On Colorings of Squares of Outerplanar Graphs

Geir Agnarsson * *Magnús M. Halldórsson* †

Abstract

We investigate the clique number, the chromatic number and the inductiveness (or the degeneracy) of the square G^2 of an outerplanar graph G, and bound as a function of the maximum degree Δ of G. Our main result is a tight bound of Δ for the inductiveness of the square of any outerplanar graph G, when $\Delta \geq 7$. This implies that a greedy algorithm yields an optimal coloring of such square graphs, and leads to an exact linear time algorithm that holds for any Δ. We then derive optimal upper bounds on the three parameters for outerplanar graphs of smaller degree $\Delta < 7$, and in the case of chordal outerplanar graphs, classify exactly which graphs have parameters exceeding the absolute minimum. A co-product of the study is a characterization of all strongly simplicial elimination orderings of an arbitrary power of a tree.

1 Introduction

The square of a graph G is the graph G^2 on the same vertex set with edges between pair of vertices of distance one or two in G. Coloring squares of graphs has been studied in relation to frequency allocation. This models the case when nodes represent both senders and receivers, and two senders with a common neighbor will interfere if using the same frequency. The problem has particularly seen much attention on planar graphs.

A conjecture of Wegner [1] dating from 1977 (see [2]), states that the square of every planar graph G has a chromatic number which does not exceed $3\Delta/2 + 1$, where $\Delta \geq 8$ is the maximum degree of G. The conjecture matches the the maximum clique number of these graphs. Currently the best upper bound known is $1.66\Delta + 78$ by Molloy and Salavatipour [16]; see [16, 3] for more history of on these problems.

An earlier paper of the current authors [3] gave a bound of $\lceil 1.8\Delta \rceil$ for the chromatic number of squares of planar graph with large maximum degree $\Delta \geq 749$. This is based on bounding the *inductiveness* of the graph, which is the maximum over all subgraphs H of the minimum degree of H. It was also shown that this was the best possible bound on the inductiveness. Borodin et al [17] showed that the bound holds for all $\Delta \geq 48$. Inductiveness has the additional advantage of also bounding the list-chromatic number.

Inductiveness leads to a natural greedy algorithm (henceforth called Greedy): Select vertex u of minimum degree, recursively color $G \setminus u$, and finally color u with the smallest available color. Alternatively, t-inductiveness leads to an *inductive ordering* $u_1, u_2, \ldots, u_n$ of the vertices such that any vertex u_i has at most t neighbors among $\{u_{i+1}, \ldots, u_n\}$. Then, if we color the vertices *first-fit* in the reverse order $u_n, u_{n-1}, \ldots, u_1$ (i.e. assigning each vertex the smallest color not used among its previously colored neighbors), the number of colors used is at most $t+1$. Implemented efficiently, the algorithm runs in time linear in the size of the graph. The algorithm has also the special advantage that it requires only the square graph G^2, and does not require information about the underlying graph G.

The purpose of the article is to further contribute to the study of various vertex colorings of squares of planar graphs, by examining an important subclass of them, the class of outerplanar graphs. Observe that the neighborhood of vertex with Δ neighbors induces a clique in the square graph. Thus, the chromatic number, and in fact the clique number, of any graph of maximum degree Δ is necessarily a function of Δ and always at least $\Delta + 1$.

Our results. We derive tight bounds on chromatic number, as well as the inductiveness and clique number, of the square of an outerplanar graph G as a function of the maximum degree Δ of G. The main result, given in Section 3, is that when $\Delta \geq 7$, the inductiveness of G^2 is exactly Δ. It follows that the clique and chromatic numbers are exactly $\Delta + 1$ and that Greedy yields an optimal coloring. We can then treat the low-degree cases separately to derive a linear-time algorithm independent of Δ.

We examine in detail in Section 4 the low-degree cases, $\Delta < 7$, and derive best possible upper bounds on the maximum clique and chromatic numbers, as well as inductiveness, of squares of outerplanar graphs. We treat the special case of chordal outerplanar graphs

*Department of Mathematical Sciences, George Mason University, MS 3F2, 4400 University Drive, Fairfax, VA 22030, `geir@math.gmu.edu`.

†Department of Computer Science, University of Iceland, Reykjavík, Iceland. `mmh@hi.is`

separately. The results are shown in Table 1. Only in the case of general outerplanar graphs of maximum degree $\Delta = 5$, is it open whether the chromatic number can exceed $\Delta + 1$ (but is known to be at most $\Delta + 2$). We further classify all chordal outerplanar graphs G for which the inductiveness of G^2 exceeds Δ or the clique or chromatic number of G^2 exceed $\Delta + 1$.

	Chordal			Non-chordal		
Δ	$\omega(G^2)$	ind	χ	ω	ind	χ
2	$\Delta+1$	Δ	$\Delta+1$	$\Delta+3$	$\Delta+2$	$\Delta+3$
3	$\Delta+1$	Δ	$\Delta+1$	$\Delta+2$	$\Delta+1$	$\Delta+2$
4	$\Delta+2$	$\Delta+1$	$\Delta+2$	$\Delta+2$	$\Delta+2$	$\Delta+2$
5	$\Delta+1$	$\Delta+1$	$\Delta+1$	$\Delta+1$	$\Delta+1$	$\Delta+2$?
6	$\Delta+1$	$\Delta+1$	$\Delta+2$	$\Delta+1$	$\Delta+1$	$\Delta+2$
7+	$\Delta+1$	Δ	$\Delta+1$	$\Delta+1$	Δ	$\Delta+1$

Table 1: Optimal upper bounds for the clique number, inductiveness, and chromatic number of the square of a chordal / non-chordal outerplanar graph G.

Since the inductiveness a biconnected outerplanar graph G is closely related to the *weak dual* $T^*(G)$, we study in some depth strong simplicial elimination orderings of an arbitrary power of a tree, and characterize all such orderings, but this analysis is omitted for lack of space. In the next section, we introduce our notation and definitions, and show how the problems reduce to the case of biconnected outerplanar graphs with no non-trivial clique separator.

Related results. It is straightforward to show that the inductiveness of a square graph of an outerplanar graph of degree Δ is at most 2Δ (see [3]), and this is attained by an inductive ordering of G. We are not aware of other work analyzing colorings of squares of outerplanar graphs.

