Categorical Properties of Topological and Differentiable Stacks

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Categorical Properties of Topological and Differentiable Stacks

Categorische Eigenschappen van Topologische en Differentieerbare Stacks

(met een samenvatting in het Nederlands)

Proefschrift

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door

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Promotor: Prof. I. Moerdijk

To my high school mathematics teacher Dennis Donovan.

The mathematics are usually considered as being the very antipodes of Poesy. Yet Mathesis and Poesy are of the closest kindred, for they are both works of the imagination.

-Thomas Hill

Contents

Introduction

What are topological and differentiable stacks?

This thesis focuses on the theory of topological and differentiable stacks. Topological and differentiable stacks behave like spaces whose points themselves posses intrinsic symmetries. The concept of symmetry is most naturally expressed through the theory of groups, and as will be explained, a natural generalization of groups called groupoids. A topological or differentiable stack is an appropriate marriage of the concept of a topological space or smooth manifold with that of a group or groupoid. Heuristically, a differentiable stack is like a manifold whose points possess intrinsic automorphism groups. They arise by "identifying" points of a manifold related to one another by a symmetry. If these points are naively glued together, the result may have singularities so it cannot be studied using differential geometry. In the topological setting, the naive quotient of a topological stack exists as a topological space, however it lacks important information about the different ways in which a point is related to itself, or other isomorphic points; roughly speaking, one may imagine the symmetry group of a point as an internal state-space for that point, and passing to the naive quotient loses this information.

Suppose that G is a Lie group acting smoothly on a manifold M. A particular example of a differentiable stack is the "stacky quotient" $M//G$. Roughly speaking, $M//G$ is like the quotient space M/G, except each point $[x]$ of M/G, which is the image of a point $x \in M$ under the canonical quotient map

$$M \to M/G,$$

has G_x, the stabilizer group of x, as an intrinsic automorphism group. Notice however that this way of assigning an automorphism group to $[x]$ is not canonically defined; it is true that any two points x and y of M which lie in the same orbit of the G-action have *isomorphic* automorphism groups, however, these isomorphisms are given by conjugation with an element g such that $g \cdot x = y$:

$$\begin{array}{ccc} G_x & \xrightarrow{\sim} & G_y \\ h & \mapsto & ghg,^{-1} \end{array}$$

and such a g is not unique unless G_x is trivial. The information of all these automorphism groups and how they patch together can be naturally described by the action groupoid $G \ltimes M$:

A groupoid is a category in which every arrow is an isomorphism. The objects of $G \ltimes M$ are the points of the manifold M, and the arrows of the groupoid are the points of the manifold $G \times M$. Here, a pair (g, x) is seen as an arrow

$$x \xrightarrow{(g,x)} g \cdot x,$$

and composition is given by the rule

$$\begin{tikzcd} x \arrow[r, "{(g,x)}"] \arrow[rr, bend right, "{(hg,x)=(h,gx)\circ(g,x)}"'] & g \cdot x \arrow[r, "{(h,gx)}"] & hg \cdot x. \end{tikzcd}$$

For each point x, its identity arrow in the category $G \ltimes M$ is the pair (e, x) and since

$$\left(g^{-1}, gx\right) \circ (g, x) = (e, x)\,,$$

every morphism of this category has in inverse, so it is a groupoid. The composition of any groupoid $\mathcal{G}$ endows each set $\mathrm{Hom}_{\mathcal{G}}(x, x)$ with the structure of a group. (From this we can see that a groupoid with only one object is the same thing as a group.) In the groupoid $G \ltimes M$, for each object x, $\mathrm{Hom}(x, x) = G_x$, the stabilizer group of x with respect to the G-action. The action groupoid contains all the information about the *set-theoretic* action of G on the set M, and how the different stabilizer groups are related to one another.

The action groupoid however possesses more structure than a set-theoretic groupoid since both the objects and the arrows of $G \ltimes M$ have the canonical structure of a smooth manifold, and the natural structure maps of the groupoid are smooth. For instance, the source map s sending an arrow (g, x) to its source x, and the analogously defined target map, are even surjective submersions. This gives $G \ltimes M$ the structure of a *Lie groupoid.* A Lie groupoid with only one object is the same thing as a Lie group. Regarding $G \ltimes M$ as a Lie groupoid, not only encodes all the set-theoretic information about the action $G \circlearrowright M$, but all of the differential-geometric information about this action as well. It describes all of the local geometry of the "stacky quotient" $M//G$ and how it patches together, in much the same way that an atlas describes the local geometry of a manifold.

Topological and differentiable stacks are 2-categorical in nature. That is, given two such stacks $\mathscr{X}$ and $\mathscr{Y}$ and two morphisms

$$f : \mathscr{X} \to \mathscr{Y}$$

and

$$f' : \mathscr{X} \to \mathscr{Y},$$

it is possible for f and f' to be related by a 2-morphism

$$\alpha : f \Rightarrow f'. \tag{1}$$

Any such 2-morphism must have an inverse, that is to say, topological and differentiable stacks form a bicategory, and for any two such stacks $\mathscr{X}$ and $\mathscr{Y}$, there exists a groupoid of maps between them, $\mathrm{Hom}\left(\mathscr{X}, \mathscr{Y}\right)$. An object of this groupoid is nothing but a map $f : \mathscr{X} \to \mathscr{Y}$ and an arrow in this groupoid is a 2-morphism as in (1). It is this 2-categorical structure which allows points of topological or differentiable stack to have automorphisms; a point of a stack $\mathscr{X}$ is the same as a map

$$x : * \to \mathscr{X}$$

from the one-point space $*$, and an automorphism of this point is simply a 2-morphism

$$\alpha : x \Rightarrow x.$$

Topological and differentiable stacks can be looked at from three seemingly different points of view:

- In the orbifold/orbispace point of view, topological and differentiable stacks may be pictured as spaces whose points have automorphism groups which themselves can have a topological or differentiable structure. This conceptual picture can be made rigorous in many cases, e.g., by describing an orbifold as a topological space equipped with an orbifold atlas.

- A second point of view is that topological and differentiable stacks are "nice" quotients of spaces by the action of a (topological or Lie) group, or more generally, groupoid. For example, if G is a Lie group acting on a manifold M, the quotient space M/G is rarely a manifold, and can be quite pathological as a space. However, there always exists a "stacky quotient" $M//G$ which makes the projection map $M \to M//G$ into a principal G-bundle over $M//G$. Effective orbifolds are precisely those differentiable stacks of the form $M//G$ where $G \circlearrowright M$ is a foliating effective action of a compact connected Lie group. More generally, topological and differentiable stacks are those stacks which arise as quotients of spaces by the action of a topological groupoid or Lie groupoid. This is the precise way in which topological and differentiable stacks arise by "identifying" points of a space related to one another by a symmetry; these symmetries are the arrows of such a groupoid.

  A seemingly pathological example of this would be to let a Lie group G act trivially on the one-point space $*$. Whereas the ordinary quotient is again a point, the "stacky quotient" $*//G$ is very different from a point. In fact, it classifies principal G-bundles in the sense that for any manifold M, there is an equivalence of groupoids

$$\operatorname{Hom}(M, *//G) \simeq Bun_G(M),$$

where $Bun_G(M)$ is the groupoid of principal G-bundles over M. That is to say, $*//G$ is the *moduli stack* of principal G-bundles.

- This highlights the third point of view on topological and differentiable stacks, which is that they are solutions to those moduli problems in topology and differential geometry which are "locally representable" as functors, that is, they are generalized moduli-spaces. To illustrate this example, the quotient map $* \to *//G$, by the Yoneda Lemma, corresponds to the trivial principal G-bundle over the point. The fact that any principal G-bundle is locally trivial means that any map

$$M \to *//G$$

locally factors through the map $* \to *//G$, up to isomorphism.

