The Discrete Wavelet Transform.

Motivation.

(a) Given a signal or sequence of data $\{c_0(k)\}_{k\in\mathbf{Z}}$, Assume that $c_0(k)$ is the kth scaling coefficient for some underlying function $f(x)$; that is, $c_0(k) = \langle f, \varphi_{0,k}\rangle$ for each $k \in \mathbf{Z}$. Why is this a reasonable assumption?

(b) The data will only allow us to construct $P_0 f = \sum_n c_0(k)\, \varphi_{0,k}$ so we cannot know the coefficients $\langle f, \varphi_{j,k}\rangle$ or $\langle f, \psi_{j,k}\rangle$ for any $j > 0$. We can however compute those coefficients for $j < 0$.

(c) For any j, k,

$$\varphi_{j,k} = \sum_n h(n-2k)\, \varphi_{j+1,n}$$

and

$$\psi_{j,k} = \sum_n g(n-2k)\, \varphi_{j+1,n}.$$

(d) Therefore, setting $c_j(k) = \langle f, \varphi_{-j,k} \rangle$ and $d_j(k) = \langle f, \psi_{-j,k} \rangle$, we have

$$c_{j+1}(k) = \sum_n c_j(n)\, \overline{h(n-2k)},$$

and

$$d_{j+1}(k) = \sum_n c_j(n)\, \overline{g(n-2k)}.$$

(e) Finally we note that

$$\begin{aligned} c_j(k) \;&=\; \sum_n c_{j+1}(n)\, h(k-2n) \\ &\qquad + \sum_n d_{j+1}(n)\, g(k-2n). \end{aligned}$$

Theorem. Let $\{V_j\}$ be an MRA with scaling filter $h(k)$ and wavelet filter $g(k)$. Then

(a) $\sum_n h(n) = \sqrt{2} \qquad (\Longleftrightarrow \int \varphi(x)\,dx \neq 0)$

(b) $\sum_n g(n) = 0 \quad (\Longleftrightarrow \int \psi(x)\,dx = 0)$

(c) $\sum_k h(k)\,\overline{h(k-2n)} = \sum_k g(k)\,\overline{g(k-2n)} = \delta(n)$
$(\Longleftrightarrow \langle \varphi_{0,0}, \varphi_{0,n}\rangle = \langle \psi_{0,0}, \psi_{0,n}\rangle = \delta(n))$

(d) $\sum_k g(k)\,\overline{h(k-2n)} = 0$ for all $n \in \mathbf{Z}$
$(\Longleftrightarrow \langle \varphi_{0,0}, \psi_{0,n}\rangle = 0$, all $n)$

(e)
$$\sum_k \overline{h(m-2k)}\,h(n-2k) + \sum_k \overline{g(m-2k)}\,g(n-2k) = \delta(n-m)$$

$(\Longleftrightarrow P_{j+1} = P_j + Q_j)$.

Some new notation.

Definition. Given a filter $h(k)$, define $g(k) = (-1)^k \overline{h(1-k)}$. Define the *approximation operator* H and *detail operator* G corresponding to $h(k)$ by

$$(Hc)(k) = \sum_n c(n)\, \overline{h(n-2k)},$$

$$(Gc)(k) = \sum_n c(n)\, \overline{g(n-2k)}.$$

Define the *approximation adjoint* H^* and *detail adjoint* G^* by

$$(H^*c)_k = \sum_n c(n)\, h(k-2n),$$

$$(G^*c)_k = \sum_n c(n)\, g(k-2n).$$

Theorem. Given $h(k)$, $g(k) = (-1)^k\,\overline{h(1-k)}$,

(a) $$\sum_k h(k)\,\overline{h(k-2n)} = \sum_k g(k)\,\overline{g(k-2n)} = \delta(n)$$
$$\Longleftrightarrow HH^* = GG^* = I$$

(b) $$\sum_k g(k)\,\overline{h(k-2n)} = 0 \Longleftrightarrow HG^* = GH^* = 0$$

(c) $$\sum_k \overline{h(m-2k)}\,h(n-2k) + \sum_k \overline{g(m-2k)}\,g(n-2k) = \delta(m-n)$$
$$\Longleftrightarrow H^*H + G^*G = I.$$

A look on the transform side.

Definition. Let $c(n)$ be a signal.

(a) Given $m \in \mathbf{Z}$, the *shift operator* τ_m is defined by $\tau_m c(n) = c(n-m)$.

(b) The *downsampling operator* $\downarrow$ is defined by $(\downarrow c)(n) = c(2n)$.

(c) The *upsampling operator* $\uparrow$ is defined by

$$(\uparrow c)(n) = \begin{cases} c(n/2) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Lemma. Given a signal $c(n)$,

(a) For every $m \in \mathbf{Z}$ $(\tau_m c)^\wedge(\gamma) = e^{-2\pi i m\gamma}\widehat{c}(\gamma)$.

(b) $(\downarrow c)^\wedge(\gamma) = \dfrac{1}{2}(\widehat{c}\left(\dfrac{\gamma}{2}\right) + \widehat{c}\left(\dfrac{\gamma+1}{2}\right))$.