Zhou, Kanari and Nishizeki [15] gave a polynomial time algorithm to find an optimal coloring of any power of a partial k-tree G, given G. Since outerplanar graphs are partial 2-trees, this solves the coloring problem we consider. For squares of outerplanar graphs, their algorithm has complexity $O(n(\Delta+1)^{2^{37}} + n^3)$, which is infeasible for moderate to large values of Δ and large for small values of Δ. When Δ is constant, one can use the observation of Krumke, Marathe and Ravi [14] that squares of outerplanar graphs have treewidth at most $k \leq 3\Delta - 1$. Thus, one can use efficient ($2^{O(k)}n$ time) algorithms for coloring partial k-trees, obtaining a linear-time algorithm when Δ is constant.

2 Definitions and preliminary results

In this section we give some basic definitions and prove results that will be used later for our main result.

Graph notation. The set $\{1, 2, 3, \ldots\}$ of natural numbers will be denoted by $\mathbb{N}$. By coloring we will always mean vertex coloring. We denote by $\chi(G)$ the chromatic number of G and by $\omega(G)$ the clique number of G. The degree of a vertex u in graph G is denoted by $d_G(u)$. Let $\delta(G)$ ($\Delta(G)$) denote the minimum (maximum) degree of a vertex in G. (When there is no danger of ambiguity, we simply write Δ instead of $\Delta(G)$.) We denote by $N_G(u)$ the *open neighborhood* of u in G, that is the set of all neighbors of u in G, and by $N_G[u]$ the *closed neighborhood* of u in G, that additionally includes u. The distance $d_G(u, v)$ between vertices u and v is the number of edges in the shortest path between them. When the graph in question is clear from context, we omit the subscript in the notation.

For $k \in \mathbb{N}$, the *power graph* G^k is a graph with the same vertex set as G, but where every pair of vertices of distance k or less in G are connected by an edge. In particular G^2 is the graph in which every two vertices with a common neighbor in G are connected with an edge. The closed neighborhood of a vertex u in G^k will be denoted by $N^k_G[u]$, and the degree of vertex u will be denoted by $d_k(u)$.

Tree terminology. The leaves of a tree T will be denoted by $L(T)$. The *diameter* of T is the number of edges in the longest simple path in T and will be denoted by $\mathrm{diam}(T)$. For a tree T with $\mathrm{diam}(T) \geq 1$ we can form the *pruned* tree $\mathrm{pr}(T) = T \setminus L(T)$. A *center* of T is a vertex of distance at most $\lceil \mathrm{diam}(T)/2 \rceil$ from all other vertices of T. A center of T is either unique, or one of two unique adjacent vertices. Clearly, the power graph T^k of a tree T, is only interesting when $k \in \{1, \ldots, \mathrm{diam}(T)\}$. When T is rooted at $r \in V(T)$, the *k-th ancestor*, if it exists, of a vertex u is the vertex on the path from u to r of distance k from u, and is denoted by $a^k_r(u)$. An *ancestor* of u is a vertex of the form $a^k_r(u)$ for some $k \geq 0$. Note that u is viewed as an ancestor of itself. When u is some vertex of T, the distance $d_T(u, r)$ to the root r will be referred to as the *level* of u and denoted by $l(u)$ when there is no danger of ambiguity. A tree is said to be *full* if it contains no degree-two vertices.

Inductiveness. The *inductiveness* or the *degeneracy* of a graph G, denoted by $\mathrm{ind}(G) \in \mathbb{N}$ is defined by

$$\mathrm{ind}(G) = \max_{H \subseteq G} \{\delta(H)\},$$

where H runs through all the induced subgraphs of G. If $k \geq \mathrm{ind}(G)$ then we say that G is *k-inductive*.

Note that for any $u \in V(G)$, the vertex set $N_G[u]$

will induce a clique in G^2, and hence $\omega(G^2), \chi(G^2) \geq \Delta + 1$. Since $\mathrm{ind}(G^2) + 1 \geq \chi(G^2)$, the upper bound of $\mathrm{ind}(G)$ is necessarily an increasing function $f(\Delta)$ of $\Delta \in \mathbb{N}$.

Cutpoints and biconnectivity. We first show that we can assume without loss of generality that G is biconnected. Let G be a graph and $\mathcal{B}$ the set of its biconnected blocks. In the same way that $\omega(G) = \max_{B\in\mathcal{B}}\{\omega(B)\}$ and $\chi(G) = \max_{B\in\mathcal{B}}\{\chi(B)\}$, we have the following.

Lemma 2.1. *For a graph G with a maximum degree Δ and set $\mathcal{B}$ of biconnected blocks we have*

$$\begin{aligned}\omega(G^2) &= \max\{\max_{B\in\mathcal{B}}\{\omega(B^2)\}, \Delta+1\},\\ \chi(G^2) &= \max\{\max_{B\in\mathcal{B}}\{\chi(B^2)\}, \Delta+1\}.\end{aligned}$$

Further, optimal $\chi(B^2)$-colorings of the squares of all the blocks B^2 can be modified to a $\chi(G^2)$-coloring of G^2 in a greedy fashion.

Proof. We use induction on $b \geq 2$, the number of blocks in G.

Let B be a block corresponding to a leaf in the block-cutpoint tree $\mathrm{BC}(G)$. In this case B has a single cut-vertex, which induces a $(\Delta + 1)$-clique in G^2. Further, any legitimate coloring of the other $b - 1$ blocks for G^2 can by a suitable permutation of colors, be extended to that of B and hence all of G. □

Note that Lemma 2.1 provides a way to extend coloring of each block of G to the whole of G, so by Lemma 2.1 we can assume our graphs are biconnected, both when considering clique and chromatic numbers of G^2.

For the inductiveness of G^2, such an extension property as Lemma 2.1, to express $\mathrm{ind}(G^2)$ directly in terms of Δ and the inductiveness of the blocks of G, is not as straightforward. There we need to consider with great care *how* the simplicial vertices of G^2 are chosen.

Note that the biconnected components for outerplanar graphs on three or more vertices are precisely those subgraphs induced by simple cycles [8, p. 240].

Duals of outerplanar graphs. To analyze the inductiveness of an outerplanar graph G, it is useful to consider the weak dual $T^*(G)$.

Lemma 2.2. *Let G be an outerplanar graph with an embedding in the plane. Let G^* be its geometrical dual, and let $u^*_\infty \in V(G^*)$ be the vertex corresponding to the infinite face of G. Then the* weak dual *graph $T^*(G) = G^* \setminus \{u^*_\infty\}$ is a forest which satisfies the following:*

1. *$T^*(G)$ is tree iff G is biconnected.*
2. *$T^*(G)$ has maximum degree at most three, if G is chordal.*

Note that for a biconnected chordal graph G, there is a one-to-one correspondence $u \leftrightarrow u^*$ between the degree-two vertices u of G, and the leaves u^* of $T^*(G)$.