Topological and differentiable stacks are stacks of torsors of topological groupoids and Lie groupoids, respectively. They are analogous to algebraic stacks, which play an important role in algebraic geometry [19], [3], [25], and have recently received great interest. Differentiable stacks have deep connections with equivariant geometry, foliation theory [41], twisted K-theory [60], and Poisson and symplectic geometry [11]. Since both differential geometry and symmetry play such an important role in theoretical physics, it is no surprise that there is a multitude of applications of differentiable stacks to physics as well. For instance, the internal state-space for a propagating spinning string has the geometry of the string group, which is a group object internal to differentiable stacks [56]. More generally, differentiable stacks play a prominent role in higher gauge theory [5], [7], [4]; essentially, higher gauge theory is an adaptation of classical gauge theory for field theories whose group of gauge symmetries are a group-object in (higher) differentiable stacks rather than in manifolds. Moreover, orbifolds, a particular type of differentiable stack, enjoy regular use in string theory and conformal field theory [33], [20].

There is also a wealth of examples of topological stacks. Any differentiable stack has an underlying topological stack, and so does any algebraic stack (locally of finite type over $\mathbb{C}$). Many topological tools have been extended to topological stacks both to study the underlying topological properties of algebraic and differentiable stacks, and also to study topological stacks in their own right. For example, Behrang Noohi recently developed a theory of fibrations for topological stacks [50], defined a functorial notion of homotopy type [51] and, together with Kai Behrend, Grégory Ginot, and Ping Xu, defined the free loop stack of a topological stack to develop string topology for differentiable stacks [8].

The content of this thesis:

Background Material

Chapter I: A primer on topological and differentiable stacks

Since the subject of topological and differentiable stacks is quite technical, the first chapter of this thesis is designed to serve as a crash-course in the necessary mathematical background for this thesis. In principle, this chapter should provide the motivated reader, who has an understanding of basic category theory and a passing acquaintance with the concept of a 2-category, with all of the needed background for the rest of the thesis.

This chapter is split into two sections. The first section is an introduction to the formalism of sheaves and stacks. It starts with very basics: the concept of a sheaf over a topological space, but quickly progresses to the general theory of stacks of groupoids over arbitrary Grothendieck sites. This level of generality is needed later in this thesis, notably in Chapter II. The second section of Chapter I is dedicated to topological and differentiable stacks themselves. It contains both their definitions, as well as many essential facts about them which are used liberally in the rest of the thesis. It is a largely self-contained introduction to the subject. In particular, many important fundamental results in the subject are proven from scratch.

New Results

Chapter II: Compactly Generated Stacks

In some sense, topological spaces are "too general" as collectively, they do not enjoy many nice categorical properties. One major defect is that the category of topological spaces is not Cartesian closed. To be Cartesian closed, in particular, one would want for any two topological spaces X and Y, there to be a topological space of maps, $\mathbb{M}\mathrm{ap}\left(X,Y\right)$, and for it to satisfy the following universal property:

For every space Z, there is a natural isomorphism

$$\mathrm{Hom}\left(Z,\mathbb{M}\mathrm{ap}\left(X,Y\right)\right)\cong\mathrm{Hom}\left(Z\times X,Y\right).$$

It has been to the embarrassment of topologists that the category of topological spaces does not have such mapping spaces. However, it does have many Cartesian closed subcategories. One of the most natural of these is the category of compactly generated Hausdorff spaces (also known as Kelley spaces). These were popularized in a 1967 paper of Normal Steenrod [57]. This paper proved to be of great importance as it is now standard practice to avoid the categorical deficiencies of the category of topological spaces by working internal to the category of compactly generated Hausdorff spaces.

Similarly, even when working only with topological stacks arising from compactly generated Hausdorff topological groupoids, the bicategory of topological stacks is plagued with categorical short-comings; not only is it not Cartesian closed, but it is also fails to be closed under many natural constructions, such as taking infinite Cartesian products. In other words, in addition to not being Cartesian closed, this bicategory is not complete. There does exist however a nicer bicategory of topological stacks, which I call "compactly generated stacks", which is Cartesian closed and complete as a bicategory. Its construction is the content of chapter II. The bicategory of compactly generated stacks provides the topologist with a convenient bicategory of topological stacks in which to work. The resulting paper, which is nearly identical to this chapter, has been accepted for publication in *Advances in Mathematics*. It is my hope that its use shall help many topologists avoid the categorical shortcomings of topological stacks in the future, much as compactly generated spaces has helped topologists avoid the categorical shortcomings of topological spaces.

How to construct a Cartesian closed bicategory of topological stacks
To construct a Cartesian closed bicategory of topological stacks, one should first determine the essential properties that make compactly generated Hausdorff spaces a Cartesian closed category. Essentially, a Hausdorff space is compactly generated if and only if its topology is determined completely by its compact subsets. The first key fact needed to prove that compactly generated Hausdorff spaces are Cartesian closed is that compact Hausdorff spaces are *exponentiable*; for any compact Hausdorff space K, and any topological space X, the mapping space $\mathtt{Map}\,(K, X)$ exists. The second key fact is that compactly generated Hausdorff spaces form a complete category. Elementary category theory shows that these two properties, together with the property of compact generation, imply that the category of compactly generated Hausdorff spaces is Cartesian closed.

The situation is quite similar for topological stacks. It is shown in [52] that compact Hausdorff spaces are exponentiable in the bicategory of topological stacks, that is, for any compact Hausdorff space K and any topological stack $\mathscr{X}$, the mapping stack $\mathtt{Map}\,(K, \mathscr{X})$ exists as a topological stack. However, as previously remarked, the bicategory of topological stacks appears to not be complete, hence we should not expect this to imply Cartesian-closure. The problem is even deeper than this, as the defining property of compactly generated Hausdorff spaces, namely that they are determined by their compact subsets, is destroyed when considering them as topological stacks; viewing spaces as stacks only allows one to reconstruct a space by from its open subsets (by gluing), not from its *compact* subsets. In technical language, this is due to the fact that topological stacks are defined to be stacks with respect to the Grothendieck topology generated by open covers. In this thesis, I show that one can modify this Grothendieck topology to take into account the compact generation of compactly generated Hausdorff spaces. This has the effect of allowing a space, when viewed as a stack for this Grothendieck topology, to be reconstructed from its compact subsets. It turns out that this completely solves all of the categorical shortcomings of the bicategory of topological stacks; the resulting bicategory of topological stacks, which I call compactly generated stacks, is Cartesian closed and complete.

Theorem. *The* 2*-category of compactly generated stacks has arbitrary products, and for any two compactly generated stacks* $\mathscr{X}$ *and* $\mathscr{Y}$*, there exists a compactly generated stack* $\mathbb{M}\mathrm{ap}\left(\mathscr{Y},\mathscr{X}\right)$*, which satisfies the following universal property:*

There is a natural equivalence of groupoids

$$\mathrm{Hom}\left(\mathscr{Z},\mathbb{M}\mathrm{ap}\left(\mathscr{X},\mathscr{Y}\right)\right)\simeq\mathrm{Hom}\left(\mathscr{Z}\times\mathscr{X},\mathscr{Y}\right),$$

for every compactly generated stack $\mathscr{Z}$*.*

Chapter III: Small Sheaves, Stacks, and Gerbes over Étale Topological and Differentiable Stacks

Chapter III of this thesis lays the foundations for the theory of small sheaves and stacks over étale topological and differentiable stacks. When one speaks of a sheaf "over a topological space (or manifold) X," this is really a sheaf over the category $\mathcal{O}\left(X\right)$ of open subsets of X (with respect to its open cover Grothendieck topology). Roughly speaking, such a sheaf F is a coherent assignment to each open subset U of X a set $F\left(U\right)$ such that these sets glue together along intersections. As an example, consider a continuous function

$$f:Y\to X,$$

and assign to each open subset U of X the set of sections of f over U. If $U\subseteq V$ and σ is a section of f over V, then $\sigma|_U$ is a section for f over U. This is just saying that F is a functor

$$F:\mathcal{O}\left(X\right)^{op}\to\mathrm{Set},$$

i.e. that F is a *presheaf.* Moreover, given an open cover (U_α) of U, having a section of f over U is the same as having a collection of sections σ_α of f over U_α such that they agree on pairwise intersections, i.e. the presheaf F is actually a *sheaf.* This construction produces a functor

$$\Gamma:\mathbb{TOP}/X\to Sh\left(X\right),$$

from the category of maps into X to the category of sheaves over X. (If X is a manifold, the category of topological spaces should be replaced with the category of smooth manifolds.) In fact, every sheaf is of this form; there is another functor

$$L : Sh(X) \to \mathbb{TOP}/X$$

which produces from a sheaf F a space

$$\underline{L(F)} \to X \tag{2}$$

over X such that the sheaf of sections of this map is isomorphic to F. It turns out that the map (2) is always a local homeomorphism, and that

$$L : Sh(X) \to Et(X),$$

is an equivalence between the category of sheaves over X, and the category of local homeomorphisms over X. This equivalence is quite fundamental. The space $\underline{L(F)}$ is called the étalé space of the sheaf F.

