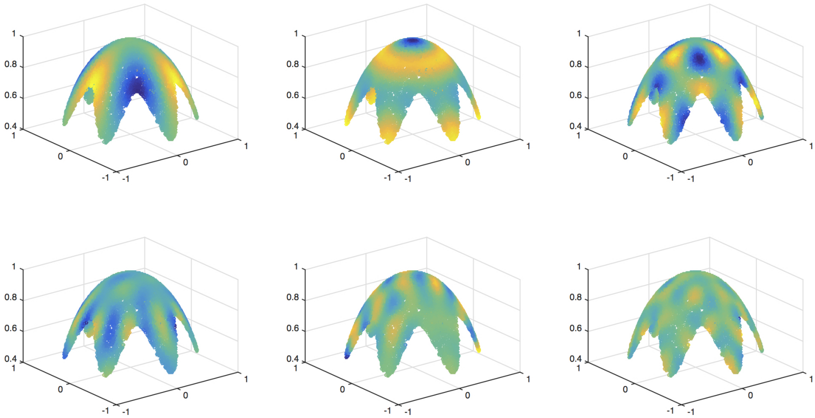
$$\lambda_i = \min_{\substack{f \perp \varphi_k \\ k=1,\ldots,i-1}} \left\{ \frac{\int_{\mathcal{M}} ||\nabla f||^2 \, dV}{\int_{\mathcal{M}} |f|^2 \, dV} \right\}$$

- Eigenfunctions of Δ are custom Fourier basis
  - Smoothest orthonormal basis for $L^2(\mathcal{M}, g)$
  - Can be used to define wavelet frame
  - Define the Sobolev spaces on $\mathcal{M}$

Harmonic Analysis on Manifolds

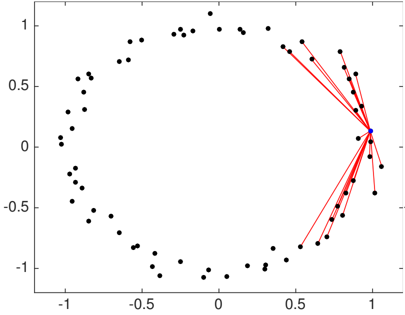


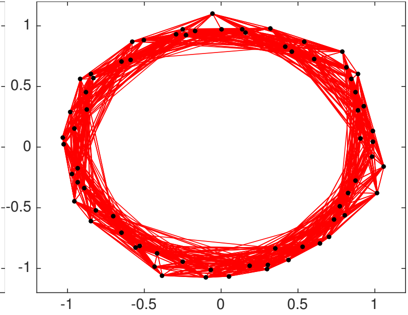
Harmonic Analysis on Manifolds



SO HOW DO WE FIND THE LAPLACIAN FROM DATA?

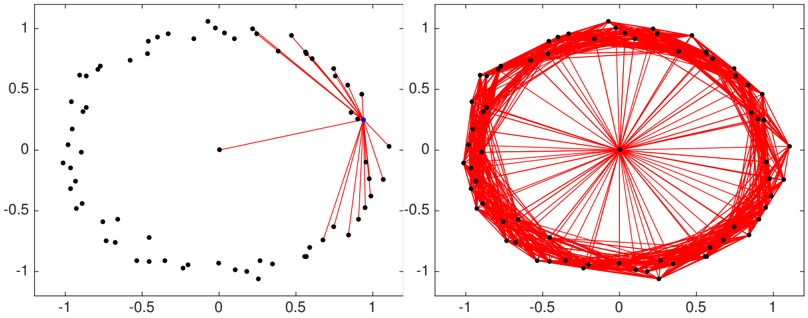
- Assume data lies on (or at least near) a manifold
- Approximate manifold with graph $\Rightarrow$ Connect nearby points





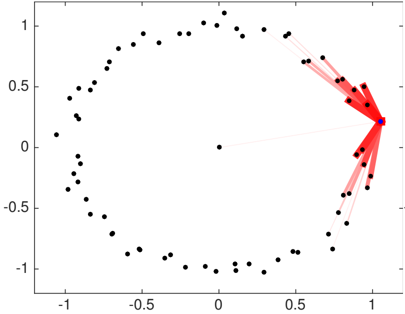
So how do we find the Laplacian from data?

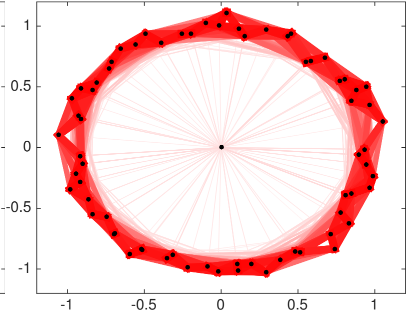
- **Problem:** Noise and outliers can lead to *bridging*



SO HOW DO WE FIND THE LAPLACIAN FROM DATA?