(c) $(\uparrow c)^\wedge(\gamma) = \widehat{c}(2\gamma)$.

Lemma.

(a) Defining $\underline{h}(n) = \overline{h(-n)}$ and $\underline{g}(n) = \overline{g(-n)}$ then

$$(Hc)(n) = \downarrow(c * \underline{h})(n)$$

and

$$(Gc)(n) = \downarrow(c * \underline{g})(n).$$

(b) Also

$$(H^*c)(n) = (\uparrow c) * h(n)$$

and

$$(G^*c)(n) = (\uparrow c) * g(n).$$

Lemma. Given $h(k)$, $g(k) = (-1)^k \, \overline{h(1-k)}$,

$$m_0(\gamma) = 2^{-1/2} \sum_k h(k) \, e^{-2\pi i k \gamma},$$

and

$$m_1(\gamma) = 2^{-1/2} \sum_k h(k) \, e^{-2\pi i k \gamma}.$$

Then for any signal $c(n)$,

$$\begin{aligned}
(Hc)^\wedge(\gamma) &= \frac{1}{\sqrt{2}}(\widehat{c}(\gamma/2) \, \overline{m_0(\gamma/2)} \\
&\qquad + \widehat{c}(\gamma/2 + 1/2) \, \overline{m_0(\gamma/2 + 1/2)}),
\end{aligned}$$

$$\begin{aligned}
(Gc)^\wedge(\gamma) &= \frac{1}{\sqrt{2}}(\widehat{c}(\gamma/2) \, \overline{m_1(\gamma/2)} \\
&\qquad + \widehat{c}(\gamma/2 + 1/2) \, \overline{m_1(\gamma/2 + 1/2)}),
\end{aligned}$$

$$(H^*c)^\wedge(\gamma) = \sqrt{2}\widehat{c}(2\gamma) m_0(\gamma),$$

$$(G^*c)^\wedge(\gamma) = \sqrt{2}\widehat{c}(2\gamma) m_1(\gamma).$$

Lemma. Given $h(k)$, $g(k)$ as usual. Then

$$m_0(\gamma)\,\overline{m_0(\gamma+1/2)} + m_1(\gamma)\,\overline{m_1(\gamma+1/2)} = 0$$

which is equivalent to

$$HG^* = GH^* = 0.$$

Theorem. Given $h(k)$, $g(k)$, $m_0(\gamma)$, $m_1(\gamma)$, and the operators H, G, H^*, and G^* as above, the following are equivalent.

(a) $|m_0(\gamma)|^2 + |m_0(\gamma+1/2)|^2 \equiv 1$.

(b) $H^*H + G^*G = I$.

(c) $HH^* = GG^* = I$.

Definition. Given $h(k)$, $m_0(\gamma)$ as before, we say $h(k)$ is a QMF (quadrature mirror filter) if

(a) $m_0(0) = 1$ and

(b) $|m_0(\gamma/2)|^2 + |m_0(\gamma/2 + 1/2)|^2 \equiv 1$.

Theorem. Suppose that $h(k)$ is a QMF. Define $g(k)$ as before. Then:

(a) $\sum_n h(n) = \sqrt{2}$,

(b) $\sum_n g(n) = 0$,

(c) $\sum_k h(k)\,\overline{h(k-2n)} = \sum_k g(k)\,\overline{g(k-2n)} = \delta(n)$.

(d) $\sum_k g(k)\,\overline{h(k-2n)} = 0$ for all $n \in \mathbf{Z}$.

(e)
$$\sum_k \overline{h(m-2k)}\,h(n-2k) + \sum_k \overline{g(m-2k)}\,g(n-2k) = \delta(n-m).$$

The Discrete Wavelet Transform (DWT)

(a) For infinite signals: Let $h(k)$ be a QMF, $g(k)$ the dual filter, and let H, G, H^*, and G^* be as above. Fix $J \in \mathbf{N}$. The DWT of a signal $c_0(n)$, is the collection of sequences

$$\{d_j(k)\colon 1 \le j \le J;\ k \in \mathbf{Z}\} \cup \{c_J(k)\colon k \in \mathbf{Z}\},$$

where

$$c_{j+1}(n) = (Hc_j)(n)$$

$$d_{j+1}(n) = (Gc_j)(n).$$

The inverse transform is

$$c_j(n) = (H^* c_{j+1})(n) + (G^* d_{j+1})(n).$$

If $J = \infty$, then the DWT of c_0 is the collection of sequences

$$\{d_j(k)\colon j \in \mathbf{N};\ k \in \mathbf{Z}\}.$$

For finite, zero-padded signals: Suppose that $c_0(n)$ has length 2^N, and that $h(n)$ and $g(n)$ have length $L > 2$, with L even. Then

(a) the sequences $c_1 = Hc_0$ and $d_1 = Gc_0$ each have length $(2^N + L - 2)/2$,

(b) c_j and d_j would have length at least $2^{N-j} + (1 - 2^{-j})(L - 2)$.

