# Dimension Reduction in Spatiotemporal Dynamics

Tyrus Berry, Rob Cressman, Tim Sauer

Department of Mathematical Sciences
George Mason University
Fairfax VA 22030

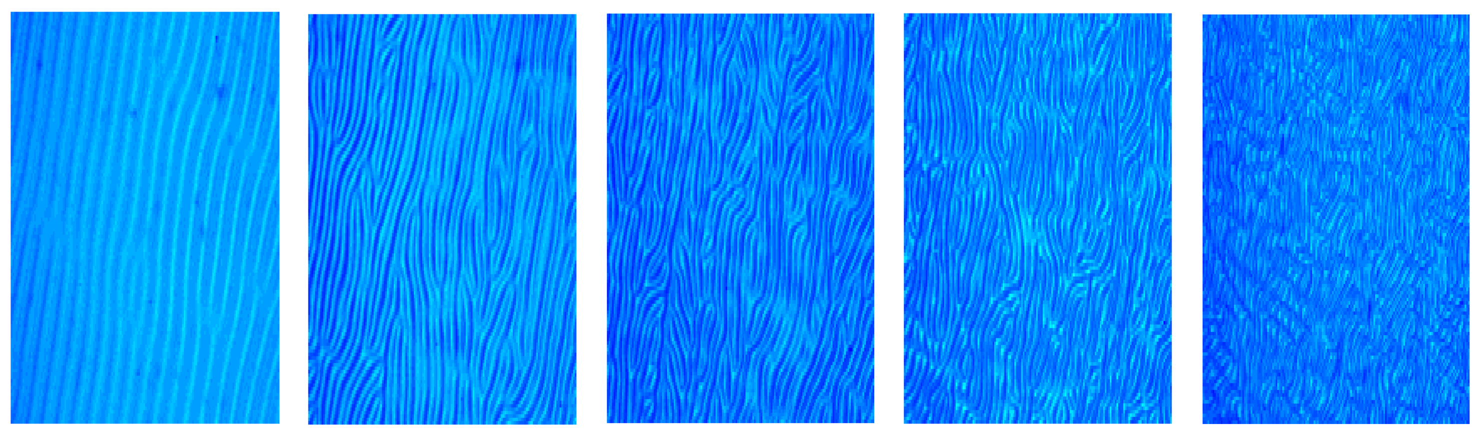
## Abstract

When a thin liquid crystal is driven by a sufficient A/C voltage, electroconvection produces complex spatiotemporal patterns. There is significant interest in quantifying this spatiotemporal complexity using measures of dimensionality and Lyapunov exponents, but these estimates are extremely difficult with experimental data because of the high dimensionality of the raw data. Simulations of models, such as Raleigh-Benard convection, indicate that there may be a low-dimensional representation of the process [2]. However, conventional techniques of dimensionality reductions such as the Karhunen-Loeve Decomposition have been unable to recover a low dimensional process even for low drivings levels [2, 3].

We have developed a new nonlinear dimensionality reduction technique to significantly decrease the number of dimensions needed to represent a given proportion of the total spatiotemporal variance. By sampling representative sub-videos we construct a low-dimension state space and a vector field that represents the dynamics of all sub-videos simultaneously. These low-dimensional representations allow estimation of Lyapunov exponents from experimental data, and may lead to new models of spatiotemporal chaotic dynamics. Furthermore, in a multi-stable system, the reduced state space naturally clusters the various attractors. Using the clustering to identify the attractors, we hope to automatically identify the basins of attraction and then efficiently steer between quasi-stable attractors.