- To prevent bridging we weight the edges
- Edges are given weights $K_\delta(x,y) = e^{-\frac{||x-y||^2}{4\delta^2}}$





SO HOW DO WE FIND THE LAPLACIAN FROM DATA?

- Data set $\Rightarrow$ *weighted graph*
- Edge Weights defined by a kernel function

$$K_\delta(x_i, x_j) = e^{-\frac{||x_i - x_j||^2}{4\delta^2}}$$

- Bandwidth δ determines localization
- 'Adjacency' matrix: $\mathbf{K}_{ij} = K(x_i, x_j)$
- 'Degree' matrix: $\mathbf{D}_{ii} = \sum_j \mathbf{K}_{ij}$
- Normalized graph Laplacian: $\mathbf{L} = \mathbf{I} - \mathbf{D}^{-1}\mathbf{K}$

POINTWISE CONVERGENCE

Theorem: (Belkin & Niyogi, 2005, Singer, 2006)
For $\{x_i\}_{i=1}^N \subset \mathcal{M} \subset \mathbb{R}^m$ uniformly sampled on a compact manifold and for $\vec{f}_i = f(x_i)$ where $f \in C^3(\mathcal{M})$

$$\frac{1}{\delta^2}\left(\mathbf{L}\vec{f}\right)_i = \Delta f(x_i) + \mathcal{O}\left(\delta^2, \frac{1}{N^{1/2}\delta^{1+d/2}}\right)$$

δ = bandwidth
N = number of points

RESTRICTIONS THAT HAVE BEEN OVERCOME TO DEAL WITH REAL DATA:

- Arbitrary sampling (Coifman & Lafon, 'Diffusion maps', ACHA 2006)
- Non-compact manifolds (Berry & Harlim, ACHA 2015)
- Other kernel functions (Thesis 2013; Berry & Sauer, ACHA 2015)
- Boundary (Coifman & Lafon, ACHA 2006; Berry & Sauer, J. Comp. Stat. 2016)
- Spectral convergence (Luxburg et al., Ann. Stat. 2008, Berry & Sauer, submitted)

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Local Kernels

- A *local kernel* has exponential decay:

$$K_\delta(x, x+\delta y) < c_1 e^{-c_2 ||y||^2}$$

- **Theorem:** Symmetric local kernels converge to Laplacians
  - Every local kernel determines a geometry
  - Every geometry accessible by a local kernel
- Explain success of 'kernel methods' in data science:
  - KPCA: Kernel Principal Component Analysis
  - KSVM: Kernel Support Vector Machines
  - KDE: Kernel Density Estimation
  - RKHS: Reproducing Kernel Hilbert Spaces
  - Spectral Clustering (KPCA)

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Tangible manifolds

- Compute ambient distance $||x - y||_{\mathbb{R}^m}$
- Need localization in $d_{\mathcal{I}}(x, y) = \inf_{\gamma} \left\{ \int_0^1 |\gamma'(t)|\, dt \right\}$
- Assume ratio $R(x, y) = \frac{||x-y||_{\mathbb{R}^m}}{d_{\mathcal{I}}(x,y)}$ bounded away from zero
- We will use the exponential map to change variables
- Assume injectivity radius $\text{inj}(x)$ bounded away from zero

Definition: A manifold is uniformly tangible if there are lower bounds on $\text{inj}(x)$ and $\inf_{y \in \mathcal{M}} R(x, y)$ independent of x

CONSISTENCY PART 1

- Matrix times vector converges to integral operator:

$$\left(\mathbf{K}\vec{f}\right)_i = \sum_{j=1}^{N} K_\delta(x_i, x_j) f(x_j) \xrightarrow{N\to\infty} \int_{\mathcal{M}} K_\delta(x_i, y) f(y)\, dV(y)$$

- Assume kernel has fast decay: $K_\delta(x,y) < e^{-||x-y||^2/\delta^2}$
- Localize integral, requires $R(x_i, y) = \frac{||x_i - y||}{d_I(x_i, y)} > 0$

$$\left(\mathbf{K}\vec{f}\right)_i \to \int_{\mathcal{M}\cap \exp_{x_i}(B_\delta(0))} K_\delta(x_i, y) f(y)\, dV(y) + \mathcal{O}(\delta^k)$$

- Change variables to the tangent space $y = \exp_{x_i}(s)$:

$$\left(\mathbf{K}\vec{f}\right)_i \to \int_{B_\delta(0)} K_\delta(x_i, \exp_{x_i}(s)) f(\exp_{x_i}(s))\, ds + \mathcal{O}(\delta^k)$$

- Requires injectivity radius $\text{inj}(x_i) > \delta > 0$

CONSISTENCY PART 2

- Taylor expansion in normal coordinates:

$$f(\exp_x(s)) = f(x) + \nabla f(x) \cdot s + \frac{1}{2} s^\top H(f \circ \exp_x)(0) s$$

