Functions and Convergence.

Definition. A piecewise continuous function $f(x)$ defined on an interval I is *bounded* (or L^∞) on I if there is a number $M > 0$ such that $|f(x)| \leq M$ for all $x \in I$. The L^∞-norm of a function $f(x)$ is defined by

$$\|f\|_\infty = \sup\{|f(x)| \colon x \in I\}.$$

Definition. A piecewise continuous function $f(x)$ defined on an interval I is *integrable* (or *of class* L^1 or simply L^1) on I if the integral

$$\int_I |f(x)|\, dx < \infty.$$

The L^1-*norm* of a function $f(x)$ is defined by

$$\|f\|_1 = \int_I |f(x)|\, dx.$$

Theorem. Let $f(x)$ be L^1 on $\mathbf{R}$, and let $\epsilon > 0$ be given. Then there exists a number R such that if $g(x) = f(x)\,\mathbf{1}_{[-R,R]}(x)$ then

$$\int_{-\infty}^{\infty} |f(x) - g(x)|\, dx = \|f - g\|_1 < \epsilon.$$

Definition. A piecewise continuous function $f(x)$ defined on an interval I is square-integrable (or of class L^2 or simply L^2) on I if the integral

$$\int_I |f(x)|^2 \, dx$$

is finite. The L^2-*norm* of a function $f(x)$ is defined by

$$\|f\|_2 = (\int_I |f(x)|^2 \, dx)^{1/2}.$$

Theorem. (Cauchy-Schwarz Inequality) Let $f(x)$ and $g(x)$ be L^2 on the interval I. Then

$$|\int_I f(x)\, g(x) \, dx| \le \|f\|_2 \, \|g\|_2.$$

Theorem. (Minkowski's Inequality) Let $f(x)$ and $g(x)$ be L^2 on the interval I. Then

$$\|f + g\|_2 \le \|f\|_2 + \|g\|_2.$$

Theorem. Let $f(x)$ be L^2 on $\mathbf{R}$, and let $\epsilon > 0$ be given. Then there exists a number R such that if $g(x) = f(x)\, \mathbf{1}_{[-R,R]}(x)$, then

$$\int_{-\infty}^{\infty} |f(x) - g(x)|^2 \, dx = \|f - g\|_2^2 < \epsilon.$$

Definition. Given $n \in \mathbf{N}$, we say that a function $f(x)$ defined on an interval I is C^n *on* I if it is n-times continuously differentiable on I. C^0 on I means that $f(x)$ is continuous on I. $f(x)$ is C^∞ on I if it is C^n on I for every $n \in \mathbf{N}$.

We say that $f(x)$ is C_c^n on I if it is C^n on I and compactly supported, C_c^0 on I if it is C^0 on I and compactly supported, and C_c^∞ on I if it is C^∞ on I and compactly supported.

Theorem. Suppose that I is a finite interval. Then
(a) $L^\infty(I) \subseteq L^1(I)$
(b) $L^1(I) \nsubseteq L^\infty(I)$
(c) $L^\infty(I) \subseteq L^2(I)$
(d) $L^2(I) \nsubseteq L^\infty(I)$
(e) $L^2(I) \subseteq L^1(I)$
(f) $L^1(I) \nsubseteq L^2(I)$

Suppose that I is an arbitrary interval. Then
(a) $L^\infty(I) \nsubseteq L^1(I)$
(b) $L^\infty(I) \nsubseteq L^2(I)$
(c) $L^2(I) \nsubseteq L^1(I)$
(d) $L^1(I) \nsubseteq L^2(I)$
(e) $L^\infty(I) \cap L^1(I) \subseteq L^2(I)$

Convergence of Sequences and Series of Functions.

Definition. The sequence $\{f_n(x)\}_{n\in\mathbf{N}}$ on I *converges pointwise* to a function $f(x)$ if for each $x_0 \in I$, the numerical sequence $\{f_n(x_0)\}_{n\in\mathbf{N}}$ converges to $f(x_0)$. We write $f_n(x) \to f(x)$ pointwise on I, as $n \to \infty$. The series $\sum_{n=1}^{\infty} f_n(x) = f(x)$ pointwise on an interval I if for each $x_0 \in I$, $\sum_{n=1}^{\infty} f_n(x_0) = f(x_0)$.

Definition. The sequence $\{f_n(x)\}_{n\in\mathbf{N}}$ *converges in L^∞ (or uniformly) on I to $f(x)$* if

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0$$

or $\lim_{n\to\infty} \|f_n - f\|_\infty$. The series $\sum_{n=1}^{\infty} f_n(x) = f(x)$ uniformly on I if the sequence of partial sums $s_N(x) = \sum_{n=1}^{N} f_n(x)$ converges in L^∞ to $f(x)$ on I.

Definition. The sequence $\{f_n(x)\}_{n\in\mathbf{N}}$ *converges in* $L^1(I)$ *to the function* $f(x)$ if

$$\lim_{n\to\infty} \int_I |f_n(x) - f(x)|\, dx = 0$$

or $\lim_{n\to\infty} \|f_n - f\|_1 = 0$. The series $\sum_{n=1}^{\infty} f_n(x) = f(x)$ in L^1 on I if the sequence of partial sums $s_N(x) = \sum_{n=1}^{N} f_n(x)$ converges in L^1 to $f(x)$.

Definition. The sequence $\{f_n(x)\}_{n\in\mathbf{N}}$ *converges in* $L^2(I)$ *to the function* $f(x)$ if

$$\lim_{n\to\infty} \int_I |f_n(x) - f(x)|^2\, dx = 0$$

or $\lim_{n\to\infty} \|f_n - f\|_2 = 0$. The series $\sum_{n=1}^{\infty} f_n(x) = f(x)$ in L^2 on I if the sequence of partial sums $s_N(x) = \sum_{n=1}^{N} f_n(x)$ converges in L^2 to $f(x)$.

Theorem. Suppose that I is a finite interval. Then
(a) Pointwise convergence $\not\Rightarrow$ L^∞, L^1, or L^2 convergence
(b) L^∞ convergence $\Rightarrow$ pointwise, L^1, and L^2 convergence
(c) Pointwise, L^1, or L^2 convergence $\not\Rightarrow$ L^∞ convergence
(d) L^1 convergence $\not\Rightarrow$ pointwise, L^2 or L^∞ convergence
(e) L^2 convergence $\Rightarrow$ L^1 convergence
(f) L^2 convergence $\not\Rightarrow$ pointwise or L^∞ convergence

Suppose that I is an arbitrary interval. Then
(a) L^∞ convergence $\Rightarrow$ pointwise convergence
(b) L^∞ convergence $\not\Rightarrow$ L^1 or L^2 convergence
(c) L^2 convergence $\not\Rightarrow$ L^1 convergence