We say that leaves of a tree are *siblings* if they are connected to a common vertex, which we call their *parent*. By Lemma 2.2, $T^*(G)$ for a chordal graph G is a tree of maximum degree three, and hence each of its leaves has at most one sibling.

Strong simplicial vertex orderings of trees. We give a characterization of strong simplicial elimination orderings of the vertices of the k-th power of a tree. Recall the following definition [11, 12].

Definition 2.3. *A vertex u in a graph G is* simplicial *if $N_G[u]$ induces a clique in G. If u is simplicial and $\{N_G[v] : v \in N_G[u]\}$ is linearly ordered by set inclusion then u is* strongly simplicial.

Any power T^k of a tree T is strongly chordal [6, 12], and hence has a strong simplicial elimination ordering of the vertices $V(T) = \{u_1, \ldots, u_n\}$ such that each vertex u_i is strongly simplicial in the subgraph of T^k induced by the previous vertices $u_1, \ldots, u_{i-1}$. Clearly, a vertex of a tree is strongly simplicial if it is a leaf, which gives us a complete description of when exactly an ordering is a strong simplicial ordering of the tree. We give a similarly complete description for higher powers of graphs.

For a tree T we can recursively define $T^{(i)}$ by

$$\begin{aligned}T^{(0)} &= T,\\ T^{(i)} &= \mathrm{pr}(T^{(i-1)}),\end{aligned}$$

as long as $T^{(i-1)}$ has leaves, that is, is neither empty nor one vertex.

With this notation we obtain the following.

Proposition 2.1. *Let T be a tree with* $\mathrm{diam}(T) = d \geq 2$. *For $u \in V(T)$ and $k \in \{1, \ldots, \lceil (d-1)/2 \rceil\}$ the following are equivalent:*

1. *For a center $c \in V(T)$, the vertex $a_c^{k-1}(u)$ is a leaf of $T^{(k-1)}$.*
2. *u is strongly simplicial in T^k.*

This proposition follows from an alternative characterization, based on the following definition.

Definition 2.4. *Let T be a tree and $k \geq 1$. We say that a vertex $u \in V(T)$ is* k-simple *if for some $r \in R_u$, the following statement holds:*

$\mathbf{P}_{u;k}(r)$ *: If the k-th ancestor* $a_r^k(u)$ *does not exist, then* T^k *is a complete graph. If* $a_r^k(u)$ *exists then* $d_T(v, a_r^{k-1}(u)) \leq k-1$ *for all* $v \in D_r[a_r^{k-1}(u)]$.

It can be shown that the truth value of $\mathbf{P}_{u;k}(r)$ is independent of $r \in R_u$.

We omit the somewhat lengthy proof of the following result.

THEOREM 2.1. *For a tree* T *and an integer* $k \geq 1$, *the vertex* $u \in V(T)$ *is k-simple in* T *iff* u *is strongly simplicial in* T^k.

3 Inductiveness of outerplanar graphs

In this section we will derive the optimal bound for the inductiveness of an outerplanar graph G of maximum degree $\Delta \geq 5$. When $\Delta \geq 7$ the inductiveness will also yield an optimal bound for both the clique number and the chromatic number.

We will first assume G to be biconnected, in which case we can assume the vertices to be labeled $\{u_1, \ldots, u_n\}$ in a clockwise order along the infinite face of G. In this way the weak dual $T^*(G)$ is a connected tree.

Consider a maximal chain of consecutive degree-two vertices $(u_i, \ldots, u_{i+\alpha})$ in G, viewed in clockwise order. For any $u \in \{u_i, \ldots, u_{i+\alpha}\}$, the first vertex to the left of u which has degree of three or more, is u_{i-1}. Likewise $u_{i+\alpha+1}$ is the first vertex to right of u which has degree three or more.

Conventions: (i) For each such a degree-two vertex u, then we denote u_{i-1} by u_l (l for "left") and we denote $u_{i+\alpha+1}$ by u_r (r for "right".) (ii) The leaf of $T^*(G)$ corresponding to the face f of G will be denoted by f^* and will be called the *dual vertex* of f. Likewise if a degree-two vertex u of G is on a boundary of a face corresponding to a leaf in $T^*(G)$, then we denote that face by f_u and the corresponding leaf of $T^*(G)$ by f_u^*, and will call it the *dual vertex* of u.

Note that when G is chordal and u is a degree-two vertex, then u_l and u_r are the left and right neighbors of u in G respectively and they are connected.

CLAIM 3.1. *A plane biconnected outerplanar graph* G *of maximum degree* $\Delta \geq 3$ *has at least two degree-two vertices* u *and* v, *such that both* f_u *and* f_v *are leaves in the weak dual tree* $T^*(G)$.

REMARK: Unlike the chordal case, where we have the one-to-one correspondence $u \leftrightarrow u^*$, the correspondence here $u \to f_u^*$ is not unique, since if a face f corresponding to a leaf f^*, is bounded by $u_{i-1}, u_i, \ldots, u_{i+\alpha}, u_{i+\alpha+1}$ where the edge $\{u_{i-1}, u_{i+\alpha+1}\}$ is the only edge not bounding the infinite face of G, then $f_{u_i}^* = f_{u_{i+1}}^* = \cdots = f_{u_{i+\alpha+1}}^*$ all represent the same leaf of $T^*(G)$.

Our first goal in this section is to prove the following theorem.

THEOREM 3.1. *Let* G *be a biconnected outerplanar graph with maximum degree* $\Delta \geq 5$. *Let* f *be a face of* G, *whose dual vertex* f^* *is strongly simplicial in* $T^*(G)^2$.

1. *If* f^* *has no sibling in* $T^*(G)$, *then* f *contains a degree-two vertex on its boundary with at most* $\Delta+1$ *neighbors in* G^2.
2. *If* f^* *has a sibling* g^* *in* $T^*(G)$, *then either* f *or* g *contains a degree-two vertex on their boundary which has at most* $\Delta+1$ *neighbors in* G^2.

In particular we have $\mathrm{ind}(G^2) \leq \Delta + 1$.

Before proving Theorem 3.1, we shall make some observations and prove a lemma.