- Symmetric kernel $\Rightarrow$ Odd terms integrate to zero

$$\left(\mathbf{K}\vec{f}\right)_i \to \int_{||s||<\delta} \left(K(||s||) + \mathcal{O}(\delta^2 s_i^4) K'(||s||)/||s|| \right) \cdot$$
$$\left(f(x_i) + \delta \nabla f(x_i) \cdot s + \frac{\delta^2}{2} s^\top H(f \circ \exp_{x_i})(0) s \right) \, ds + \mathcal{O}(\delta^4)$$
$$= f(x_i) + m\delta^2 (f(x_i)\omega(x) + \Delta f(x_i)) + \mathcal{O}(\delta^4)$$

- Normalize: $\mathbf{D}^{-1}\mathbf{K}\vec{f} = \frac{\mathbf{K}\vec{f}}{\mathbf{K}\vec{1}} \to \vec{f} + m\delta^2 \overrightarrow{\Delta f} + \mathcal{O}(\delta^4)$
- **Consistency:** $\frac{1}{m\delta^2}(\mathbf{D}^{-1}\mathbf{K} - \mathbf{I})\vec{f} \to \overrightarrow{\Delta f} + \mathcal{O}(\delta^2)$

CONSISTENCY IS NOT ENOUGH!

- Extend to arbitrary sampling $x_i \sim q$ (Coifman & Lafon)
- **Variance:** $\mathbb{E}[((L\vec{f})_i - \Delta f(x_i))^2] = \mathcal{O}\left(\frac{q(x_i)^{3-4d}}{N\delta^{2+d}}\right)$
- Negative exponent: $3 - 4d < 0$
- As density q approaches zero the variance blows up!
- **Solution:** Variable bandwidth

Berry and Harlim (ACHA, 2015)

Variable Bandwidth Kernels

We introduced the variable bandwidth kernel:

$$K_{\delta,\beta}(x,y) = K\left(\frac{||x-y||}{\delta\sqrt{q(x)^{\beta}q(y)^{\beta}}}\right)$$

Theorem (Berry and Harlim, ACHA, 2015):

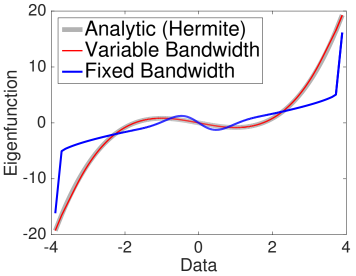
$$\mathbf{L}_{\delta,\alpha,\beta}\vec{f} = \Delta f + c_1 \nabla f \cdot \nabla \log q + \mathcal{O}\left(\delta^2, \frac{q^{-c_2}}{\sqrt{N}h^{1+d/2}}\right)$$

- Operator defined by: $c_1 = 2 - 2\alpha + d\beta + 2\beta$
- Variance determined by: $c_2 = 1/2 - 2\alpha + 2d\alpha + d\beta/2 + \beta$

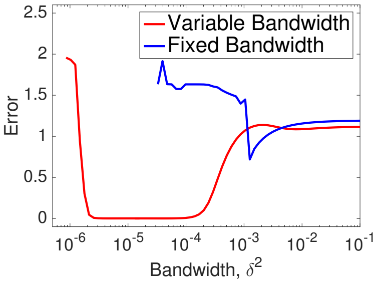
Example: Variable Bandwidth Kernel

Gaussian data: Brownian motion in quadratic potential

Eigenfunctions (Hermite)



Error vs. Bandwidth



Berry and Harlim (ACHA, 2015)

SUMMARY OF MANIFOLD LEARNING

- Manifold learning $\Leftrightarrow$ Estimating Laplace-Beltrami
- Can estimate Laplace-Beltrami with a graph Laplacian
- For a non-compact manifold:
  - Manifold must be tangible
  - Requires a variable bandwidth kernel
- My other contributions:
  - Access any desired geometry (local kernels)
  - Manifolds with boundary
  - Spectral convergence

CONTINUOUS K-NEAREST NEIGHBORS (CKNN)

Building unweighted graphs from data (TDA)

CkNN Graph: Edge $\{x, y\}$ added if $\frac{||x-y||}{\sqrt{||x-x_k|| \; ||y-y_k||}} < \delta$

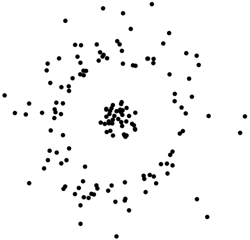
- $x_k = k$-th nearest neighbor of x
- Unnormalized graph Laplacian: $\mathbf{L}_{\mathrm{un}} = \mathbf{D} - \mathbf{K}$
- **Corollary:** $\mathbf{L}_{\mathrm{un}} \vec{f} \to \overrightarrow{\Delta_{\tilde{g}} f}$ where $(\tilde{g} = q^{2/d} g, d\tilde{V} = q\, dV)$
- **New result:** Spectral convergence $\mathbf{L}_{\mathrm{un}} \to \Delta_{\tilde{g}}$
- Consistency of CkNN clustering:
  - Conn. comp. of graph $\Leftrightarrow$ Kernel of L_{un}
  - Conn. comp. of $\mathcal{M}$ $\Leftrightarrow$ Kernel of $\Delta_{\tilde{g}}$ (Hodge theorem)