(c) The total length of the DWT for c_0 would be at least

$$\begin{aligned} &(2^N 2^{-J} + (1 - 2^{-J})(L - 2)) \\ &\qquad + \sum_{j=1}^{J} (2^N 2^{-j} + (1 - 2^{-j})(L - 2)) \\ = \;& 2^N + J(L - 2), \end{aligned}$$

where $J \in \mathbf{N}$ indicates the depth chosen for the DWT.

MATLAB illustration.

```
>> x=[0 1 2 3 4 5 6 7 8 7 6 5 4 3 2 1];
>> dwtmode('zpd')

****************************************
**  DWT Extension Mode: Zero Padding  **
****************************************

>> [h g h1 g1]=wfilters('db2');
>> h
h =
   -0.1294    0.2241    0.8365    0.4830
>> [c1 d1]=dwt(x,'db2')
c1 =
   -0.1294    0.8966    3.7250    6.5534    9.6407
   10.4171    7.5887    4.7603    1.8024
d1 =
   -0.4830   -0.0000   -0.0000   -0.0000    0.9659
   0.0000    0.0000    0.0000   -0.4830
>> length(x)
ans =
    16
>> length([c1 d1])
ans =
    18
>> [C L]=wavedec(x,4,'db2');
>> length(C)
ans =
    25
```

For periodic signals:

Lemma. Let $c(n)$ have period 2^N, $h(k)$ a QMF, Then $(Hc)(n)$ and $(Gc)(n)$ have period 2^{N-1}, and $(H^*c)(n)$ and $(G^*c)(n)$ have period 2^{N+1}.

MATLAB illustration.

```
>> dwtmode('per')
*****************************************
**  DWT Extension Mode: Periodization  **
*****************************************
>> [c1 d1]=dwt(x,'db2')
c1 =
    0.4483    2.3108    5.1392    7.9676
    10.8654    9.0029    6.1745    3.3461
d1 =
   -0.2588   -0.0000   -0.0000   -0.0000
   0.2588    0.0000    0.0000    0.0000
>> length(x)
ans =
    16
>> length([c1 d1])
ans =
    16
>> [C L]=wavedec(x,4,'db2');
>> length(C)
ans =
    16
```

Notice that all of the numbers are different for the two different extension modes. Why is this?

Scaling functions from scaling filters.

Let $\varphi(x)$ be the scaling function of an MRA. Then

$$\widehat{\varphi}(\gamma) = m_0(\gamma/2)\,\widehat{\varphi}(\gamma/2),$$

and we may write

$$\begin{aligned}
\widehat{\varphi}(\gamma) &= m_0(\gamma/2)\,\widehat{\varphi}(\gamma/2) \\
&= m_0(\gamma/2)\,m_0(\gamma/4)\,\widehat{\varphi}(\gamma/4) \\
&= \cdots \\
&= \prod_{j=1}^{n} m_0(\gamma/2^j)\,\widehat{\varphi}(\gamma/2^n).
\end{aligned}$$

Letting $n \to \infty$, and assuming $\widehat{\varphi}(0) = 1$,

$$\widehat{\varphi}(\gamma) = \prod_{j=1}^{\infty} m_0(\gamma/2^j).$$

How can we use this formula to get a picture of the scaling function associated with a QMF filter? There are two ways.

(1) Define

$$\widehat{\mu_\ell}(\gamma) = \prod_{j=1}^{\ell} m_0(\gamma/2^j)\, \mathbf{1}_{[-2^{\ell-1},2^{\ell-1}]}(\gamma).$$

Theorem. Let $h(k)$ be a finite QMF, and suppose that there is a number $c > 0$ such that

$$|m_0(\gamma)| \geq c \qquad \text{for} \qquad |\gamma| \leq 1/4.$$

Then:
(a) $\widehat{\mu_\ell} \to \widehat{\varphi}$ in $L^2(\mathbf{R})$, and by Plancherel's formula $\mu_\ell \to \varphi$ in $L^2(\mathbf{R})$.
(b) $\|\widehat{\varphi}\|_2 = \|\varphi\|_2 = 1$.

(2) This method is called the Cascade Algorithm. The idea here is to define an operator T by

$$f(x) \longmapsto Tf(x) = \sum_n h(n)\, 2^{1/2} f(2x-n).$$

The scaling function φ will be a function that satisfies the *fixed point formula* $T\varphi = \varphi$.

The cascade algorithm sets up an iteration scheme to find this fixed point, viz., fix some initial function $\eta_0(x)$ and define for all $\ell \in \mathbf{N}$,

$$\eta_\ell(x) = \sum_n h(n)\, 2^{1/2}\, \eta_{\ell-1}(2x-n).$$

Theorem. Let $h(k)$ be a finite QMF, and suppose that there is a number $c > 0$ such that

$$|m_0(\gamma)| \geq c \qquad \text{for} \qquad |\gamma| \leq 1/4.$$

Then: Let $\eta_0(x) = \mathbf{1}_{[-1/2,1/2]}(x)$. Then:
(a) $\eta_\ell \to \varphi$ in $L^2(\mathbf{R})$, and
(b) $\{T_n\varphi(x)\}_{n\in\mathbf{Z}}$ is an orthonormal system of translates.