Recall that since G is a plane graph, then so is the dual tree $T^*(G)$. If there is a leaf f^* of $T^*(G)$ such that f is bounded by a cycle of length five or more in G, then there is a degree-two vertex with both its neighbors of degree two, and hence its degree in G^2 is four or less. We will therefore henceforth assume that each face corresponding to a leaf in $T^*(G)$ is bounded by a cycle of length three or four.

Suppose we have leaves $f_1^*, \ldots, f_p^*$, where $p \geq 3$, which are all siblings, and where their listing is clockwise w.r.t. their common parent in the plane embedding of $T^*(G)$. In this case each degree-two vertex bounding the *internal* faces $f_2, \ldots, f_{p-1}$ has degree at most six in G^2, since its neighbors have at most degree four in G. Hence their degree in G^2 is also at most six. We will therefore henceforth assume that every leaf in $T^*(G)$ has at most one sibling.

Convention: For an arbitrary plane tree T, and a leaf x of T, denote by ${}_lx$ and ${}_rx$ the left and right neighbor leaves of x respectively, in a clockwise preorder traversal of the vertices of T. Denote by $\partial_l(x)$ and $\partial_r(x)$ the distances in T from x to ${}_lx$ and ${}_rx$ respectively.

LEMMA 3.2. *Let* G *be a biconnected outerplanar graph of maximum degree* $\Delta \geq 3$ *and let* u *be a degree-two vertex* u *in* G *such that* f_u^* *is a leaf in* $T^*(G)$. *In this case we have*

$$d_G(u_l) \leq \partial_l(f_u^*) + 2, \tag{3.1}$$

$$d_G(u_r) \leq \partial_r(f_u^*) + 2. \tag{3.2}$$

In particular, $n_l = |N[u_l]| \leq \partial_l(f_u^*) + 3$ *and* $n_r = |N[u_r]| \leq \partial_r(f_u^*) + 3$. *Equality holds in (3.1), and hence also for* n_l, *iff* $u_l = v_r$ *for some degree-two vertex* v

bounding ${}_l f_u^*$. Similarly, equality holds in (3.2), and hence also for n_r, iff $u_r = w_l$ for some degree-two vertex w bounding ${}_r f_u^*$.

Proof. The simple path in $T^*(G)$ from f_u^* to ${}_l f_u^*$ has length at least $d_G(u_l) - 2$. The length is precisely $d_G(u_l) - 2$ iff $u_l = v_r$ for some degree-two vertex on the boundary of ${}_l f_u^*$. In the same way we obtain the result for f_r^*. □

The following proof of Theorem 3.1 consists of dispatching several cases, treating the most involved case last.

Proof. (Theorem 3.1) Let f^* be a leaf which is strongly simplicial in $T^*(G)^2$, and let u be a degree-two vertex on the boundary of the corresponding face f. In this case $f^* = f_u^*$.

If f^* has no sibling, then either u_l or u_r has degree three, say $d_G(u_r) = 3$, in which case $n_r = 4$. If f is bounded by four vertices, then the degree-two neighbor of u_r has at most four neighbors in G^2. If f is bounded by three vertices, then for the unique degree-two vertex u bounding f we have $n_r = 4$, $n_l \leq \Delta + 1$ and $\{u, u_l, u_r\} \subseteq N[u_l] \cap N[u_r]$, and hence by the inclusion/exclusion (*I/E*) principle we obtain

$$\begin{aligned} d_2(u) &= |N[u_l] \cup N[u_r]| - 1 \\ &= (n_l + n_r - 3) - 1 \leq \Delta + 1. \end{aligned} \tag{3.3}$$

If f^* has a sibling g^* (which we can assume is to the right of f^* in the planar embedding of $T^*(G)$), let v be a degree-two vertex on the corresponding face g, and hence in this case $g^* = f_v^*$. Since f^* and g^* are strongly simplicial in $T^*(G)^2$, then by Proposition 2.1 their common parent $f'^* = g'^* = f_u'^* = f_v'^*$ is a leaf in the pruned tree $T^*(G)^{(1)}$, which is proper since $\Delta \geq 5$. All the vertices u_l, u_r, v_l, v_r are on the boundary of f'^*, in this clockwise order. Note that both u_r and v_l are bounding at most three faces f, g and f', and therefore both of them have degree at most four. We now consider some cases.

First case: u_r and v_l are distinct. In this case we have $d_G(u_r) = d_G(v_l) = 3$ and hence $|N[u_r]| = |N[v_l]| = 4$. If either f or g are bounded by four vertices, say f, then the degree-two neighbor vertex of u_r has four neighbors in G^2. If both f and g are bounded by exactly three vertices, u_l, u, u_r and v_l, v, v_r respectively, then since $\{u, u_l, u_r\} \subseteq N[u_l] \cap N[u_r]$ we obtain (3.3) by the I/E principle, as for the case where f^* had no sibling.

Second case: $u_r = v_l$. In this case we have $d_G(u_r) = d_G(v_l) = 4$. If either f or g are bounded by four vertices, say f, then the degree-two neighbor of u_r has at most five neighbors in G^2. We will for the rest of this proof assume that both f and g are bounded by exactly three vertices. Since the degree of f'^* in $T^*(G)$ is three, there are exactly three edges on the boundary of f' which do not face the infinite face, two of them being $\{u_l, u_r\}$ and $\{v_l, v_r\}$. If f' is bounded by four or more vertices, then there is an edge with either u_l or v_r as an endvertex, bounding f' and the infinite face, say u_l. In this case $d_G(u_r) = 4$ and $d_G(u_l) = 3$ and hence u has at most five neighbors in G^2. Therefore we can assume f' to be bounded by exactly three vertices, which in this case must consist of u_l, $u_r = v_l$ and v_r. Since f'^* is a leaf of $T^*(G)^{(1)}$, then by Lemma 3.2 we have $n_r \leq 5$, and further we have $n_l \leq \Delta + 1$. Since now $N[u_l] \cap N[u_r] = \{u, u_l, u_r, v_r\}$, we finally have by the I/E principle that

$$d_2(u) = |N[u_l] \cup N[u_r]| - 1 = (n_l + n_r - 4) - 1 \leq \Delta + 1,$$

which completes the proof of the theorem. □

When $\Delta \geq 7$ we can obtain sharper results. The underlying reason for this is that when $\Delta \geq 7$, then $T^*(G)$ has a path of length five or more, and hence the pruned tree $T^*(G)^{(2)}$ is proper with leaves.