## Thin Liquid Crystals

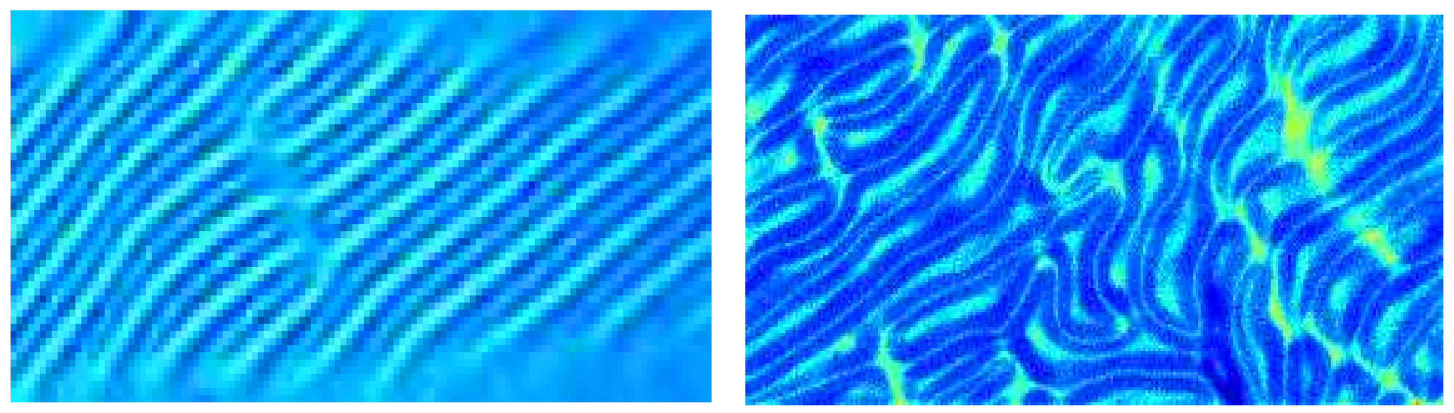
Applying an A/C voltage to a thin liquid crystal produces complex spatiotemporal patterns shown below. By varying the applied voltage we observe a continuous range of dynamic behaviors.



Liquid crystal driven at 10V, 12V, 14V, 16V, and 20V (left to right).

## Ridges and Defects

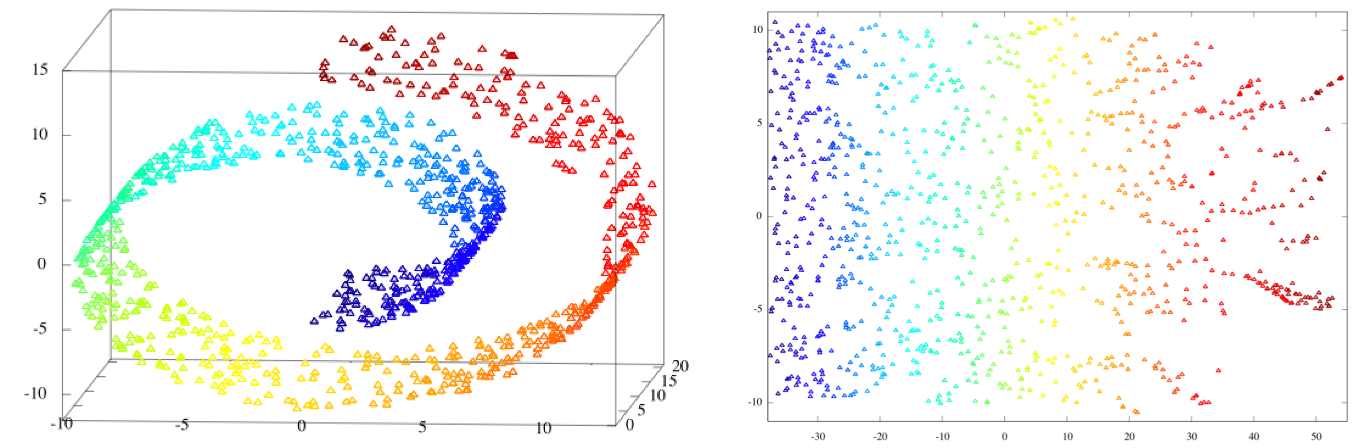
The two important features to note in the above images are the continuous ridges and the defects. At low driving voltages, the ridges move slowly across the image and only deviate slightly from their preferred orientation. The defects are discontinuities in the ridges and move more quickly and chaotically around the image. At the higher driving voltages, the ridges begin to have larger deviations from their preferred orientation and their movement becomes more complex. The concentration of defects becomes higher and the defects move very quickly and sporadically. Below we see enlarged images of the respective defects.