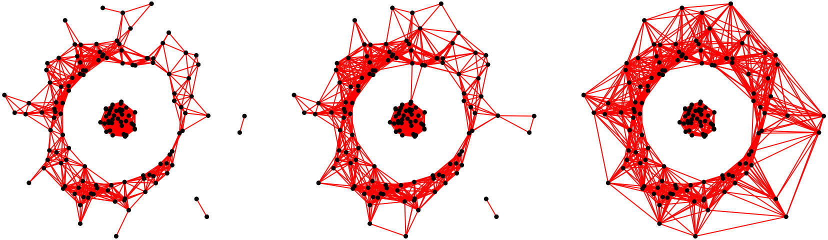
(Berry & Harlim (ACHA, 2015); Berry & Sauer (in review))

CkNN yields improved graph construction

2D Gaussian with annulus removed:

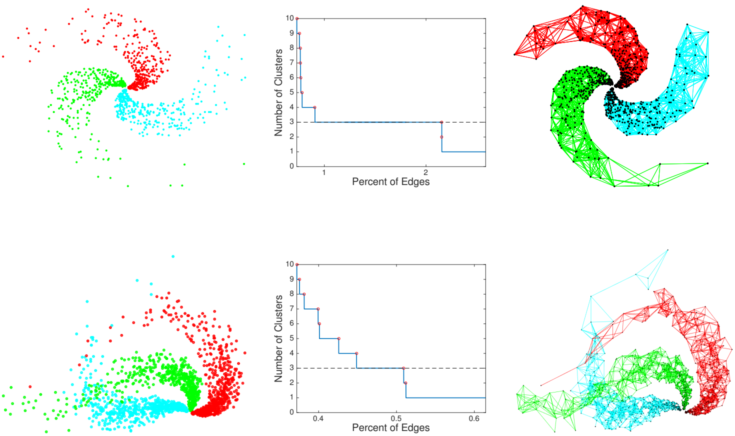
Persistent vs. consistent homology



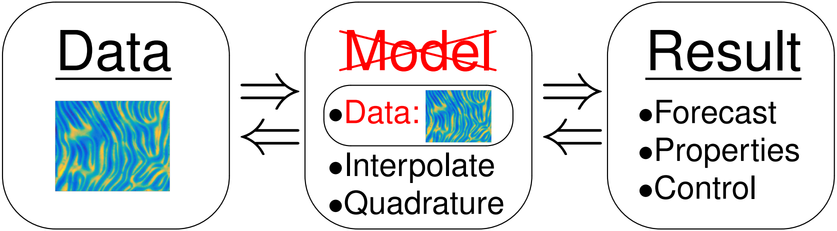


Small bandwidth | Large bandwidth | CkNN

IMPROVED CLUSTERING USING CKNN



Nonparametric Modeling



- **Tools:** Geometry and Harmonic/Functional Analysis
  - Interpolate: $f(x) = \sum_j \langle f, \varphi_j \rangle \varphi_j(x)$
  - Quadrature: $\langle f, \varphi_i \rangle \approx \sum_i f(x_i)\varphi(x_i)$
  - Operator Representation: $\mathbf{A}_{jk} = \langle \varphi_j, \mathcal{A}\varphi_k \rangle$
- All require a basis $\{\varphi_j\}$!

DIFFUSION FORECAST

- Autonomous SDE: $dx = a(x)\,dt + b(x)\,dW_t$
- Density solves Fokker-Planck PDE: $\frac{\partial}{\partial t} p = \mathcal{L}^* p$
- Shift map: $S(f)(x_i) = f(x_{i+1})$ estimates $\mathbb{E}[S(f)] = e^{\tau\mathcal{L}} f$
- $\vec{c}(t)$ are the custom Fourier coefficients of p

$$p(x,t) \xrightarrow{\text{Diffusion Forecast}} p(x, t+\tau) = e^{\tau\mathcal{L}^*} p(x,t)$$

$$p(x,t) \xrightarrow{\langle p,\varphi_j\rangle} \vec{c}(t)$$

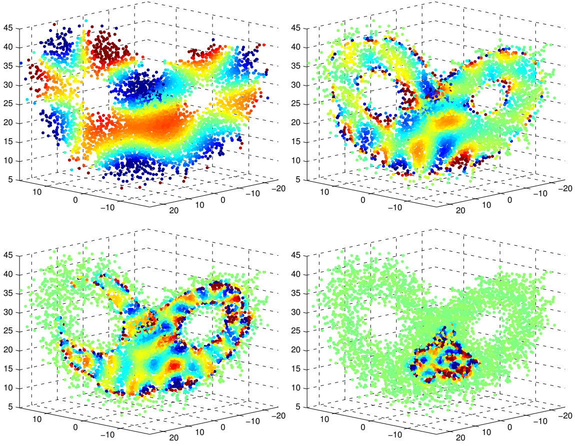
$$\vec{c}(t) \xrightarrow{A_{lj} \equiv \mathbb{E}[\langle \varphi_j, S\varphi_l\rangle_q]} \vec{c}(t+\tau) = A\vec{c}(t).$$