Let f^* be a leaf in $T^*(G)$. As mentioned, we can assume that corresponding face f is bounded by three or four vertices. We can further assume that f^* has at most one sibling. The next lemma shows that we can make the same assumption for the parent f'^* of f^*. Recall that for a group of three or more siblings, a leaf is *internal* if it has at least one sibling to its left and another to its right, when viewed from the their common parent in the plane embedding of $T^*(G)$.

Lemma 3.3. *Assume G is a biconnected outerplanar graph with maximum degree $\Delta \geq 7$. Let f^* be a leaf which is strongly simplicial in $T^*(G)^3$. Assume that its parent f'^* is an internal one among its three or more siblings in $T^*(G)^{(1)}$. Then for any degree-two vertex u bounding f we have $|N[u_l]| = n_l \leq 7$ and $|N[u_r] = n_r \leq 5$, or vice versa.*

Proof. By Proposition 2.1 the common parent f''^* of the siblings is a leaf in $T^*(G)^{(1)}$. Hence if f^* has a sibling, then $\partial_l(f_u^*) \leq 4$ and $\partial_r(f_u^*) \leq 2$ or vice versa, and the lemma follows from Lemma 3.2. If f^* has no sibling we have further that either $d_G(u_l) = 3$ or $d_G(u_r) = 3$. □

Let f^* be strongly simplicial in $T^*(G)^3$. By Lemma 3.3 we can assume $n_l \leq 7$ which is the same as $d_G(u_l) \leq 6$. Treating $d_G(u_l)$ as the maximum degree, we can now apply exactly the same arguments as in the proof of Theorem 3.1 and obtain the following observation.

Observation 3.4. *Let G be a biconnected outerplanar graph with maximum degree $\Delta \geq 7$. Let f be a face of G, whose dual vertex f^* is strongly simplicial in $T^*(G)^3$, and assume further that its parent f'^* is an internal sibling of three and more siblings in $T^*(G)^{(1)}$. Then we have the following.*

1. *If f^* has no sibling in $T^*(G)$, then f contains a degree-two vertex on its boundary with at most seven neighbors in G^2.*
2. *If f^* has a sibling g^* in $T^*(G)$, then either f or g contains a degree-two vertex on their boundary which has at most seven neighbors in G^2.*

When consider the inductiveness of G^2 for a biconnected graph G of maximum degree $\Delta \geq 7$, we can by Observation 3.4 assume further that every leaf f'^* has at most one sibling in $T^*(G)^{(1)}$, in addition to assuming that each leaf face f is bounded by three or four vertices and that f^* has at most one sibling. Call such a G *restricted*, if it satisfies all these mentioned assumptions. In this case, we prove the following theorem.

Theorem 3.2. *Let G be a restricted biconnected outerplanar graph with maximum degree $\Delta \geq 7$. Let f be a face, whose dual vertex f^* in $T^*(G)$ is a leaf which is strongly simplicial in $T^*(G)^3$. Then one of the following holds:*

1. *f^* has a sibling g^* and either f or g contains a degree-two vertex on their bounding cycle, which has at most seven neighbors in G^2.*
2. *f^* has no sibling and its parent f'^* is bounded by three edges, in which case f contains a degree-two vertex on its bounding cycle with at most Δ neighbors in G^2.*
3. *f^* has no sibling and its parent f'^* is bounded by four or more vertices, in which case either f or f' contains a degree-two vertex with at most seven neighbors in G^2.*

Proof. Let u be a degree-two vertex in G, whose dual vertex $f_u^* = f^*$ is a leaf which is strongly simplicial in $T^*(G)^3$. By Proposition 2.1 the parent f'^* of f^* in $T^*(G)$ is a leaf in the pruned tree $T^*(G)^{(1)}$, which we can by Observation 3.4 assume to be one of at most two siblings. If f'^* has a sibling then assume it to be to the right of f'^* viewed from their common parent in the plane embedding of $T^*(G)$.

Assume f^* has a sibling $g^* = {}_rf^*$ to its right, which then is also strongly simplicial in $T^*(G)^3$. Since $\partial_l(g^*) = 2$ and $\partial_r(g^*) \leq 4$, we have by Lemma 3.2 that the only possibility of g not to contain a degree-two vertex with at most seven neighbors in G^2, is for g to be bounded by exactly three vertices v, v_l and v_r, where $v_l = u_r$, $d_G(v_l) = 4$ and $d_G(v_r) = 6$, where u_l and v_r are not connected, since f' is bounded by at least four vertices. In this case, however, $d_G(u_l) = 3$ must hold, since the degree of f'^* in $T^*(G)$ is exactly three and hence there are precisely three edges on the boundary of f' which do not face the infinite face. By Lemma 3.2 we have $d_G(u_r) \leq 4$, and hence u has at most six neighbors in G^2. This proves the first part of three.

Assume next that f^* has no sibling. In this case either $d_G(u_l)$ or $d_G(u_r)$ is three, say $d_G(u_r) = 3$. If f is bounded by four vertices, then the degree-two neighbor of u_r has at most four neighbors in G^2. Otherwise f is bounded by three vertices u, u_l and u_r. The only possibility for u to have more than Δ neighbors in G^2, is for u_l and u_r to have only u as a common neighbor and $d_G(u_l) = \Delta$. In this case f' is bounded by at least four vertices, and there is a degree-two neighbor w of u_r on the boundary of f', whose other neighbor (to its right) is either another degree-two vertex, in which case w has at most four neighbors in G^2, or a vertex u' which is a neighbor of u_l, where w, u', u_l, u_r are precisely the bounding vertices of f'. Since f^* is strongly simplicial in $T^*(G)^3$, then by Proposition 2.1 the parent f''^* of f' is a leaf in the pruned tree $T^*(G)^{(2)}$. From this we see that $d_G(u') \leq 6$. In addition f^* has no siblings and therefore we have further that $d_G(u') \leq 5$. Since $u' \in N[u_r] \cap N[v_l]|$,

$$d_2(w) = |N[u_r] \cup N[u']| - 1 \leq 4 + 5 - 1 - 1 = 7,$$

showing that f' contains a degree-two vertex on its boundary with at most $\Delta \geq 7$ neighbors. This proves the theorem. □

By Observation 3.4 and Theorem 3.2 we obtain the following corollary.

Corollary 3.1. *For a biconnected outerplanar graph G with maximum degree $\Delta \geq 7$, we have* $\mathrm{ind}(G^2) = \Delta$.

