Vanishing Moments.

Relation to Smoothness.

Theorem. Suppose that $\{\psi_{j,k}(x)\}_{j,k\in\mathbf{Z}}$ is an orthogonal system on $\mathbf{R}$ and that $\psi(x)$ and $\widehat{\psi}(\gamma)$ are both L^1 on $\mathbf{R}$. Then $\int_{\mathbf{R}} \psi(x)\,dx = 0$.

Theorem. Let $\psi(x)$ be such that for some $N \in \mathbf{N}$, both $x^N\psi(x)$ and $\gamma^{N+1}\widehat{\psi}(\gamma)$ are in $L^1(\mathbf{R})$. If $\{\psi_{j,k}(x)\}_{j,k\in\mathbf{Z}}$ is an orthogonal system on $\mathbf{R}$, then $\int_{\mathbf{R}} x^m\,\psi(x)\,dx = 0$ for $0 \leq m \leq N$.

Relation to approximation of smooth functions.

Theorem. Given $N \in \mathbf{N}$, assume that the function $f \in C^N(\mathbf{R})$, and that $f^{(N)} \in L^\infty(\mathbf{R})$. Assume that the function $\psi(x)$ has compact support, that $\int_{\mathbf{R}} x^m\, \psi(x)\, dx = 0$, for $0 \leq m \leq N-1$ and that $\int_{\mathbf{R}} |\psi_{j,k}(x)|^2\, dx = 1$ for all j, $k \in \mathbf{Z}$. Then there is a constant $C > 0$ depending only on N and $f(x)$ such that for every j, $k \in \mathbf{Z}$,

$$|\langle f, \psi_{j,k}\rangle| \leq C 2^{-jN}\, 2^{-j/2}.$$

Reproduction of polynomials.

Theorem. Let $\varphi(x)$ be a compactly supported scaling function associated with an MRA, and let $\psi(x)$ be the wavelet. If $\psi(x)$ has N vanishing moments, then for each integer $0 \leq k \leq N-1$, there are coefficients $\{q_{k,n}\}_{n\in\mathbf{Z}}$ such that

$$\sum_n q_{k,n}\, \varphi(x+n) = x^k.$$

Equivalent conditions for vanishing moments.

Theorem. Let $\varphi(x)$ be a compactly supported scaling function associated with an MRA with finite scaling filter $h(n)$. Let $\psi(x)$ be the corresponding wavelet. Then for each $N \in \mathbf{N}$, the following are equivalent.

(a) $\displaystyle\int_{\mathbf{R}} x^k \, \psi(x) \, dx = 0$ for $0 \le k \le N-1$.

(b) $m_0^{(k)}(1/2) = 0$, for $0 \le k \le N-1$.

(c) $m_0(\gamma)$ can be factored as

$$m_0(\gamma) = \left(\frac{1 + e^{-2\pi i \gamma}}{2}\right)^N \mathcal{L}(\gamma),$$

for some period 1 trigonometric polynomial $\mathcal{L}(\gamma)$.

(d) $\displaystyle\sum_n h(n) \, (-1)^n \, n^k = 0$ for $0 \le k \le N-1$.

The Daubechies Polynomials.

(1) We want to construct a trig polynomial $m_0(\gamma) = \frac{1}{\sqrt{2}} \sum_k h(k)\, e^{-2\pi i k\gamma}$ satisfying

$$m_0(\gamma) = (\frac{1 + e^{-2\pi i\gamma}}{2})^N \mathcal{L}(\gamma).$$

and satisfying the QMF conditions.

(2)

$$\begin{aligned} |m_0(\gamma)|^2 &= |\frac{1 + e^{-2\pi i\gamma}}{2}|^{2N} |\mathcal{L}(\gamma)|^2 \\ &= \cos^{2N}(\pi\gamma)\, L(\gamma). \end{aligned}$$

(3) Since $L(\gamma)$ is a real-valued trig polynomial with real coefficients, we arrive at

$$L(\gamma) = P(\sin^2(\pi\gamma))$$

for some polynomial P.

(4) This polynomial P must satisfy

$$1 = (1-y)^N P(y) + y^N P(1-y)$$

with $P(y) \geq 0$ for all $0 \leq y \leq 1$.

(5) We arrive at finally the definition

$$P_{N-1}(y) = \sum_{k=0}^{N-1} \begin{pmatrix} 2N-1 \\ k \end{pmatrix} y^k (1-y)^{N-1-k}.$$

For example,

$$\begin{aligned} P_0(y) &= 1, \\ P_1(y) &= 1 + 2y, \\ P_2(y) &= 1 + 3y + 6y^2, \\ P_3(y) &= 1 + 4y + 10y^2 + 20y^3. \end{aligned}$$

Spectral Factorization.