$$\vec{c}(t+\tau) \xrightarrow{\sum_j c_j \varphi_j q} p(x, t+\tau)$$

Berry and Harlim (SIAM J. Uncertainty Quantification, 2014)
Berry, Harlim, and Giannakis (Physical Review E, 2015)

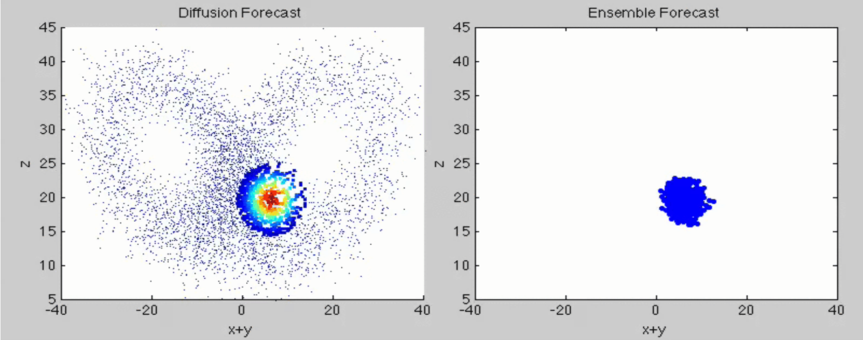
MANIFOLD LEARNING ⇒ CUSTOM 'FOURIER' BASIS

- **Optimal basis:** Minimum variance $A_{lj} \equiv \mathbb{E}[\langle \varphi_j, S\varphi_l \rangle_q]$



DIFFUSION FORECAST EXAMPLE

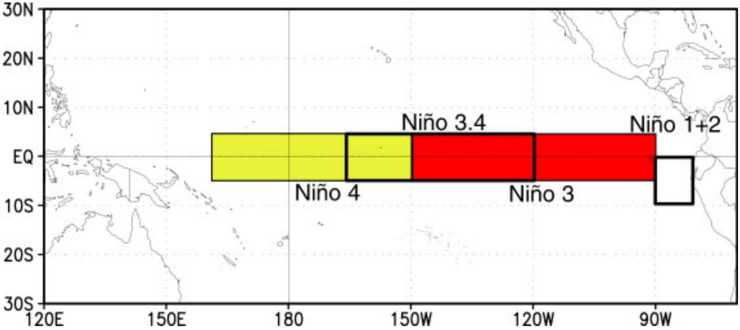
No Model Perfect Model



Berry, Harlim, and Giannakis (PRE, 2015)

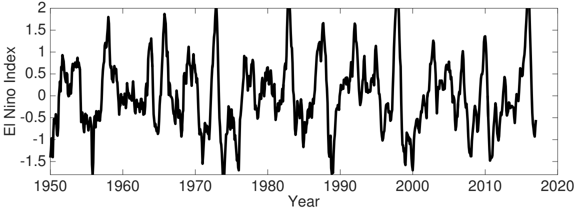
FORECASTING THE EL NIÑO INDEX

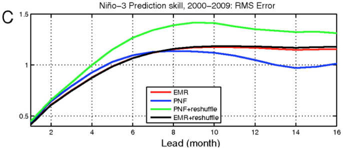
Sea surface temperatures (SST) in the Niño indices:



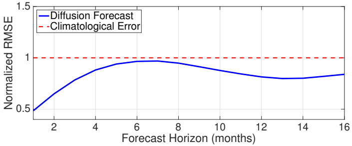
Index: 3-month running average SST anomaly

FORECASTING THE EL NIÑO INDEX





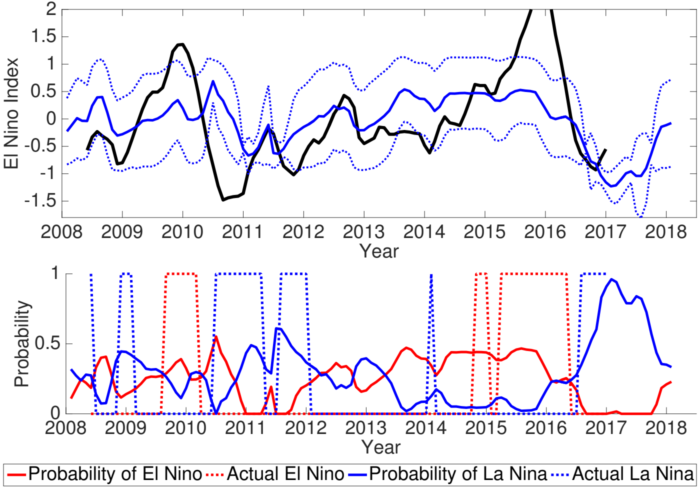
Chekrouna, Kondrashov, and Ghil, PNAS 2011,108,no.29