Enlarged image showing defects at low voltage (left) and high voltage (right).

## Nonlinear Dimensionality Reduction

Nonlinear Dimensionality Reduction (NLDR) techniques can simplify high-dimensional data sets by imposing implicit regularity assumptions on the data. Diffusion Maps [1] and Isomap [4] are NLDR techniques which will be used below. Both techniques preserve the local data structure while changing non-local relationships in order to simplify the data [5]. The methods are interchangeable below; while Diffusion Maps is faster and more robust to noise it requires tuning additional nuisance parameters.



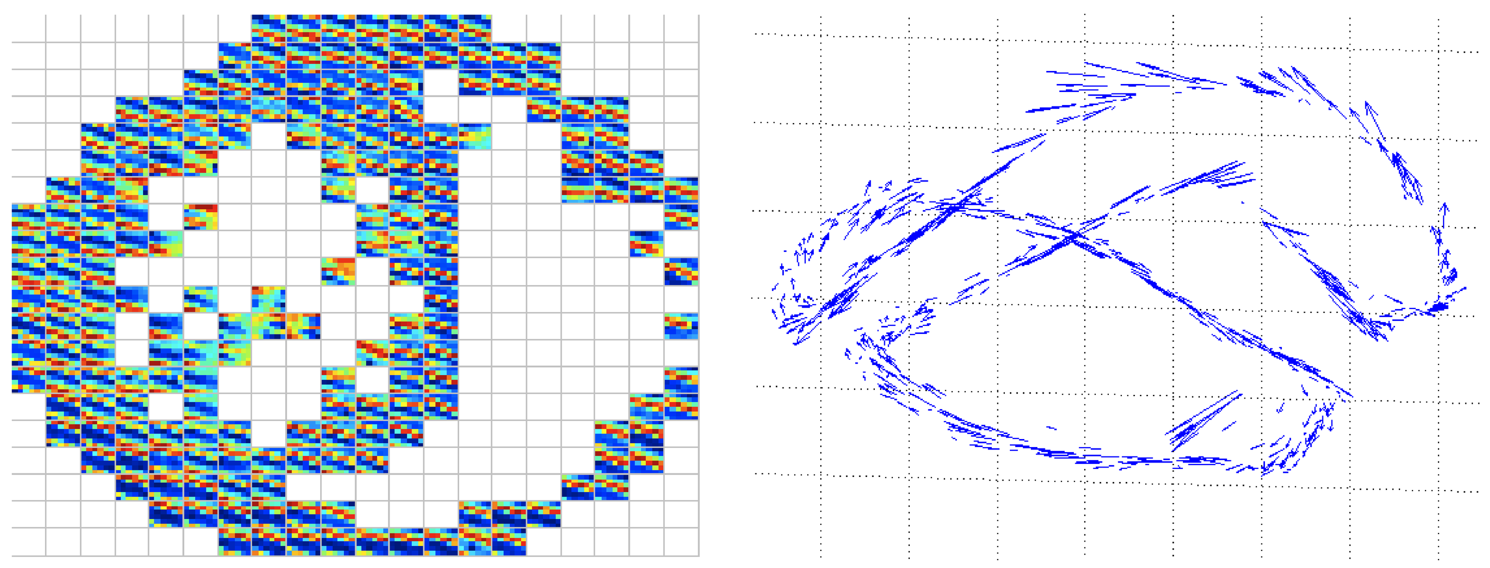
Isomap applied to a 2D manifold (left) produces the intrinsic coordinates (right)

## Dimensionality Reduction Algorithm for Spatiotemporal Attractors

1. Choose a representative sample of images (Vector Quantization).
2. For each image form a state vector, $\overline{y}_i$ by appending several time delayed images.
3. Compute the covariance matrix, $S$, of the coordinates of the state vector.
4. Compute the initial distance matrix as $d(i,j) = \sqrt{(\overline{y}_i - \overline{y}_j)^T S(\overline{y}_i - \overline{y}_j)}$.
5. For each column of $d$ set all distances to "inf" except for the $k$ nearest neighbors.
6. Recompute the long distance, thereby filling in all "inf" values:
   (a) Diffusion Maps: Compute long distances as expected commute times.
   OR
   (b) Isomap: Compute long distances as graph geodesic distances.
7. Find the Singular Value Decomposition of the resulting matrix of distances.
8. Use the Singular Values to determine the dimension of the manifold.
9. Find the reduced coordinates by projecting on the appropriate Singular Vectors.