We make the following definitions:

(a) $\mathcal{P}_{2N-1}(y) = (1-y)^N \, P_{N-1}(y)$

(b) $\mathbf{P}_{2N-1}(z) = \mathcal{P}_{2N-1}(1/2 - (z + z^{-1})/4)$

(c) $\widetilde{\mathbf{P}}_{4N-2}(z) = z^{2N-1} \, \mathbf{P}_{2N-1}(z) = \sum_{m=0}^{4N-2} \widetilde{a}_m \, z^m,$

Some examples.

(a) $N = 1$

$$
\begin{aligned}
P_0(y) &= 1, \\
\mathcal{P}_1(y) &= (1-y), \\
\mathbf{P}_1(z) &= \mathcal{P}_1(1/2 - (z + z^{-1})/4) \\
&= -\frac{1}{4}\,z^{-1} + \frac{1}{2} - \frac{1}{4}\,z, \\
\widetilde{\mathbf{P}}_2(z) &= z\,\mathbf{P}_1(z) = -\frac{1}{4} + \frac{1}{2}\,z - \frac{1}{4}\,z^2.
\end{aligned}
$$

(b) $N = 2$

$$
\begin{aligned}
P_1(y) &= 1 + 2y, \\
\mathcal{P}_3(y) &= (1-y)^2\,(1+2y), \\
\mathbf{P}_3(z) &= \mathcal{P}_3(1/2 - (z + z^{-1})/4) \\
&= \frac{1}{32}(-z^{-3} + 9\,z^{-1} + 16 + 9\,z - z^3), \\
\widetilde{\mathbf{P}}_6(z) &= z^3\,\mathbf{P}_3(z) \\
&= \frac{1}{32}(-1 + 9\,z^2 + 16\,z^3 + 9\,z^4 - z^6).
\end{aligned}
$$

(c) $N = 3$

$$
\begin{aligned}
P_2(y) &= 1 + 3y + 6y^2, \\
\mathcal{P}_5(y) &= (1-y)^3\,(1 + 3y + 6y^2), \\
\mathbf{P}_5(z) &= \mathcal{P}_5(1/2 - (z + z^{-1})/4) \\
&= \frac{1}{512}(3\,z^{-5} - 25\,z^{-3} + 150\,z^{-1} \\
&\quad +256 + 150\,z - 25\,z^3 + 3\,z^5), \\
\widetilde{\mathbf{P}}_{10}(z) &= z^5\,\mathbf{P}_5(z) \\
&= \frac{1}{512}(3 - 25\,z^2 + 150\,z^4 + 256\,z^5 \\
&\quad +150\,z^6 - 25\,z^8 + 3\,z^{10}).
\end{aligned}
$$

(d) $N = 4$

$$
\begin{aligned}
P_3(y) &= 1 + 4y + 10y^2 + 20y^3, \\
\mathcal{P}_7(y) &= (1-y)^4\,(1 + 4y + 10y^2 + 20y^3), \\
\mathbf{P}_7(z) &= \mathcal{P}_7(1/2 - (z + z^{-1})/4) \\
&= \frac{1}{4096}(-5\,z^{-7} + 49\,z^{-5} - 245\,z^{-3} \\
&\quad +1225\,z^{-1} + 2048 + 1225\,z \\
&\quad -245\,z^3 + 49\,z^5 - 5\,z^7), \\
\widetilde{\mathbf{P}}_{14}(z) &= z^7\,\mathbf{P}_7(z) \\
&= \frac{1}{4096}(-5 + 49\,z^2 - 245\,z^4 + 1225\,z^6 \\
&\quad +2048\,z^7 + 1225\,z^8 \\
&\quad -245\,z^{10} + 49\,z^{12} - 5\,z^{14}).
\end{aligned}
$$

Theorem. For each $N \in \mathbf{N}$, $\mathbf{P}_{2N-1}(z)$ satisfies:

(a) $\mathbf{P}_{2N-1}(z) = \sum_{m=-2N+1}^{2N-1} a_m \, z^m$ for some real-valued coefficients a_m.

(b) $\mathbf{P}_{2N-1}(z) + \mathbf{P}_{2N-1}(-z) = 1$ for all $z \in \mathbf{C}$, $z \neq 0$.

(c) $\mathbf{P}_{2N-1}(z) \geq 0$ for $|z| = 1$.