Diffusion Forecast

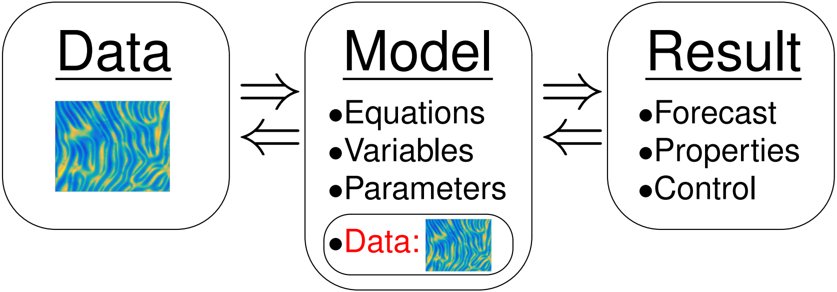
Berry, Harlim, and Giannakis (PRE, 2015)

Your 13-month Forecast



El Niño $= \text{Index} > 0.5$ La Niña $= \text{Index} < -0.5$

SEMIPARAMETRIC MODELING



- **Data becomes part of the model:**
  - Start with imperfect parametric model
  - Assimilate data (adaptive), collect residual errors
  - Build nonparametric model for the residuals

Semiparametric Forecast Model

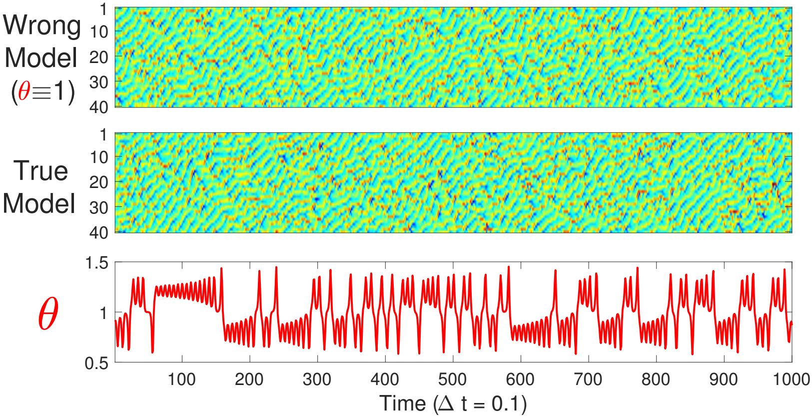
- Partially known model $\dot{x} = f(x, \theta)$
- No equations for θ!
- Apply the Diffusion Forecast to $p(\theta, t)$
- Sample $\theta^k(t) \sim p(\theta, t)$ and pair with ensemble $x^k(t)$

$$
\begin{array}{ccc}
(x^k(t), \theta^k(t)) & \xrightarrow{\dot{x}=f(x,\theta)} & (x^k(t+\tau), \theta^k(t+\tau)) \\
\uparrow \theta^k(t) & & \uparrow \theta^k(t+\tau) \\
p(\theta, t) & \xrightarrow{\text{Diffusion Forecast}} & p(\theta, t+\tau)
\end{array}
$$

Berry and Harlim (J. Computational Physics, 2016)

Example: 40-dimensional Lorenz-96 system driven by Lorenz-63

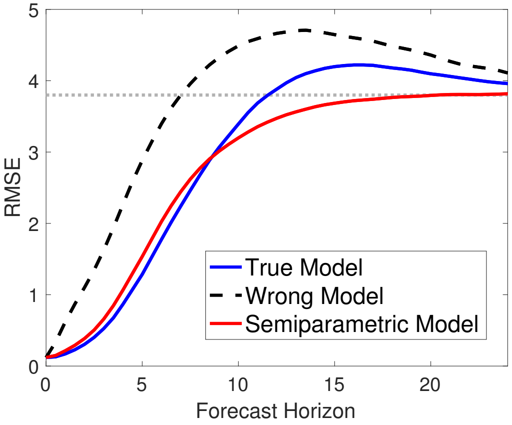
$$\dot{x}_i = \theta x_{i-1}x_{i+1} - x_{i-1}x_{i-2} - x_i + 8$$



Berry and Harlim (J. Computational Physics, 2016)

EXAMPLE: 40-DIMENSIONAL LORENZ-96 SYSTEM DRIVEN BY LORENZ-63

$$\dot{x}_i = \theta x_{i-1}x_{i+1} - x_{i-1}x_{i-2} - x_i + 8$$



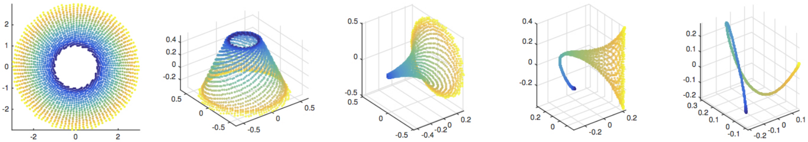
Berry and Harlim (J. Computational Physics, 2016)