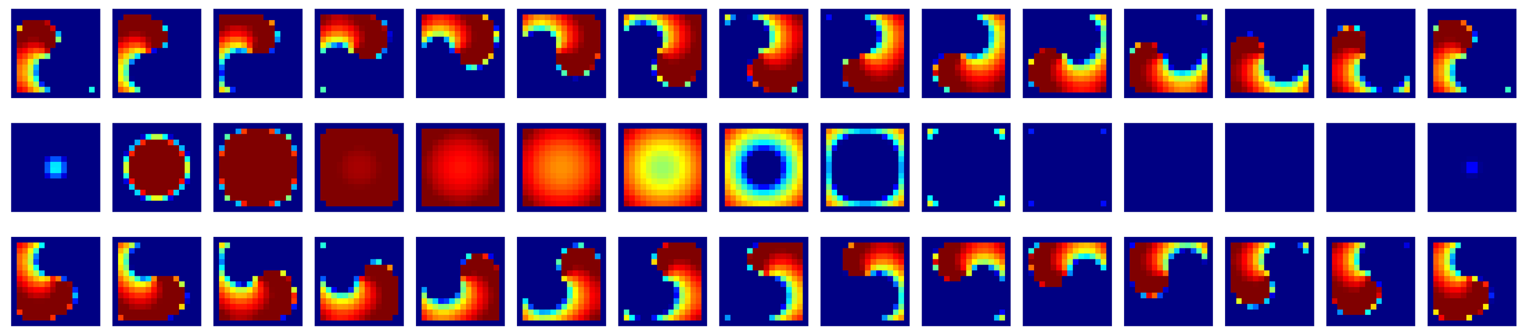
## Application to Liquid Crystal Data

We applied our algorithm to the set of $56^2$ overlapping 8x8-pixel sub-images of the liquid crystal data at 10V. At lower left, the sub-images are arranged according to the low-dimensional projection. The image on the right shows each sub-image being projected into the reduced state space. The arrows indicate the trajectory from the previous time period to the current. The alignment of the trajectories shows that the low voltage spatiotemporal dynamics have a large deterministic component.



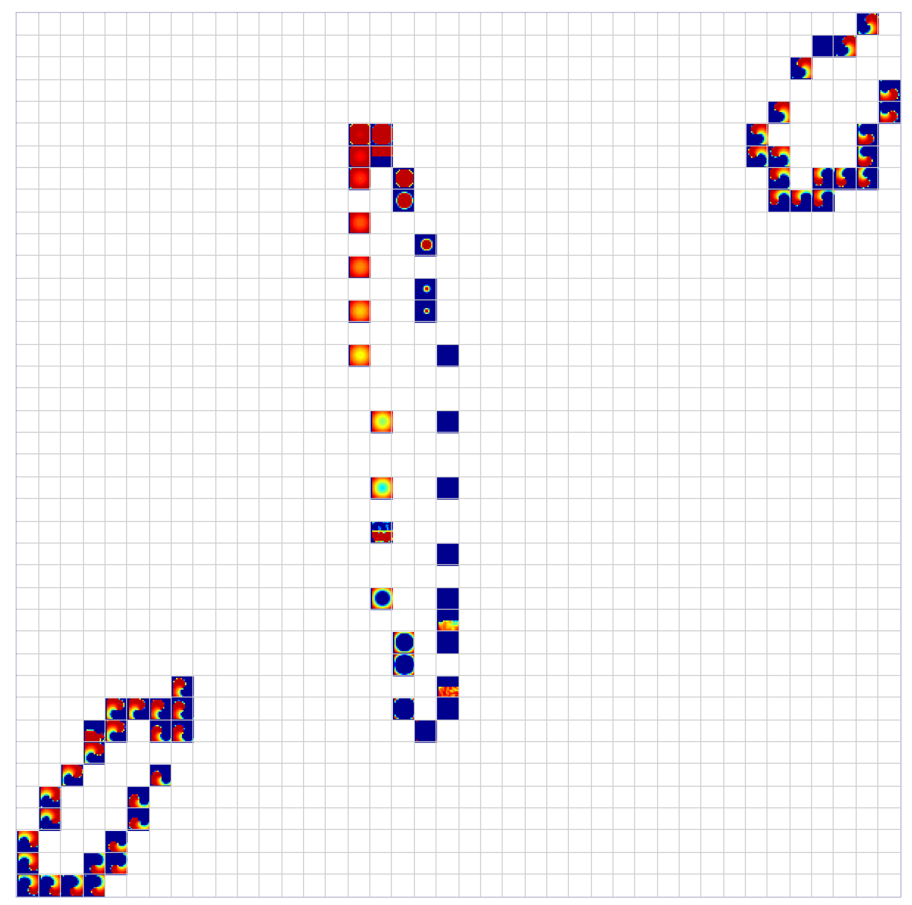
## Application to Excitable Media Simulation