So far we have only discussed the case where G is a biconnected and outerplanar, and we have given a complete description on how to locate the simplicial vertices of G^2, in terms of the corresponding weak dual tree $T^*(G)$ and the strongly simplicial vertices of its second or third power, depending whether we consider the case $\Delta \geq 5$ or $\Delta \geq 7$.

For a biconnected outerplanar graph G with maximum degree $\Delta \geq 5$ we have that the proper tree $T^*(G)^{(1)}$ has at least two leaves. We have derived the existence of degree-two vertices with at most $\Delta + 1$ neighbors in G from just one leaf of $T^*(G)^{(1)}$. We can of course obtain *another* such degree-two vertex by working from another leaf of $T^*(G)^{(1)}$. Likewise, if $\Delta \geq 7$

we have that $T^*(G)^{(2)}$ is a proper tree with at least two leaves.

From the proof of Theorem 3.2 we note that if f^* is a leaf of $T^*(G)$ satisfying condition 3, where its parent f'^* is the face containing the degree-two vertex w with at most seven neighbors in G^2, then the other degree-two vertex with at most Δ neighbors in G^2 is clearly of distance three or more from w in G, since the degree two vertices of distance two form w are on the boundary of faces that correspond to leaves of $T^*(G)$ which are descendants of f''^*, the grandparent of f^*. In this case G^2 has two simplicial degree-two vertices of distance three or more from each other.

Assume that all the simplicial degree-two vertices of G^2, where G has a maximum degree $\Delta \geq 5$, are of distance two from each other in G. If they are more than three, then clearly for any vertex x of G there is one simplicial vertex of in G^2 of distance two or more from x in G. If however, there are only two simplicial vertices in G^2 and they are of distance two from each other in G, then from the above, they must each be on the boundary of distinct faces f and h where f^* and h^* are strongly simplicial in $T^*(G)^3$ (or in $T^*(G)^2$ in case $\Delta \leq 6$.) If now f^* and h^* are endpoints of a longest path in $T^*(G)$, then both f^* and h^* are strongly simplicial in $T^*(G)^k$ for any $k \geq 2$, in particular for $k = 2, 3$. By considering their possible siblings we may assume that these faces f and h contain the two degree-two vertices x and y which are simplicial in G^2, that is, have at most Δ neighbors in G^2 (or $\Delta+1$ neighbors in the case $\Delta \leq 6$.) Since there is a path in $T^*(G)$ of length $\Delta - 2$, the distance between f^* and h^* is at least $\Delta - 2$. If z is the common neighbor of x and y in G, then there is a path between f^* and h^* of length $d_G(z) - 2$, but by the our choice of f^* and h^* we have $d_G(z) = \Delta$.

From this we can deduce the following lemma.

Lemma 3.5. *In a biconnected outerplanar graph G with maximum degree $\Delta \geq 5$, we have either of the two conditions.*

1. *For any vertex x there is a simplicial vertex u of G^2 of distance two or more from x in G.*
2. *There is a vertex x such that the only two simplicial vertices of G^2 have x as a common neighbor in G which is of degree Δ in G.*

Let now G be a general outerplanar graph of maximum degree $\Delta \geq 5$, and let B be a block which corresponds to a leaf of the block-cutpoint tree $\mathrm{BC}(G)$. Here B contains only one cut-vertex x.

If there is a simplicial vertex u of B^2 of distance two or more from x, then u is also a simplicial vertex of G^2 with at most Δ neighbors in G^2 (or $\Delta + 1$ neighbors if $\Delta \leq 6$.)

If the cut-vertex x is the neighbor of both the simplicial vertices x and y of B, then $d_B(x), d_B(y) \leq \Delta(B)$ (or $\leq \Delta(B) + 1$ if $\Delta(B) \leq 6$,) and hence both x and y have at most Δ neighbors in G^2, since $N_G[x]$ induces a $(\Delta+1)$-clique in G, in the same way as $N_B[x]$ induces a clique of size $\Delta(B) + 1$ in B.

From this we obtain the main result of the paper.

Theorem 3.3. *For an outerplanar graph G of maximum degree $\Delta \geq 5$, we have* $\mathrm{ind}(G^2) \leq \Delta+1$. *If further, $\Delta \geq 7$, then* $\mathrm{ind}(G^2) = \Delta$.

Choosability and algorithmic concerns. As mentioned in the introduction, the bound on the inductiveness of Theorem 3.3 implies that Greedy finds an optimal coloring of squares of outerplanar graphs of degree $\Delta \geq 7$. When $\Delta < 6$, we can also obtain an efficient time algorithm from the observation of Krumke, Marathe and Ravi [14] that squares of outerplanar graphs have treewidth at most $3\Delta - 1$. This allows for the use of $2^{O(k)}n$ time algorithm for coloring graphs of bounded treewidth.

Theorem 3.4. *There is a linear time algorithm to color squares of outerplanar graphs.*

We conclude this section with an application of our results on inductiveness to choosability.

Definition 3.6. *A graph G is k-*choosable*, if for every collection or lists $\{S_v : v \in V(G)\}$ of colors, $S_v \subseteq \{1, 2, 3, \ldots\}$ where $|S_v| = k$ for every $v \in V(G)$, there is a color assignment $c : V(G) \to \bigcup_{v \in V(G)} S_v$, such that*

- *$c(v) \in S_v$ for each $v \in V(G)$, and*
- *if $c(v) = c(u)$ then v and u are not neighbors in G.*

The minimum such k is called the choosability *or the* list-chromatic number *of G, and is denoted by* $\mathrm{ch}(G)$.

Note that if a graph is k-choosable, then it is k-colorable. Also, by an easy induction, we see that if a graph is k-inductive then it is $(k+1)$-choosable. For any graph G we therefore have $\chi(G) \leq \mathrm{ch}(G) \leq \mathrm{ind}(G) + 1$ and hence the following.

Corollary 3.2. *For any outerplanar graph with maximum degree $\Delta \geq 7$, we have* $\mathrm{ch}(G^2) = \Delta + 1$.

4 Characterizations for the small degree case

We conclude by characterizing bounds on the clique number, chromatic number and inductiveness for the squares of outerplanar graphs of maximum degree $\Delta \in \{2, 3, 4, 5, 6\}$. In the case of clique and chromatic number, we may assume by Lemma 2.1 that the graphs are biconnected. We start with the chordal case.

Conventions: (i) Let G be a given biconnected outerplanar on n vertices of maximum degree Δ, with a fixed planar embedding. The graph obtain from G by connecting an additional vertex to each pair of endvertices of an edge bounding the infinite face, will be denoted by $\widehat{G}$. Clearly $\widehat{G}$ will be an outerplanar graph on $2n$ vertices of maximum degree $\Delta + 2$. (ii) By the *rigid n-ladder* or just the *rigid ladder* RL_n on $n = 2k$ vertices we will mean the graph given by

$$\begin{aligned} V(RL_n) &= \{u_1, \ldots, u_k\} \cup \{v_1, \ldots, v_k\}, \\ E(RL_n) &= \{\{u_i, v_i\}, \{u_i, u_{i+1}\}, \{u_i, v_{i+1}\}, \{v_i, v_{i+1}\} : \\ &\quad i \in \{1, \ldots, n-1\}\} \cup \{\{u_k, v_k\}\}. \end{aligned}$$

For odd n, the graph RL_n will mean $RL_{n+1} - u_{(n+1)/2}$.
(iii) Let $F_4 = \widehat{K_3}$, $F_5 = \widehat{RL_4}$, and $F_6 = \widehat{\widehat{K_3}}$.

We characterize exactly the chromatic and clique numbers of squares of chordal outerplanar graphs in terms of forbidden subgraphs F_Δ.

Theorem 4.1. *Let G be a chordal outerplanar graph of maximum degree Δ. Then,*

1. $\chi(G^2) = \Delta + 1$, *unless* $\Delta \in \{4, 6\}$ *and* $F_\Delta \subseteq G$ *in which case* $\chi(G^2) = \Delta + 2$.
2. $\omega(G^2) = \Delta + 1$, *unless* $\Delta = 4$ *and* $F_\Delta \subseteq G$ *in which case* $\omega(G^2) = \Delta + 2$.

We consider each maximum degree Δ separately (with $\Delta > 6$ treated in the preceding section.) The case $\Delta = 2$ is trivial since there is only one chordal outerplanar biconnected graph $G = K_3$. The case $\Delta \leq 3$ is easy, since there are only three biconnected chordal outerplanar graphs with $\Delta \leq 3$: $RL_2 = P_2$, the 2-path, $RL_3 = C_3 = K_3$, the 3-cycle, and RL_4 is the 4-cycle with one diagonal. From this we deduce the following tree-like structure of G in this case.

Lemma 4.1. *Let G be a chordal outerplanar graph of maximum degree $\Delta \leq 3$. Then the blocks of G are among $\{RL_2, RL_3, RL_4\}$, where any two blocks from $\{RL_3, RL_4\}$ are separated by at least one RL_2 block.*

Considering the blocks of G that represent the leaves in the block cut-point tree $\mathrm{BC}(G)$ we obtain from the structure given in Lemma 4.1 the following.

Observation 4.2. *For a chordal outerplanar graph G with $\Delta \in \{2, 3\}$, we have $\omega(G^2) = \chi(G^2) = \mathrm{ind}(G^2) + 1 = \Delta + 1$.*

The case $\Delta = 4$ is more interesting, since it is the first case involving a "forbidden subgraph" condition for both the clique and the chromatic number of G^2. By considering the removal of a degree-two vertex from G, we obtain the following by induction on n, the number of vertices of G.

Lemma 4.3. *A graph G is a biconnected chordal outerplanar graph with $\Delta = 4$ if, and only if, $G \in \{F_4\} \cup \{RL_n : n \geq 5\}$.*

Proof. Clearly each graph in $\{F_4\} \cup \{RL_n : n \geq 5\}$ is biconnected and outerplanar. Conversely, let G be a biconnected outerplanar graph on $n \geq 5$ vertices with maximum degree four. By removing a vertex u of degree two from G, we obtain a biconnected outerplanar graph $G-u$ with $\Delta(G-u) \in \{3, 4\}$, and hence equal to RL_4 or, by induction, from the set $\{F_4\} \cup \{RL_n : n \geq 5\}$. Since G is of maximum degree $\Delta = 4$ it is impossible that $G - u = F_4$. For the same reason if $G - u = RL_4$, then $G = RL_5$. Also, $G-u = RL_5$ only when $G \in \{RL_6, F_4\}$, and lastly if $G - u = RL_n$ for some $n \geq 6$, then $G = RL_{n+1}$ must hold, thereby proving the lemma. $\square$

Note that for $G = F_4$ we have $G^2 = K_6$. Hence both $\omega(G^2)$ and $\chi(G^2)$ equal $\Delta + 2 = 6$, while $\mathrm{ind}(G^2) = 5$.

Observe that $\mathrm{ind}(RL_n^2) = 4$, for any $n \geq 5$, since removing the last vertex in the square graph leaves the graph RL_{n-1}^2. Thus, $\omega(RL_n^2) = \chi(RL_n^2) = 5$. By Lemma 2.1 we have proved Theorem 4.1 for $\Delta = 4$.

The last two cases to consider are $\Delta \in \{5, 6\}$. For these we provide optimal bounds of $\omega(G^2)$ and $\chi(G^2)$ in terms of the maximum degree Δ.

The proof of the following lemma is omitted.

Lemma 4.4. *Let G be a chordal biconnected outerplanar graph with such that the dual tree of G is full. If $\Delta \in \{5, 6\}$, then $G = F_\Delta$.*

Note that F_5 has $\Delta = 5$, and there are exactly two disjoint pair of vertices of distance 3 from each other. Each pair can form a monochromatic pair in the square of F_5, and the remaining vertices can receive a unique color, which shows that $\chi(F_5^2) = 6$. We shall use the following definition.

Definition 4.5. *Let G be a graph. Call a subgraph $H \subseteq G$ on h vertices an* h-separator*, or just a* separator *if it induces a clique in G^2 whose removal breaks G^2 into disconnected components.*

The following lemma shows that it suffices to bound the chromatic number for graphs without separators.

Lemma 4.6. *If a graph G has a separator H with $G = G' \cup G''$ and $H = G' \cap G''$, then $\chi(G^2) = \max\{\chi(G'^2), \chi(G''^2)\}$.*

Proof. Since $G^2 = G'^2 \cup G''^2$ and $G'^2 \cap G''^2 = H^2$, which is a clique, we have the lemma. $\square$

We note that if the dual tree $T^*(G)$ is not full, then G contains a $(\Delta + 1)$-separator. This separator is in fact induced by the neighborhood of a single vertex. By Lemma 4.4 we have proved Theorem 4.1 for $\Delta = 5, 6$.

For the clique number we have the following.

LEMMA 4.7. *For an outerplanar graph G with $\Delta \geq 5$, we have $\omega(G^2) = \Delta + 1$. Further, any clique with $\Delta + 1$ vertices is the closed neighborhood of some vertex.*

Proof. We show that the only way to form a clique on 6 vertices in an outerplanar graph is via the closed neighborhood of a vertex.

Consider an induced subgraph S_t of G with $t + 1$ vertices: a vertex u and its neighbors $u_1, u_2, \ldots, u_t$ in a clockwise order in the plane embedding of G. Then only adjacent pairs u_i and u_{i+1}, for $i = 1, \ldots, t-1$, may be connected by an edge. Consider now a vertex w that is not a neighbor of u. Then, w can be adjacent to at most two neighbors of u and only consecutive ones, by the outerplanarity property. Then, if $t \geq 5$, w cannot be adjacent to both one of u_1 and u_2 and to one of u_{t-1} and u_t. Thus, it must be of distance at least 3 from either u_1 or u_t. Hence, $S \cup \{w\}$ is not a clique.

Consider instead when $t = 4$ and we have two vertices w_1 and w_2 that are non-neighbors of u. In order to be of distance at most 2 from both u_1 and u_4, a vertex must be adjacent to u_2 and u_3. But, in an outerplanar graph, not both w_1 and w_2 can be so. Hence, $S_4 \cup \{w_1, w_2\}$ does not form a clique.

Finally, suppose an induced subgraph H of maximum degree three induces a 6-clique in G. By Lemma 2.1 we can assume H to be biconnected. There can be at most two chords in H and they must be disjoint since $\Delta(H) = 3$. Then there are two vertices in H of degree two that are of distance 3 in H. Further, since all vertices of H lie on the outer face, there can be no vertex outside H connecting them. Hence, the lemma. $\square$

Using F_4, F_5 and F_6 as building blocks, we can obtain the following (proof omitted).

OBSERVATION 4.8. *For each $\Delta \in \{4, 5, 6\}$, there are infinitely many biconnected chordal outerplanar graphs G of maximum degree Δ with* $\text{ind}(G^2) = \Delta + 1$.

Non-chordal graphs of small degree. Non-chordal graphs become quickly harder to characterize. We illustrate first the instances of non-chordal graphs that give higher values than for chordal graphs.

For $\Delta = 2$, we give a complete characterization, albeit not very compact. Let P_k (C_k) be the path (cycle) on k vertices, respectively. It can be verified that

$$\chi(G^2) = \begin{cases} 3 & G = P_k \cup C_{3k} \\ 4 & G = C_{3k+1} \cup C_{3k+2} \setminus C_5 \\ 5 & G = C_5 \end{cases}$$

$$\omega(G^2) = \begin{cases} 3 & \text{otherwise} \\ 4 & G = C_4 \\ 5 & G = C_5 \end{cases}$$

$$\text{ind}(G^2) = \begin{cases} 2 & G = P_k \cup C_3 \\ 3 & G = C_4 \\ 4 & G = C_k, k \geq 5. \end{cases}$$

Further, Greedy obtains an optimal coloring even when inductiveness is not a tight bound on the chromatic number.

For $\Delta = 3$, consider the graph G consisting of C_5 with an additional chord. Then, $\text{ind}(G^2) + 1 = \omega(G^2) = \chi(G^2) = \Delta + 2 = 5$. For $\Delta = 4$, the graph $G = \widehat{C_4}$ satisfies $\text{ind}(G^2) = \Delta + 2 = 6$. Other lower bounds follow from the chordal case.

We now turn to upper bounds in the non-chordal case. Recall the upper bounds on ind and ω given by Theorem 3.1 and Lemma 4.7 for $\Delta \geq 5$. The following two results fill all the remaining gaps, but one.

LEMMA 4.9. *If G is an outerplanar graph with maximum degree $\Delta = 4$, then $\chi(G^2) \leq 6$.*

Proof. By Lemma 2.1 we may assume G to be biconnected. By induction on n, we may assume that each face f corresponding to a leaf f^* of the dual tree $T^*(G)$ is bounded by exactly three vertices, since otherwise there is a degree-two vertex on f with at most $\Delta + 1 = 5$ neighbors in G^2. Let u, v, w be the three vertices bounding f, where u is of degree 2. Then we can further assume that we always have $d_G(v) = d_G(w) = 4$, since otherwise u has at most five neighbors in G^2 also in this case. Now, by contracting u, v, w into a single vertex, the same assumptions hold for the resulting graph. Hence, by induction on $n = |V(G)|$, we see that G must have the form $G = \widehat{C_n}$ for some $n \geq 3$.

Since $\widehat{C_n} \subseteq C_{2n}^2$, we have $\widehat{C_n}^2 \subseteq C_{2n}^4$, which can be colored cyclically in a greedy fashion by colors 1 through 5, or 1 through 6. Hence, $\chi(\widehat{C_n}^2) \leq 6$, which completes the proof. $\square$

The proof shows that when G is biconnected with $\Delta = 4$, then G^2 is 5-inductive unless $G = \widehat{C_n}$, for some $n \geq 3$.

LEMMA 4.10. *If G is an outerplanar graph with maximum degree $\Delta = 3$ or $\Delta = 4$, then* $\text{ind}(G^2) \leq 2\Delta - 2$.

Proof. Proof by induction on the number n of vertices. The base case is immediate. We show that there exists a degree-two vertex with at most $2\Delta - 2$ neighbors in G^2. Contracting an edge incident to this vertex yields a graph on $n-1$ vertex of maximum degree at most Δ. By the inductive hypothesis, this graph is $2\Delta-2$-inductive, thus yielding the same for G^2.

Consider a face f of G whose dual vertex is a leaf of $T^*(G)$. If f is bounded by a 5-cycle or larger, then there is a degree-two vertex whose neighbors are also of degree 2. Hence, the degree of that vertex in G^2 is $4 \leq 2\Delta - 2$. If f is bounded by a 4-cycle, then there are two adjacent degree-two vertices on the cycle, and both of them have $\Delta + 1 < 2\Delta - 2$ neighbors in G^2. Finally, if f is bounded by a 3-cycle, then the degree-two vertex on the cycle has at most $2\Delta - 2$ neighbors in G^2. □

As a corollary, $\omega(G^2) \leq \chi(G^2) \leq \Delta+2 = 5$ when $\Delta = 3$.

The case of the chromatic number of squares of non-chordal outerplanar graphs when $\Delta = 5$ remains open; we conjecture that it is always $\Delta + 1 = 6$.

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