FUTURE DIRECTION #1: FEATURE MAPS

- Want to represent map $\mathcal{H} : \mathcal{M} \to \mathcal{N}$
- For $\mathcal{H}$ a diffeomorphism: pull-back metric
- Otherwise: Apply the Iterated Diffusion Map (IDM)

$$\frac{dg}{dt} = \frac{1}{2}\left((D\mathcal{H}^\top D\mathcal{H} - I)g + g(D\mathcal{H}^\top D\mathcal{H} - I)\right)$$

- Example: $\mathcal{H}(x, y) = \sqrt{x^2 + y^2}$



Berry & Sauer, (ACHA, 2015)

Berry & Harlim (ACHA, 2016)

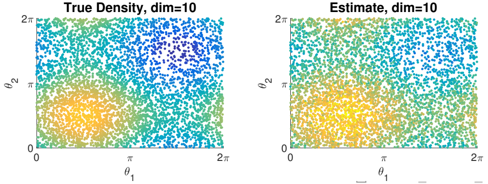
FUTURE DIRECTION #2: CONSISTENCY OF TOPOLOGICAL DATA ANALYSIS (TDA)

- **Topological Consistency:** VR homology $\to H_k(\mathcal{M})$
- Spectral convergence proves consistency of $H_0(\mathcal{M})$
- Discrete Exterior Calculus (DEC):
  - TDA uses simplicial complexes to compute homology
  - Weighted simplices correspond to differential forms
  - Kernel on simplices can define Laplacians on forms
  - Which kernels recover the Laplace de-Rham operator?
- Smooth Exterior Calculus (SEC):
  - Start with the smooth eigenfunctions $\Delta\varphi_i = \lambda_i\varphi_i$
  - Define a frame for 1-forms: $b^{ij} = \varphi_i d\varphi_j - \varphi_j d\varphi_i$
  - Define Laplace-de Rham operators on b^{ij}
  - $\left\langle b^{kl}, \Delta^1(b^{ij})\right\rangle = \sum_r (c_{kir}c_{ljr} - c_{kjr}c_{lir})(\lambda_r^2 - \lambda_r(\lambda_k + \lambda_i + \lambda_l + \lambda_j)) + c_{ijr}c_{klr}(\lambda_j - \lambda_i)(\lambda_l - \lambda_k)$

Future Direction #3: Smoothness Priors

- Manifold learning suffers from the curse-of-dimensionality
  - Bias-squared: $\mathcal{O}(\delta^4)$
  - Variance: $\mathcal{O}(N^{-1}\delta^{-2-d})$
  - Optimal bandwidth: $\delta = \mathcal{O}(N^{-1/(6+d)})$
  - Minimal Error: $\mathcal{O}(N^{-2/(6+d)})$
- Richardson Extrapolation: Combine multiple δ's
  - Reduces bias to $\mathcal{O}(\delta^{2k})$
  - Increases variance by a constant
  - Requires $\mathcal{M}$ to be C^k
- 'Solves' curse-of-dimensionality by assuming smoothness

- 5000 points
- 10-dim torus
- In $\mathbb{R}^{20}$



SUMMARY

- Manifold learning $\Leftrightarrow$ Estimating Laplace-Beltrami
- Can estimate Laplace-Beltrami with a graph Laplacian
- Need an appropriate kernel (variable bandwidth)
- Results imply better method for graph construction (CkNN)
- Spectral convergence gives us a custom Fourier basis
- Allows model-free forecasting and correcting model error

A bit of geometry

- Let $\iota : \mathcal{M} \to \mathbb{R}^m$ be the embedding into data space
- Tangent space $T_x\mathcal{M}$ inherits an inner product

$$g_x(v, w) = \langle D\iota(x)v, D\iota(x)w \rangle_{\mathbb{R}^m}$$

- g is called the Riemannian metric
- If $e_1, ..., e_d \in T_x\mathcal{M}$ is a basis, define $g_{ij}(x) = g_x(e_i, e_j)$
- Define the volume form $dV(x) = \sqrt{\det(g(x))}$
- $\mathrm{vol}(\mathcal{M}) = \int_{x \in \mathcal{M}} 1 \, dV(x)$