In this simulation we built a network of 256 neurons based on a simplified FitzHugh-Nagumo model. The neurons were arranged in a 16x16 planar grid with connection strengths decaying with distance. The system was found to be multi-stable, with three dominant attractors. The simulation was run several times with random initial conditions and our algorithm was applied to the resultant data.



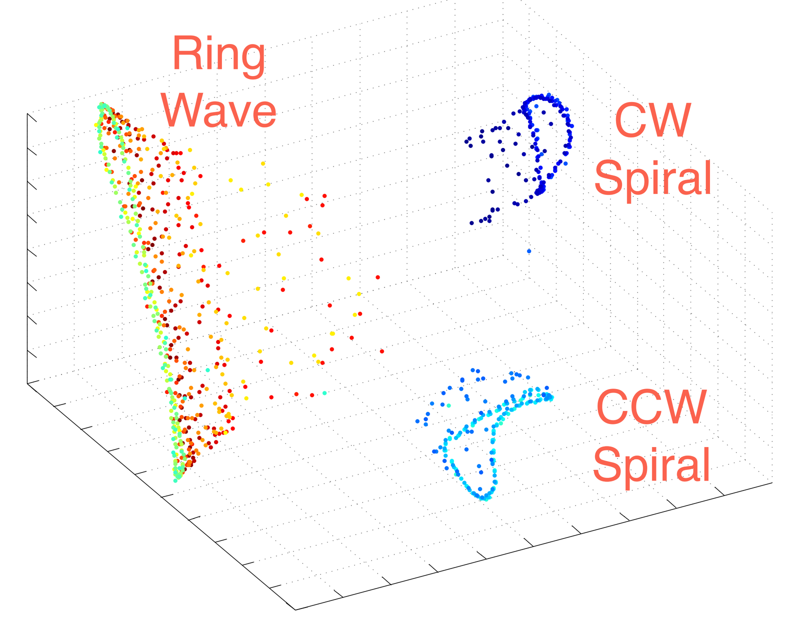
Top to bottom: a clockwise spiral wave, a ring wave, and a counter-clockwise spiral.

## Attractor Clustering in Multi-stable Systems



Since the system spends most of its time very close to an attractor, there are very few paths between attractors, so the expected time for a diffusion process to move between attractors is long. Thus, the Diffusion Maps version of our algorithm naturally constructs a low dimensional state space in which the attractors are separated. Moreover, since the algorithm maintains the local data structure, the dynamics of the attractors are preserved in the low dimensional state space. The image to the left shows how individual images are embedded in the low dimensional state space.

The image to the right shows the same attractor structure from another angle, where now we can see the transient trajectories from the randomized initial conditions. In the future we plan to apply this technique to more complex systems with many more quasi-stable attractors. Attractors that are nearby in the structure will be easier to transition between, thus creating clusters of attractors. This representation of the multi-stable structure may lead to new techniques for steering in complex systems.

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