(d) $\mathbf{P}_{2N-1}(z) = \mathbf{P}_{2N-1}(z^{-1})$ for all $z \in \mathbf{C}$, $z \neq 0$.

(e) $a_m = a_{-m}$ for $-2N + 1 \leq m \leq 2N - 1$.

(f) $a_m = 0$ if m is even and $m \neq 0$, and $a_0 = 1/2$.

Remark. The zeros of $\widetilde{\mathbf{P}}_{4N-2}(z)$ fall into three categories.

(1) The **zero at** -1 which must have multiplicity $2N$. Note also that always $\widetilde{\mathbf{P}}_{4N-2}(1) = 1$.

(2) The **real zeros not equal to** -1. These come in pairs, (z_0, z_0^{-1}). Since $z_0 \neq \pm 1$, one of the pair must have absolute value less than 1 and the other absolute value greater than 1.

Define $Z_{\mathbf{R}}$ by $Z_{\mathbf{R}} = \{z_0 \in \mathbf{R}\colon \widetilde{\mathbf{P}}_{4N-2}(z_0) = 0,\, |z_0| < 1\}$.

(3) The **nonreal zeros**. These zeros come in clusters of four, namely $(z_0, z_0^{-1}, \overline{z_0}, \overline{z_0}^{-1})$. Only one of these zeros can lie within the unit circle and in the upper half–plane.

Define $Z_{\mathbf{C}}$ by $Z_{\mathbf{C}} = \{z_0 \in \mathbf{C}\colon \widetilde{\mathbf{P}}_{4N-2}(z_0) = 0,\, |z_0| < 1,\, \Im(z_0) > 0\}$.

Theorem. Let $N \in \mathbf{N}$. Then there exists a polynomial $B_{2N-1}(z)$ of degree $2N-1$ with real coefficients such that

$$\widetilde{\mathbf{P}}_{4N-2}(z) = |B_{2N-1}(z)|^2.$$

Moreover, $B_{2N-1}(z) = (z+1)^N \, C_{N-1}(z)$ for some degree $N-1$ polynomial $C_{N-1}(z)$ with real coefficients.

Proof:

$$\begin{aligned} B_{2N-1}(z) &= |\alpha|^{1/2} \, (z+1)^N \\ &\quad \times \prod_{z_0 \in Z_{\mathbf{R}}} |z_0|^{-1/2} \, (z - z_0) \\ &\quad \times \prod_{z_0 \in Z_{\mathbf{C}}} |z_0|^{-1} \, (z - z_0) \, (z - \overline{z_0}). \end{aligned}$$

Examples.

(a) With $N = 2$,

$$\widetilde{\mathbf{P}}_6(z) = \frac{1}{32}(-1 + 9z^2 + 16z^3 + 9z^4 - z^6).$$

We factor

$$\begin{aligned}\widetilde{\mathbf{P}}_6(z) \quad &= \frac{1}{32}(z+1)^4(-z^2+4z-1) \\ &-\frac{1}{32}(z+1)^4(z-(2-\sqrt{3}))(z-(2+\sqrt{3})).\end{aligned}$$

Therefore,

$$\begin{aligned}B_3(z) &= \frac{1}{4\sqrt{2}}(z+1)^2(2-\sqrt{3})^{-1/2}(z-(2-\sqrt{3})) \\ &= \frac{1+\sqrt{3}}{8}(z+1)^2(z-(2-\sqrt{3}))\end{aligned}$$

$$= \frac{1+\sqrt{3}}{8}z^3 + \frac{3+\sqrt{3}}{8}z^2 + \frac{3-\sqrt{3}}{8}z + \frac{1-\sqrt{3}}{8}.$$

(b) With $N = 3$,

$$\widetilde{\mathbf{P}}_{10}(z) = \frac{1}{512}(3 - 25\,z^2 + 75\,z^4 + 256\,z^5 + 75\,z^6 - 25\,z^8 + 3\,z^{10}).$$

We factor

$$\begin{aligned}\widetilde{\mathbf{P}}_{10}(z) &= \frac{1}{512}(z+1)^6 (3z^4 - 18z^3 + 38z^2 - 18z + 3) \\ &= \frac{3}{512}(z+1)^6 (z-\alpha)(z-\overline{\alpha})(z-\alpha^{-1})(z-\overline{\alpha}^{-1}),\end{aligned}$$

where $\alpha \approx .2873 + .1529\,i$ and

$$B_5(z) = \frac{\sqrt{3}}{|\alpha| 16\sqrt{2}}(z+1)^3 (z-\alpha)(z-\overline{\alpha})$$