A bit more geometry: The exponential map

- The exponential map takes tangent vectors to the manifold

$$\exp_x : T_x\mathcal{M} \to U \subset \mathcal{M}$$

- Let $\gamma : [0,1] \to \mathcal{M}$ be geodesic curve with $\gamma'(t) = 1$
- If $\gamma'(0) = s/||s||$ then $\exp_x(s) = \gamma(1)$ and $\exp_x(0) = x$ so

$$y = x + s + \frac{1}{2} II(s,s) + \mathcal{O}(s_i^3)$$

- **Fact 1:** $||y - x||^2 = ||s||^2 + \mathcal{O}(s_i^4)$
- **Fact 2:** Natural volume element, $dV(y) = ds$
- **Fact 3:** Gradient, $D_s(f \circ \exp_x) = \nabla f$
- **Fact 4:** Laplace-Beltrami operator, $\sum_{i=1}^{d} \frac{d^2(f \circ \exp_x)}{ds_i^2} = \Delta f$

DIFFUSION MAPS: ALLOWING ARBITRARY SAMPLING

- For $X_i \sim q$

$$\mathbb{E}[Kf(x)] = f(x)q(x) + mh^2(f(x)q(x)\omega(x) + \Delta(fq)(x)) + \mathcal{O}(h^4)$$

$$D(x) = K1(x) = q(x) + mh^2(q(x)\omega(x) + \Delta q(x)) + \mathcal{O}(h^4)$$

- Right normalize:

$$\hat{K}f \equiv K\left(\frac{f}{D}\right) = f(x) + mh^2\left(\Delta f(x) - f(x)\frac{\Delta q(x)}{q(x)}\right)$$

- Left normalize: $\hat{D} \equiv \hat{K}1 = 1 - mh^2\frac{\Delta q(x)}{q(x)}$

$$\frac{\hat{K}f}{\hat{D}} = f(x) + mh^2\Delta f(x)$$

Continuous k-Nearest Neighbors (CkNN)

- Let x_k denote the kth nearest neighbor of x

CkNN: Edge between the points x, y if $\frac{||x-y||}{\sqrt{||x-x_k|| \; ||y-y_k||}} < \delta$

- Corresponds to variable bandwidth kernel with $\beta = -1/d$
- **Corollary:** $L_{\mathrm{un}}\vec{f} \to \overrightarrow{\Delta_{\tilde{g}} f}$
- For fixed k, $||x - x_k|| \propto q(x)^{-1/d}$ so $\beta = -1/d$
- This is a variable bandwidth kernel with $K(t) = 1_{\{t<1\}}$ so

$$K\left(\frac{||x-y||}{\delta\sqrt{q(x)^{-1/d}q(y)^{-1/d}}}\right) = 1_{\left\{\frac{||x-y||}{\sqrt{||x-x_k|| \, ||y-y_k||}}<\delta\right\}}$$

CKNN CONVERGENCE RESULT

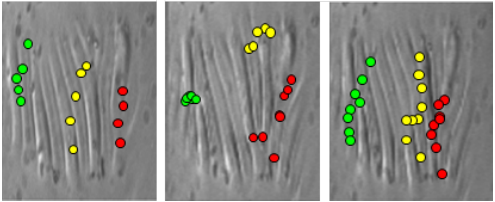
- Define the unnormalized graph Laplacian $L_{\mathrm{un}} = D - K$
- **Corollary:** $L_{\mathrm{un}}\vec{f} \to \overrightarrow{\Delta_{\tilde{g}} f}$
- Only $\beta = -1/d$ yields a Laplace-Beltrami operator
- $\tilde{g} \equiv q^{2/d} g$ is a conformal change of metric on $\mathcal{M}$
- Natural volume form:

$$d\tilde{V} = \sqrt{|\tilde{g}|} = \sqrt{|q^{2/d} g|} = q\sqrt{|g|} = q\,dV$$

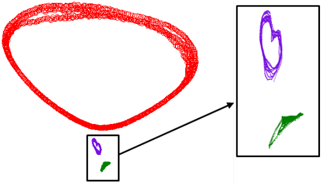
- $\mathrm{vol}_{\tilde{g}}(\mathcal{M}) = \int_{\mathcal{M}} d\tilde{V} = \int_{\mathcal{M}} q\,dV = 1$

Attractor Clustering

Multi-stability in Nematic Liquid Crystals:

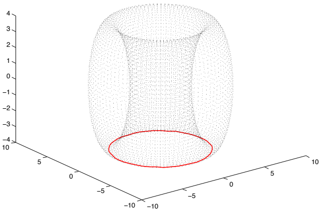


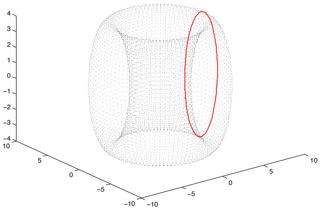
Finding good metrics/coordinates:



THE DISCRETE EXTERIOR CALCULUS (DEC)

- Estimate Laplace-de Rham: $\Delta^k = \delta^{k+1} d^k + d^{k-1} \delta^k$
- Compute Betti numbers: $H^k(\mathcal{M}) \cong \text{Kernel}(\Delta^k)$
- Eigenforms in the kernel of Δ^1 on T^2:





- Representatives of $H^1(\mathcal{M})$ on a genus two surface: