Sample Problems for Final Exam

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Topos Theory

July 21, 2013

Problem 1. Topoi are Sheaves over Themselves

Let $\mathscr{E}$ be a Grothendieck topos in the Grothendieck universe $\mathcal{U}$. Equip $\mathscr{E}$ with the following Grothendieck pretopology:
A collection of arrows

$$(E_i \to E)_{i \in I}$$

is declared to be a cover if the induced morphism

$$\coprod_{i \in I} E_i \to E$$

is an epimorphism. Call the associated Grothendieck topology the **epimorphism topology**.

(a) Show that the above definition of cover indeed defines a Grothendieck pretopology.

(b) Choose a subcanonical Grothendieck site $(\mathscr{C}, J)$ such that $\mathbf{Sh}_J(\mathscr{C}) \simeq \mathscr{E}$. (Why can we arrange J to be subcanonical?). Let $\mathcal{V}$ be a Grothendieck universe such that $\mathcal{U} \in \mathcal{V}$, so that $\mathscr{E}$ is $\mathcal{V}$-small. Show that there exists a Grothendieck topology K on $\mathscr{E}$ such that

$$\widehat{\mathbf{Sh}_K}(\mathscr{E}) \simeq \widehat{\mathbf{Sh}_J}(\mathscr{C}),$$

where the "hat" notation means sheaves of $\mathcal{V}$-small sets (as opposed to $\mathcal{U}$-small sets, in which case we omit the "hat").

(c) Show that the above equivalence restricts to equivalences

$$\mathscr{E} \simeq \mathbf{Sh}_J(\mathscr{C}) \simeq \mathbf{Sh}_K(\mathscr{E}),$$

where $\mathbf{Sh}_K(\mathscr{E})$ is the full subcategory of $\widehat{\mathbf{Sh}_K}(\mathscr{E})$ spanned by sheaves of $\mathcal{U}$-small sets.

(d) Show that K is the epimorphism topology. (It is hence also the same as the canonical topology).

Problem 2. An Exercise about Sheaves on the Circle

Consider the unit circle

$$S^1 \subset \mathbb{C}$$

and let z denote the global complex coordinate of $\mathbb{C}$. Let n be a positive integer. Consider the following poset P:
The objects consist of pairs (U, φ) where $U \subseteq S^1$ is an open subset, and

$$\varphi : U \to S^1$$

is a continuous function such that for all $z \in U$,

$$\varphi (z)^n = z.$$

We declare

$$(U, \varphi) \leq (V, \psi)$$

if $U \subseteq V$ and $\psi|_U = \varphi$.
Equip P with a Grothendieck pretopology by declaring a family of arrows

$$((U_\alpha, \varphi_\alpha) \leq (U, \varphi))_{\alpha \in A}$$

to be a cover if

$$U = \bigcup_{\alpha \in A} U_\alpha.$$

Denote the associated Grothendieck topology by J.
Show that

$$\mathbf{Sh}_J (P) \simeq \mathbf{Sh} \left(S^1\right).$$

Problem 3. Characterizing Localic Topoi

Let $\mathscr{E}$ be a Grothendieck topos. Show that the following properties are equivalent:

i) $\mathscr{E}$ is localic.

ii) There exists a poset P equipped with a Grothendieck topology J such that $\mathscr{E} \simeq \mathbf{Sh}_J(P)$.

iii) The inclusion $\mathbf{Sub}_{\mathscr{E}}(1) \hookrightarrow \mathscr{E}$ of the poset of subobjects of the terminal object is strongly generating.

Remark. Recall that a full and faithful functor

$$i : \mathscr{C} \hookrightarrow \mathscr{D}$$

is strongly generating if for ever object D, the canonical map

$$\varinjlim \pi_D \to D$$

is an isomorphism, where

$$\pi_D : \mathscr{C}/D \to \mathscr{D}$$

is the functor

$$(i(C) \to D) \mapsto i(C).$$