A *stack* $\mathscr{Z}$ over X is roughly the same thing as a sheaf over X, except that it assigns each open subset U a *groupoid* $\mathscr{Z}(U)$ instead of just a set, and the way in which these groupoids must glue along intersections is more subtle. For a given a stack $\mathscr{Z}$ over X, we cannot hope to find an étalé *space* for $\mathscr{Z}$, but we can hope to find an "étalé topological stack," that is a(n étale) topological stack $\mathscr{Y}$ together with a local homeomorphism

$$\mathscr{Y} \to X$$

such that its stack of sections is equivalent to $\mathscr{Z}$.

We can generalize this further by allowing X itself to be an étale topological or differentiable stack $\mathscr{X}$. Such étale stacks behave much more like spaces than general topological and differentiable stacks. In the differentiable setting, étale stacks are those stacks all of whose automorphism groups are discrete and these includes all orbifolds, and more generally, all "stacky" leaf-spaces of foliated manifolds. They are an important generalization of spaces. For example, it is not true that every foliation of a manifold M arises from a submersion $f : M \to N$ of manifolds, however, it is true that every foliation on M arises from a submersion $M \to \mathscr{X}$, where $\mathscr{X}$ is allowed to be an étale differentiable stack [41]. Similarly, it is not true that every Lie algebroid over a manifold M integrates to a Lie groupoid $\mathcal{G} \rightrightarrows M$ [18], however it is true when the arrow space $\mathcal{G}$ is allowed to be an étale differentiable stack [59]. In Chapter III of this thesis, we define the notion of a sheaf and stack "over" an étale topological or differentiable stack in much the same way as for topological spaces and smooth manifolds. For example, if G is a discrete group acting on a space X, the stacky quotient $X//G$ is an étale topological stack, and a *small* sheaf over $M//G$ is the same as a G-equivariant sheaf over M, which can be described as a space E equipped with an action of G

and a local homeomorphism $E \to M$ which is equivariant with respect to the two G-actions. Our notion of small sheaf over an étale stack $\mathscr{X}$ agrees with the existing notion of a sheaf over an orbifold, when $\mathscr{X}$ is an orbifold. Small sheaves over an étale stack are an extension of the concept of a sheaf "over" a space, to allow the space to be an étale stack. Stacks "over" a space can also be generalized in this way, and stacks "over" an étale topological or differentiable stack, in this sense, are what I call *small* stacks. One of the main results of Chapter III is the following theorem:

Theorem. *For any étale topological or differentiable stack $\mathscr{X}$, there is an adjoint-equivalence of 2-categories*

$$\mathrm{St}(\mathscr{X}) \underset{L}{\overset{\Gamma}{\rightleftarrows}} Et(\mathscr{X}),$$

between small stacks over $\mathscr{X}$ and local homeomorphisms over $\mathscr{X}$, of the form $\mathscr{Y} \to \mathscr{X}$, with $\mathscr{Y}$ another étale stack.

The 2-functor L associates to each small stack $\mathscr{Z}$ over $\mathscr{X}$ its *étalé realization.*

Small gerbes and ineffective data The rest of Chapter III is dedicated to unraveling the mystery behind "ineffective data" of étale stacks. Suppose that $\mathscr{X}$ is a smooth orbifold. Then for each point $x : * \to \mathscr{X}$ of $\mathscr{X}$, the automorphism group $Aut(x)$ is finite and there exists a manifold V_x and a (representable) local homeomorphism $p : V_x \to \mathscr{X}$ such that:

i) the point x factors (up to isomorphism) as $* \xrightarrow{\tilde{x}} V_x \xrightarrow{p} \mathscr{X}$, and

ii) the automorphism group $Aut(x)$ acts on V_x.

The orbifold $\mathscr{X}$ is obtained by gluing together each of the "stacky-quotients" $V_x//Aut(x)$. If for each point x, these actions of $Aut(x)$ are faithful, i.e. the induced map

$$\rho_x : Aut(x) \to Diff(V_x)$$

is a monomorphism, where $Diff(V_x)$ is the group of diffeomorphisms of V_x, then the orbifold is called effective. In general, for each x, the subgroup $Ker(\rho_x)$ of $Aut(x)$, is called the *ineffective isotropy group* of x. These kernels may be killed off to obtain an underlying effective orbifold, called its *effective part.* For a general étale stack, the automorphism group of a point may not induce an action on some manifold, but there is a single manifold V and a (representable) local homeomorphism

$$p : V \to \mathscr{X}$$

such that

i) each point x factors (up to isomorphism) as $* \xrightarrow{\tilde{x}} V \xrightarrow{p} \mathscr{X}$, and

ii) there is a canonical homomorphism $\tilde{\rho}_x : Aut\,(x) \to Diff_{\tilde{x}}\,(V)$,

where $Diff_{\tilde{x}}\,(V)$ is the group of *germs* of locally defined diffeomorphisms of V that fix $\tilde{x}$. Again, the kernel of each ρ_x is called the *ineffective isotropy group* of x, and these groups can be killed off to obtain the effective part of $\mathscr{X}$. Ineffective isotropy groups can be thought of as data that "artificially inflates" the automorphism groups of the underlying effective étale stack. For example, if G is a finite group acting trivially on a manifold M, then the "stacky quotient" $M//G$ looks like M except each point x, instead of having a trivial automorphism group, has G as an automorphism group. These automorphisms groups are somehow artificial as the action sees nothing of the group G. Indeed, the effective part of the stack $M//G$ is M, and all the isotropy data is purely ineffective. This is an example of what is known as a *purely ineffective orbifold*. In general, an étale stack $\mathscr{X}$ is called *purely ineffective* if its effective part is equivalent to a manifold.

It is claimed in [27] that purely ineffective étale stacks are the same as manifolds equipped with a (small) gerbe. A gerbe over M is a stack $\mathscr{G}$ over M such that over each point x of M, the stalk $\mathscr{G}_x$ is equivalent to a group. From such a gerbe, one can construct an étale stack which looks just like M except each point x, now instead of having a trivial automorphism group, has (a group equivalent to) $\mathscr{G}_x$ as its automorphism group. The effective part of such an étale stack is M, hence it is a purely ineffective étale stack. This construction was eluded to in [27]. In Chapter III of this thesis, I show that this result extends to general étale stacks, namely that any étale stack $\mathscr{X}$ encodes a small gerbe (in the sense I define in this thesis) over its effective part $\mathrm{Eff}\,(\mathscr{X})$, and moreover, every small gerbe over an effective étale stack $\mathscr{Y}$ arises uniquely from some étale stack $\mathscr{Z}$ whose effective part is equivalent to $\mathscr{Y}$. The construction of an étale stack $\mathscr{Z}$ out of an effective étale stack $\mathscr{Y}$ equipped with a small gerbe $\mathscr{G}$, is precisely the étalé realization of the gerbe $\mathscr{G}$, and the ineffective isotropy groups are given by the stalks of the gerbe $\mathscr{G}$.

In Section III.7, I introduce the 2-category of gerbed effective étale stacks. The objects of this 2-category are effective étale stacks equipped with a small gerbe. I then show that when restricting to nice enough classes of maps, this 2-category is equivalent étale stacks. In particular, I prove:

Theorem. *There is an equivalence of 2-categories between gerbed effective étale differentiable stacks and submersions, $\mathscr{Gerbed}\,(\mathfrak{EffEt})_{subm}$, and the 2-category of étale differentiable stacks and submersions, $\mathfrak{Et}_{subm}$.*

This has applications to foliation theory. Roughly speaking, a foliation of a manifold M is a smooth partitioning of M into (immersed) connected submanifolds, all of the same dimension, called leaves. A submersion

$$f : M \to \mathscr{X}$$

from a manifold M to an étale differentiable stack induces a foliation on M. If $\mathscr{X}$ is effective, then this is essentially all the information encoded by this map. If $\mathscr{X}$ is not effective however, there is some information lost by considering only this induced foliation. The results of Chapter III imply that the extra structure induced on M from f, besides the foliation, is a gerbe which is compatible with the foliation. The theory of gerbed foliations and their holonomy will be developed further in another paper.

Chapter I

A primer on topological and differentiable stacks

I.1 The Language of Sheaves and Stacks

In order to delve into the theory of topological and differentiable stacks, one most first have a firm grasp on the mathematical framework on which they are built, namely the theory of stacks. Stacks, although developed in the field of algebraic geometry [3], [19], [25], in and of themselves, as algebraic objects, lie purely in category theory. It is because of this that they lend themselves to various fields outside of algebraic geometry, such as topology and differential geometry. Stacks are nothing more than a categorification of the concept of a sheaf. Therefore, we first turn our attention to the theory of sheaves.

I.1.1 Sheaves over a space

Historically, the first sheaves to be considered were sheaves *over a topological space*. We will briefly recall the definition of such a sheaf.

Definition I.1.1. Let X be a topological space. A **presheaf** (of sets) over X consists of the data of

i) an assignment to each open subset $U \subseteq X$ a set $F(U)$, and

ii) to each inclusion $U \hookrightarrow V$ of open subsets, a **restriction function** $r_{U,V} : F(V) \to F(U)$

subject to the conditions that

i) for each open subset U, $r_{U,U} = id_{F(U)}$, and

ii) if $U \hookrightarrow V \hookrightarrow W$, then $r_{W,V} \circ r_{V,U} = r_{W,U}$.

More concisely, we may regard the poset of open subsets of X as a category $\mathcal{O}(X)$, in which the arrows are given by inclusion. A presheaf is nothing more than a functor $F : \mathcal{O}(X)^{op} \to \mathrm{Set}$. Presheaves themselves form a category, where the morphisms are given by natural transformations of functors.

Before giving the definition of a sheaf, we shall provide a prototypical example. Consider the presheaf $C(\,\cdot\,, \mathbb{R})$ which assigns an open subset $U \subseteq X$ the set of continuous real-valued functions, with the restriction functions given by actual restriction. Suppose we are given any open cover $(U_i)_i$ of an open subset U. Given a real-valued function $f : U \to \mathbb{R}$, denote by f_i the restriction of f to U_i. Then, for all i and j,

$$f_i|_{U_{ij}} = f_j|_{U_{ij}},$$

where U_{ij} denotes the pairwise intersection. Moreover, if we are given two real-valued functions f and g on U such that for each i, $f_i = g_i$, then we must have that f and g are equal. This condition can be phrased more abstractly by saying that the canonical map

$$C(U, \mathbb{R}) \to \varprojlim \left[\prod C(U_i, \mathbb{R}) \rightrightarrows \prod C(U_{ij}, \mathbb{R})\right] \tag{I.1}$$

is a monomorphism of sets. The limit above is just the collection of functions $h_j : U_j \to \mathbb{R}$, with j running over all the open subsets in the cover, such that for all i and j

$$h_i|_{U_{ij}} = h_j|_{U_{ij}}.$$

Notice however, by continuity, if we are given such a collection of functions (h_i) which agree on pairwise intersections, then they necessarily conglomerate to a function on U,

$$h : U \to \mathbb{R}.$$

That is to say, the canonical map (I.1) is not just a monomorphism, but is an *isomorphism.*

Definition I.1.2. A presheaf $F : \mathcal{O}(X)^{op} \to \mathrm{Set}$ is a **sheaf** if for every open subset U and for every open cover $(U_i)_i$ of U, the canonical map

$$F(U) \to \varprojlim \left[\prod F(U_i) \rightrightarrows \prod F(U_{ij})\right] \tag{I.2}$$

is an isomorphism of sets. If the map is a monomorphism, then F is called a **separated** presheaf.

Nothing is special about the real line $\mathbb{R}$, in that we could have equally well considered an arbitrary topological space Y and looked at continuous functions into Y to get the presheaf $C(\,\cdot\,, Y)$ of Y-valued continuous functions. This presheaf is easily seen to be a sheaf.

Example 1. Consider the presheaf which assigns an open subset U of X the set of embeddings of U (when regarded as an abstract topological space) into X. Given two embeddings of U into X which agree when restricted to every element of some open cover (U_i) of U, one can conclude that they must be equal, so this presheaf is separated. However, given an open cover $(U_i)_i$ of U and a collection of embeddings $h_i : U_i \hookrightarrow X$ which agree on pairwise intersections, they need not glue to an embedding $h : U \hookrightarrow X$. For example, if $h : U \to X$ is only a local homeomorphism and not an embedding, we can nonetheless find a cover (U_i) of U such that each f_i is an embedding, but the only function that the h_i can conglomerate to is h, which is not an embedding. Hence this presheaf is not a sheaf.

There are various examples of sheaves over a space, for example, if X is the underlying topological space of a smooth manifold, the assignment to each open subset U the set of differential q-forms $\Omega^q(U)$ is a sheaf, and the same is true for the set of *smooth* functions $f : U \to \mathbb{R}$, as well as the for C^k functions for any k. For a smooth manifold X, the sheaf of smooth functions is called the **structure sheaf** of X and is sometimes denoted by $\mathcal{O}_X$. In fact, whereas there can be more than one non-isomorphic structure of a smooth manifold on the same topological space X, if two smooth structures on X yield the same structure sheaf, they must be the same [1]. From this point of observation, we can view a sheaf *over a space* as a way of encoding extra algebraic or geometric structure attached to space.

As we shall see in Section I.1.3, any presheaf can be turned into a sheaf in a canonical way. More precisely, there is a functorial assignment to each presheaf F a sheaf aF and a morphism

$$\eta_F : F \to aF$$

such that for any morphism $\alpha : F \to G$ with G a sheaf, there is a unique arrow making the following diagram commute:

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[d, "\eta_F"'] & G . \\ aF \arrow[ur, dotted] & \end{tikzcd}$$

Denoting by $Sh(X)$ the full subcategory of presheaves $\mathrm{Set}^{\mathcal{O}(X)^{op}}$ consisting of those presheaves which are sheaves, we can state this abstractly by saying a is left adjoint to the inclusion

$$Sh(X) \hookrightarrow \mathrm{Set}^{\mathcal{O}(X)^{op}}$$

of sheaves into presheaves. The functor a is called the **sheafification** functor.

[1] We are slightly cheating here. For this statement to be correct, we need to regard $\mathcal{O}_X$ as a sheaf of rings, rather than a sheaf of sets.

I.1.2 Sheaves over a category of spaces

There are two points of view regarding sheaves. Sheaves *over a space* can be thought of as a way of encoding extra algebraic or geometric data. However, sheaves *over a category of spaces* play a different role as they may be regarded as geometric objects in their own right. In this subsection, we will define sheaves over the category of all spaces and explain this distinction. We will talk about sheaves over the category of all topological spaces $\mathbb{TOP}$ [2] but everything we say holds equally well for the category of smooth manifolds.

Definition I.1.3. A **presheaf** over the category of topological spaces is a contravariant functor $F : \mathbb{TOP}^{op} \to \mathrm{Set}$. Presheaves over topological spaces form a category where the morphisms are given by natural transformations.

Such a presheaf is a **sheaf** if and only if for every topological space X, and every open cover (U_i) of X, the induced map

$$F(X) \to \varprojlim \left[\prod F(U_i) \rightrightarrows \prod F(U_{ij})\right] \tag{I.3}$$

is an isomorphism. If the induced map is a monomorphism, then F is called **separated**.

Remark. Let X be a topological space. There is canonical functor

$$j_X : \mathcal{O}(X) \to X$$

which sends an open subset U of X to U considered as an object in $\mathbb{TOP}$ and sends each inclusion of open subsets to the associated embedding of topological spaces. This induces a functor between presheaf categories

$$j_X^* : \mathrm{Set}^{\mathbb{TOP}^{op}} \to \mathrm{Set}^{\mathcal{O}(X)^{op}}$$

which sends a presheaf $F : \mathbb{TOP}^{op} \to \mathrm{Set}$ to the presheaf $F \circ j_X^{op}$. Less abstractly,

$$j_X^*(F)(U) = F(U)$$

for each open subset U of X, and the restriction maps are defined in the obvious way. With this notation established, it is easy to see that F is a sheaf over $\mathbb{TOP}$ if and only if $j_X^*(F)$ is a sheaf over X for every space X. In other words, F is a sheaf over all spaces if and only if its restriction to each space is a sheaf. Similarly for a presheaf being separated.

Fix a topological space Y. Then Y determines a presheaf on $\mathbb{TOP}$ by the rule $\mathrm{Hom}_{\mathbb{TOP}}(\,\cdot\,, Y)$, that is, it assigns each space X the set of continuous functions from X to Y. We denote this presheaf by $y(Y)$. A presheaf of the form $y(Y)$ is called **representable**. By definition, we see that $j_X^*(y(Y))$ is

[2] For technical reasons, we should actually restrict to a Grothendieck universe of topological spaces.

the sheaf $C(\cdot, Y)$ over X from the previous subsection. In particular this implies that $j_X^*(y(Y))$ is a sheaf for all spaces X, hence $y(Y)$ is a sheaf over $\mathbb{TOP}$.

We digress into a small piece of category theory:

Lemma I.1.1. ***The Yoneda Lemma:*** *The assignment $T \mapsto y(T)$ is functorial and the functor $y : \mathbb{TOP} \to \mathrm{Set}^{\mathbb{TOP}^{op}}$ is full and faithful. Furthermore, for any space X and any presheaf F, there is a natural bijection*

$$\mathrm{Hom}_{\mathrm{Set}^{\mathbb{TOP}^{op}}}(y(T), F) \cong F(T).$$

Since y is fully faithful, we may simply denote $y(T)$ by T and "view" T as a sheaf over $\mathbb{TOP}$. Explicitly, T is a sheaf since for any space X, and any open cover $(U_i)_i$ of X, having a continuous function $f : X \to T$ is the same as having a collection of functions $f_i : U_i \to T$ which agree on pairwise intersections. In other words, the canonical map

$$\mathrm{Hom}(X, T) \to \varprojlim \left[\prod \mathrm{Hom}(U_i, T) \rightrightarrows \prod \mathrm{Hom}(U_{ij}, T) \right] \tag{I.4}$$

is an isomorphism. Now, consider an arbitrary sheaf F over $\mathbb{TOP}$. In light of the Yoneda lemma, we may rewrite the sheaf condition (I.3) to resemble (I.4), that is, it is equivalent to demanding that the canonical map

$$\mathrm{Hom}(X, F) \to \varprojlim \left[\prod \mathrm{Hom}(U_i, F) \rightrightarrows \prod \mathrm{Hom}(U_{ij}, F) \right] \tag{I.5}$$

is an isomorphism, where $\mathrm{Hom}(X, F)$ denotes the morphisms of presheaves from $y(X)$ to F, etc. So in light of the Yoneda lemma, a presheaf F is a sheaf if and only if having a morphism $f : X \to F$ is the same as having a collection of functions $f_i : U_i \to F$ which agree on pairwise intersections. In this sense, we may view sheaves over $\mathbb{TOP}$ as geometric objects which we can map into in a continuous way.

Remark. The fact that sheaves over $\mathbb{TOP}$ can be seen as geometric objects can be taken a bit further. For example, given a sheaf F over $\mathbb{TOP}$, we can define a topological space $u(F)$ as follows. As a set,

$$u(F) = F(*),$$

where $*$ is the one-point space. This assignment is clearly functorial and assembles into a functor

$$u : Sh(\mathbb{TOP}) \to \mathrm{Set}.$$

Note that for a topological space X, $u(X)$ is just its underlying set. Define a subset U of $u(F)$ to be open if for all topological spaces X and for all maps $f : X \to F$, $u(f)^{-1}(U)$ is open in X. By the special adjoint functor theorem, a presheaf F over topological spaces is representable if and only if is a so-called continuous functor, that is, if and only if it sends colimits in $\mathbb{TOP}$ to limits in Set. For such a functor F, it is easy to check that F is represented by the space $u(F)$.

Example 2. Just as we can speak of sheaves over the category of all topological spaces, we can just as well speak of sheaves over the category of locally compact Hausdorff spaces, $\mathbb{LCH}$. Any topological space X defines a sheaf $y_{\mathbb{LCH}}(X)$ which assigns a locally compact Hausdorff space T the set of continuous maps from T to X. However, not every space X can be faithfully represented by its sheaf over locally compact Hausdorff spaces. For any sheaf F over $\mathbb{LCH}$, one can mimic the construction of the topological space $u(F)$ as in the proceeding remark in the obvious way. The spaces of the form $u(F)$ for some sheaf F over $\mathbb{LCH}$ are precisely compactly generated topological spaces. If X is an arbitrary topological space, then $u(y_{\mathbb{LCH}}(X))$ is homeomorphic to its Kelley-completion $k(X)$, which is X endowed with the final topology with respect to all maps into it from compact Hausdorff spaces, and in this case

$$y_{\mathbb{LCH}}(X) = y_{\mathbb{LCH}}(k(X)).$$

If X is already compactly generated, then $k(X) \cong X$. Moreover, if X and Y are compactly generated, then

$$\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(y_{\mathbb{LCH}}(X), y_{\mathbb{LCH}}(Y)).$$

Hence, regarding compactly generated spaces as topological spaces, or as their induced sheaves over $\mathbb{LCH}$, are equivalent. This example illustrates how sheaves may be used as a formal tool to represent different types of topological spaces.

Example 3. Another example of how sheaves can be used to model geometric objects is the following. Let *Mfd* denote the category of smooth manifolds, then we can equally well speak of sheaves over *Mfd*. Given a sheaf F over *Mfd*, we may associate a topological space $u(F)$ in an analogous way, namely the points are $F(*)$ and the open subsets are those subsets $U \subseteq F(*)$ such that for all manifolds M and for all maps $f : M \to F$, $u(f)^{-1}(U)$ is open in M. The topological spaces of the form $u(F)$, for some sheaf over manifolds, are precisely those which are homeomorphic to the topological quotient of a possibly infinite disjoint union of Cartesian manifolds (i.e. manifolds of the form $\mathbb{R}^n$ for some n). However, one can recover more than just a topological space from a sheaf F over *Mfd*. Given a such a sheaf F, the topological space $u(F)$ comes equipped with a generalization of a differentiable structure, called a diffeology. Such a space is called a diffeological space, and just as for manifolds, the topology is determined by the diffeology, so, one often speaks of such a space simply as a set together with a diffeology on it. A sheaf F over *Mfd* is determined by its underlying diffeological space if and only if it is a so-called concrete sheaf, that is, if and only if for every manifold M, the assignment

$$\begin{array}{rcl} \operatorname{Hom}(M,F) & \to & \operatorname{Hom}(u(M),u(F)) \\ f & \mapsto & u(f) \end{array}$$

is injective.

Diffeological spaces are a category of generalized manifolds and are useful since this category has all limits and colimits, is (locally) Cartesian closed, and includes all infinite dimensional manifolds. In particular, for any two manifolds M and N, even if the smooth maps from M to N fail to form a Banach or Fréchet manifold, they will always form a diffeological space.

I.1.3 Sheaves over a Grothendieck site

We now turn to the general notion of a sheaf. This is an axiomatization of the two notions of sheaves already discussed which allows for a more general notion of cover. We will need this generality later in this thesis. General references for the content of this subsection are [23],[36], and [53].

Definition I.1.4. Let $\mathscr{C}$ be a small category. A **presheaf** on $\mathscr{C}$ is a contravariant functor from $\mathscr{C}$ to Set. Presheaves on $\mathscr{C}$ form a category $\mathrm{Set}^{\mathscr{C}^{op}}$, where the arrows are given by natural transformations.

Definition I.1.5. The **Yoneda embedding** is the functor

$$y:\mathscr{C}\to\mathrm{Set}^{\mathscr{C}^{op}}$$

from $\mathscr{C}$ into the category of presheaves on $\mathscr{C}$ given by

$$C\mapsto\operatorname{Hom}_{\mathscr{C}}(\,\cdot\,,C).$$

Such a presheaf $y(C)$ is called **representable**.

Lemma I.1.2. ***The Yoneda Lemma:*** *The functor y is fully-faithful. Furthermore, for any object C and any presheaf F, there is a natural bijection*

$$\operatorname{Hom}_{\mathrm{Set}^{\mathscr{C}^{op}}}(y(C),F)\cong F(C).$$

In light of this lemma, we will often denote $y(C)$ simply by C.

Remark. The category $\mathrm{Set}^{\mathscr{C}^{op}}$ is both complete and co-complete; limits are computed "point-wise":

$$\left(\varprojlim F_i\right)(X)=\varprojlim F_i(X)$$

where the limit to the right is computed in Set. Similarly for colimits.

Definition I.1.6. A **sieve** on an object C of $\mathscr{C}$ is a subobject of $y(C)$.

If R is a sieve on C and $f: D \to C$, we denote by $f^*(R)$ the subobject of D obtained by pulling back along f.

Definition I.1.7. A Grothendieck topology J on a small category $\mathscr{C}$ is an assignment to every object C of $\mathscr{C}$, a set $Cov(C)$ of **covering sieves** on C, such that

i) The maximal subobject $y(C)$ is a covering sieve on C.

ii) If R is a covering sieve on C and $f: D \to C$, then $f^*(R)$ is a covering sieve on D.

iii) If R is a sieve on C and S is a covering sieve on C such that for each object D and every arrow

$$f \in S(D) \subseteq \mathrm{Hom}_{\mathscr{C}}(D, C)$$

$f^*(R)$ is a covering sieve on D, then R is a covering sieve on C.

Definition I.1.8. A pair $(\mathscr{C}, J)$ of a small category $\mathscr{C}$ with a Grothendieck topology J is called a **site** .

Definition I.1.9. A **basis** for a Grothendieck topology is an assignment to each object C of $\mathscr{C}$ a set $\mathcal{B}(C)$ of families of arrows $(U_i \to C)_{i \in I}$, called **covering families**, such that

i) If $D \to C$ is an isomorphism, then $(D \to C)$ is in $\mathcal{B}(C)$.

ii) If $(U_i \to C)_i$ is in $\mathcal{B}(C)$ and $f: D \to C$, then the fibered products $U_i \times_C D$ exist and the set of the induced maps $U_i \times_C D \to D$ is in $\mathcal{B}(D)$.

iii) If $(U_i \to C)_i$ is in $\mathcal{B}(C)$ and for each i, $(V_{ij} \to U_i)_j$ is in $\mathcal{B}(U_i)$, then $(V_{ij} \to C)_{ij}$ is in $\mathcal{B}(C)$.

To each covering family $\mathcal{U} = (U_i \to C)_i$, there is an associated sieve

$$S_{\mathcal{U}}(D) := \{f: D \to C \text{ such that } f \text{ factors through } U_i \text{ for some } i\}.$$

The Grothendieck topology generated by the basis $\mathcal{B}$ is given by saying a sieve R on C is in $Cov(C)$ if there exists a covering family $\mathcal{U}$ such that $S_{\mathcal{U}} \subseteq R$.

Example 4. The small site of a topological space: If X is a topological space, we denote its poset of open subsets by $\mathcal{O}(X)$. Considering this poset as a category, there is an obvious choice for a basis for a Grothendieck topology on $\mathcal{O}(X)$. Covering families in this basis are given by open covers, that is families of the form

$$(U_\alpha \hookrightarrow U)_\alpha,$$

where the inclusions $U_\alpha \hookrightarrow U$ constitute an open cover of U. Consequently, the Grothendieck topology generated by this basis is called the **open cover topology**. The site consisting of $\mathcal{O}(X)$ together with this topology is called the **small site** for the space X.

Example 5. By choosing a Grothendieck universe, we can choose a small category of topological spaces and continuous maps, $\mathbb{TOP}$. There is an analogously defined basis for a Grothendieck topology on this category, where

$$(U_\alpha \hookrightarrow Y)_\alpha,$$

is a covering family if and only if $(U_\alpha)_\alpha$ is an open cover of Y. By abuse of terminology, we will also refer to the generated Grothendieck topology as the **open cover topology**.

For X a fixed topological space, one may define the **open cover topology** on the slice category $\mathbb{TOP}/X$ in a completely analogous manner. $\mathbb{TOP}/X$ together with this topology is called the **large site** for the space X.

Remark. If $\mathscr{C}$ is a small category, then every Grothendieck topology on $\mathscr{C}$ has a generating basis:

Let $\mathcal{B}_{max(J)}(C)$ be the set of all families $\mathcal{U} = (U_i \to C)_{i \in I}$ such that $S_{\mathcal{U}}$ is in $Cov(C)$. This is the so-called maximal basis and it generates J.

Definition I.1.10. A morphism $f : D \to C$ in $\mathscr{C}$ is said to **admit local sections** with respect to the topology J (generated by the basis $\mathcal{B}$) if there exists a covering family $\mathcal{U} = (U_i \to C)_i$ of C and morphisms $\sigma_i : U_i \to D$ called **local sections** such that the following diagram commutes for all i:

$$\begin{tikzcd} & D \arrow[dr, "f"] & \\ U_i \arrow[ur, "\sigma_i"] \arrow[rr] & & C. \end{tikzcd}$$

Definition I.1.11. A presheaf F in $\mathrm{Set}^{\mathscr{C}^{op}}$ is a **sheaf** if for every object C, if S is a covering sieve on C, then the map

$$\mathrm{Hom}_{\mathrm{Set}^{\mathscr{C}^{op}}}(C, F) \to \mathrm{Hom}_{\mathrm{Set}^{\mathscr{C}^{op}}}(S, F),$$

induced by composition, is a bijection.

If the induced map is injective, the presheaf is called **separated**.

We denote the full-subcategory of $\mathrm{Set}^{\mathscr{C}^{op}}$ consisting of those presheaves which are sheaves by $Sh_J(\mathscr{C})$. A Grothendieck topology is called **subcanonical** if every representable presheaf $y(C)$ is a sheaf.

It is immediate from the definition that the limit of any small diagram of sheaves is again a sheaf.

If $\mathcal{B}$ is a basis for the topology J, then it suffices to check the sheaf condition for every sieve of the form $S_{\mathcal{U}}$. This is equivalent to saying a presheaf is a sheaf if and only if for every covering family $(U_i \to C)_i$, the induced map

$$F(C) \to \varprojlim \left[\prod F(U_i) \rightrightarrows \prod F(U_{ij})\right] \tag{I.6}$$

is a bijection, where U_{ij} denotes the fibered product $U_i \times_C U_j$.

Similarly, F is separated if and only if the induced map is injective.

Remark. For presheaves over $\mathcal{O}(X)$, this notion of sheaf and separated presheaf agrees with the previously defined notions, and similarly for presheaves over $\mathbb{TOP}$.

Proposition I.1.1. *If J is subcanonical and $(U_i \to C)_i$ is a covering family for an object C, then,* in the category of sheaves,

$$C \cong \varinjlim \left[\coprod U_{ij} \rightrightarrows \coprod U_i\right] \tag{I.7}$$

Proof. Let F be a sheaf. By the Yoneda lemma,

$$\begin{aligned}
\mathrm{Hom}_{Sh_J(\mathscr{C})}(C, F) &\cong F(C) \\
&\cong \varprojlim \left[\prod F(U_i) \rightrightarrows \prod F(U_{ij})\right] \\
&\cong \varprojlim \left[\prod \mathrm{Hom}_{Sh_J(\mathscr{C})}(U_i, F) \rightrightarrows \prod \mathrm{Hom}_{Sh_J(\mathscr{C})}(U_{ij}, F)\right] \\
&\cong \mathrm{Hom}_{Sh_J(\mathscr{C})}\left(\varinjlim \left[\coprod U_{ij} \rightrightarrows \coprod U_i\right], F\right).
\end{aligned}$$

□

Definition I.1.12. A morphism $\varphi : F \to G$ of sheaves is called **representable** if for any object $C \in \mathscr{C}$ and any morphism $C \to G$, the pullback $C \times_G F$ in the category of sheaves is (isomorphic to) an object D of $\mathscr{C}$.

For any object C, the covering sieves $Cov(C)$ form a category by reverse inclusion. Hence, we can define a functor

$$(\,\cdot\,)^+ : \mathrm{Set}^{\mathscr{C}^{op}} \to \mathrm{Set}^{\mathscr{C}^{op}}$$

by

$$F^{+}(C) := \varinjlim_{S \in Cov(C)} \mathrm{Hom}_{\mathrm{Set}^{\mathscr{C}^{op}}}(S, F). \tag{I.8}$$

This functor is called the **plus-construction.** One can also define $(\cdot)^{+}$ by using a basis $\mathcal{B}$ and arrive at a naturally isomorphic functor. To do this, let $cov(C)$ (not to be confused with $Cov(C)$) denote the category of covering families on C, where we define an arrow between two covering families

$$(U_i \to C)_{i \in I} \to (V_j \to C)_{j \in I}$$

to be a function $\lambda : I \to J$ and a collection of maps $f_i : U_i \to V_{\lambda(i)}$ such that

$$\begin{tikzcd} U_i \arrow[rr, "f_i"] \arrow[dr] & & V_{\lambda(i)} \arrow[dl] \\ & C & \end{tikzcd}$$

commutes.

We can then define

$$F^{+}(C) := \varinjlim_{\mathcal{U} \in cov(C)} \left[\varprojlim \left[\prod F(U_i) \rightrightarrows \prod F(U_{ij}) \right] \right]. \tag{I.9}$$

A presheaf F is separated if and only if the canonical map $F \to F^{+}$ is a monomorphism. It is a sheaf if and only if this map is an isomorphism.

If F is separated, then F^{+} is a sheaf. Furthermore, F^{+} is always separated, for any F. Hence, F^{++} is always a sheaf. We denote by a_J the functor

$$F \to F^{++},$$

and call it the **sheafification functor**. If F is separated, $a_J(F) \cong F^{+}$, and if F is a sheaf, $a_J(F) \cong F$. The functor a_J is left adjoint to the inclusion

$$i : Sh_J(\mathscr{C}) \hookrightarrow \mathrm{Set}^{\mathscr{C}^{op}}$$

and is **left-exact** , i.e. preserves finite limits.

Remark. The category $Sh_J(\mathscr{C})$ is both complete and co-complete. Since i is a right adjoint, it follows that the computation of limits of sheaves can be done in the category $\mathrm{Set}^{\mathscr{C}^{op}}$, hence can be done “point-wise”. To compute the colimit of a diagram of sheaves, one must first compute it in $\mathrm{Set}^{\mathscr{C}^{op}}$ and then apply the sheafification functor a_J.

Definition I.1.13. Let F be a presheaf on $\mathscr{C}$. Define its **category of elements** to be full subcategory of the slice category $\mathrm{Set}^{\mathscr{C}^{op}}/F$ consisting of those objects of the form $C \to F$ with C an object of $\mathscr{C}$. We denote this category by $\mathscr{C}/F$.

For such a presheaf F, define the functor $\theta_F : \mathscr{C}/F \to \mathrm{Set}^{\mathscr{C}^{op}}$ to be the composite of the forgetful functor

$$\mathrm{Set}^{\mathscr{C}^{op}}/F \to \mathrm{Set}^{\mathscr{C}^{op}}$$

with the inclusion

$$\mathscr{C}/F \to \mathrm{Set}^{\mathscr{C}^{op}}/F,$$

i.e. it sends $C \to F$ to C.

Proposition I.1.2. *For any presheaf F, F is the colimit of the functor θ_F.*

Proof. This follows immediately from the Yoneda Lemma. □

Remark. Often this proposition is stated by saying "F is a colimit of representables," and is denoted by

$$F \cong \varinjlim_{C \to F} C.$$

Now suppose that $\mathscr{D}$ is any co-complete category, i.e. one which has all small colimits, and $f : \mathscr{C} \to \mathscr{D}$ is any functor. Then f induces a pair of adjoint functors

$$Set^{\mathscr{C}^{op}} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathscr{D}\ ,$$

with $f^* \dashv f_*$. Explicitly

$$f_*(D)(C) = \mathrm{Hom}_{\mathscr{D}}(f(C), D),$$

and f is uniquely determined by the fact that it is colimit preserving and

$$f^*(y(C)) = f(C).$$

It follows that f^* is given explicitly by

$$f^*(F) = \varinjlim_{C \to F} f(C).$$

Definition I.1.14. The functor f^* is the **left Kan extension** of f along the Yoneda embedding.

In fact, this construction induces an equivalence of categories between the category of functors $\mathscr{D}^{\mathscr{C}}$ and the category of adjunctions between $\mathrm{Set}^{\mathscr{C}^{op}}$ and $\mathscr{D}$.

The following proposition can be easily checked:

Proposition I.1.3. *Fix F in* $\mathrm{Set}^{\mathscr{C}^{op}}$ *and let* $\mathscr{C}' = \mathscr{C}/F$, *and* $\mathscr{D} = \mathrm{Set}^{\mathscr{C}^{op}}/F$. *Let*

$$f : \mathscr{C}' \to \mathscr{D}$$

be given by the canonical inclusion. Then the pair $f^* \dashv f_*$ is an adjoint equivalence

$$Set^{(\mathscr{C}/F)^{op}} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathrm{Set}^{\mathscr{C}^{op}}/F\ .$$

Moreover, suppose that J is a Grothendieck topology on $\mathscr{C}$ coming from a basis $\mathcal{B}$, and that F is a J-sheaf. Then there is an induced Grothendieck topology J_F on $\mathrm{Set}^{\mathscr{C}^{op}}/F$ *given by saying a family of maps in $\mathscr{C}/F$ is a covering family if and only if it is a covering family on $\mathscr{C}$ after applying the forgetful functor, and the adjoint equivalence $f^* \dashv f_*$ restricts to an adjoint equivalence*

$$Sh_{J_F}(\mathscr{C}/F) \underset{f_*}{\overset{f^*}{\leftrightarrows}} Sh_J(\mathscr{C})/F\ .$$

Remark. If $\mathscr{C} = \mathbb{TOP}$, J is the open cover topology, and $F = X$ is a representable sheaf, then this implies that a sheaf over $\mathbb{TOP}/X$, a so-called *large sheaf over* X, is the same as a sheaf G over $\mathbb{TOP}$ together with a map $G \to X$.

I.1.4 Cartesian closed categories

In this subsection, we introduce the concept of Cartesian closed categories. This concept is quite central to this thesis, as one of the main results of Chapter II is the construction of a Cartesian closed bicategory of topological stacks.

Definition I.1.15. Let $\mathscr{C}$ be a small category. An object X is said to be **exponentiable** if the functor

$$\begin{array}{rcl} (\,\cdot\,) \times X : \mathscr{C} & \to & \mathscr{C} \\ Y & \mapsto & Y \times X \end{array}$$

has a right adjoint

$$\begin{array}{rcl} (\,\cdot\,)^X : \mathscr{C} & \to & \mathscr{C} \\ Y & \mapsto & Y^X. \end{array}$$

In other words, for every object Y of $\mathscr{C}$, there exists an "object of maps" from Y to X, denoted by Y^X such that for any object Z, there is a natural isomorphism

$$\mathrm{Hom}\left(Z, Y^X\right) \cong \mathrm{Hom}\left(Z \times X, Y\right).$$

Definition I.1.16. A category $\mathscr{C}$ is **Cartesian closed** if it has binary products, and every object X of $\mathscr{C}$ is exponentiable.

Example 6. The category Set of sets is Cartesian closed. In this case, Y^X is given by the set of all functions from Y to X.

Example 7. The category $\mathbb{TOP}$ of topological space is *not* Cartesian closed, however, the full subcategory of compactly generated Hausdorff spaces, $\mathbb{CGH}$ is Cartesian closed, where Y^X is the set of continuous maps from Y to X endowed with an appropriate topology.

Proposition I.1.4. *Let $\mathscr{C}$ be any small category. Then $\mathrm{Set}^{\mathscr{C}^{op}}$ is Cartesian closed.*

Proof. Suppose that F and G are two presheaves. Note that, for every object C of $\mathscr{C}$ we must have

$$\mathrm{Hom}\left(C, F^G\right) \cong \mathrm{Hom}\left(C \times G, F\right).$$

By the Yoneda Lemma, this implies that we must have

$$F^G\left(C\right) \cong \mathrm{Hom}\left(C \times G, F\right).$$

Hence, we can *define* F^G to the presheaf $C \mapsto \mathrm{Hom}\left(C \times G, F\right)$. It suffices to show that F^G satisfies the correct universal property for arbitrary presheaves G, not just representables. Let H be such a presheaf. Then we have the following chain of isomorphisms:

$$\begin{aligned}
\mathrm{Hom}\left(H, F^G\right) &\cong \mathrm{Hom}\left(\varinjlim_{C \to H} C, F^G\right) \\
&\cong \varprojlim_{C \to H} \mathrm{Hom}\left(C, F^G\right) \\
&\cong \varprojlim_{C \to H} \mathrm{Hom}\left(C \times G, F\right) \\
&\cong \mathrm{Hom}\left(\varinjlim_{C \to H} \left(C \times G\right), F\right).
\end{aligned}$$

However, since limits in $\mathrm{Set}^{\mathscr{C}^{op}}$ are computed point-wise in Set, and binary products preserve colimits in Set, we have

$$\varinjlim\left(C \times G\right) \cong \varinjlim\left(C\right) \times G$$

in $\mathrm{Set}^{\mathscr{C}^{op}}$. Since $\mathrm{Set}^{\mathscr{C}^{op}}$ has all limits, this shows $\mathrm{Set}^{\mathscr{C}^{op}}$ is Cartesian closed. $\square$

Lemma I.1.3. *Let $(\mathscr{C}, J)$ be a Grothendieck site and let C be an object of $\mathscr{C}$ and F a J-sheaf. Then F^C, is a J-sheaf.*

Proof. Let $S \rightarrowtail D$ be a covering sieve. It suffices to show the induced map

$$\mathrm{Hom}\left(D, F^C\right) \to \mathrm{Hom}\left(S, F^C\right)$$

is an isomorphism. Notice that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}\left(D, F^C\right) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Hom}\left(S, F^C\right) \arrow[d, "\wr"] \\ \mathrm{Hom}\left(D \times C, F\right) \arrow[r] & \mathrm{Hom}\left(S \times C, F\right), \end{tikzcd}$$

where the isomorphisms are given by the Yoneda Lemma. However, since S is a covering sieve of D, it follows that $S \times C$ is a covering sieve of $D \times C$. Since F is a sheaf, it follows that

$$\mathrm{Hom}\left(D \times C, F\right) \to \mathrm{Hom}\left(S \times C, F\right)$$

an isomorphism, so we are done since the diagram commutes. □

Theorem I.1.4. *Let $(\mathscr{C}, J)$ be any Grothendieck site. Then $Sh_J(\mathscr{C})$ is Cartesian closed.*

Proof. Let F be a J-sheaf and G any presheaf. We will show that F^G is a J-sheaf. Consider the presheaf

$$\varprojlim_{C \to G} F^C.$$

Since each F^C is a sheaf and limits of sheaves are sheaves, it follows that this presheaf is a sheaf as well. Let D be any object of $\mathscr{C}$. Consider the chain of isomorphisms

$$\begin{aligned} \left(\varprojlim_{C \to G} F^C\right)(D) &\cong \mathrm{Hom}\left(D, \varprojlim_{C \to G} F^C\right) \\ &\cong \varprojlim_{C \to G} \mathrm{Hom}\left(D, F^C\right) \\ &\cong \varprojlim_{C \to G} \mathrm{Hom}\left(D \times C, F\right) \\ &\cong \mathrm{Hom}\left(\varinjlim_{C \to G}\left(D \times C\right), F\right) \\ &\cong \mathrm{Hom}\left(D \times \varinjlim_{C \to G} C, F\right) \\ &\cong \mathrm{Hom}\left(D \times G, F\right) \\ &= F^G(D). \end{aligned}$$

It follows that $F^G \cong \varprojlim_{C \to G} F^C$, and since the latter is a sheaf, we are done. □

I.1.5 2-category theory

So far, we have gotten by with ordinary category theory. However, in order to study the theory of stacks, one must use 2-category theory. This subsection is a quick summary of this subject and leaves out most of the details. For details see [10].

We first start with the notion of a 2-category:

Definition I.1.17. A 2-category is a category enriched in $(\mathcal{CAT}, \times)$, the monoidal category of small categories. (See for instance [31].)

Spelling this out, this means that for any two objects C and D of a 2-category, instead of having a *set* of morphisms, they have a *category*

$$\operatorname{Hom}_{\mathscr{C}}(C, D)$$

of morphisms. An object f of the category $\operatorname{Hom}_{\mathscr{C}}(C, D)$ will be denoted by

$$f : C \to D$$

and will be called a **1-morphism** of $\mathscr{C}$ (or sometimes simply an arrow or morphism). If $g : C \to D$ is another object of $\operatorname{Hom}_{\mathscr{C}}(C, D)$, and α is an arrow from f to g in this morphism-category, we call this a **2-morphism** (or sometimes a **2-cell**) and denote it by

$$\alpha : f \Rightarrow g$$

or sometimes more pictorially as

$$\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & D. \end{tikzcd}$$

Since $\operatorname{Hom}_{\mathscr{C}}(C, D)$ is a category, it comes equipped with a composition. We will denote this composition by $\bullet$. If

$$\alpha : f \Rightarrow g$$

and

$$\beta : g \Rightarrow h,$$

the composite

$$\beta \bullet \alpha : f \Rightarrow h$$

is called the **vertical composition** of the two 2-morphisms α and β.

Since $\mathscr{C}$ is a $(\mathcal{CAT}, \times)$-enriched *category*, it comes equipped with its own composition. That is, for any triple of objects C, D, and E of $\mathscr{C}$, there exists a composition functor

$$c_{C,D,E} : \mathrm{Hom}_{\mathscr{C}}(C, D) \times \mathrm{Hom}_{\mathscr{C}}(D, E) \to \mathrm{Hom}_{\mathscr{C}}(C, E).$$

If $f : C \to D$ and $g : D \to E$ are 1-morphisms of $\mathscr{C}$, we denote their composition $c_{C,D,E}(f, g)$ by $g \circ f$, just like we do in a 1-category. We adopt a similar notation for 2-cells. If we are given

$$\begin{tikzcd} C \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\Downarrow\beta"] & E, \end{tikzcd}$$

we denote $c_{C,D,E}(\alpha, \beta)$ by $\beta \circ \alpha$ which can be expressed pictorially as

$$\begin{tikzcd} C \arrow[r, bend left, "g_1 \circ f_1"] \arrow[r, bend right, "g_2 \circ f_2"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & E. \end{tikzcd}$$

We should note that it is customary to abuse some of this notation sometimes. For example, if we are given a diagram of the form

$$\begin{tikzcd} C \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \arrow[r, "h"] & E, \end{tikzcd}$$

$id_h \circ \alpha$ is often written as $h \circ \alpha$ or simply $h\alpha$. We will use the latter of these in this thesis.

As $\mathscr{C}$ is an enriched category, the composition $\circ$ is associative, both on 1-morphisms and 2-morphisms.

Again, since $\mathscr{C}$ is an enriched category, it has identities. This means, for every object C of $\mathscr{C}$, there is a 1-morphism $id_C : C \to C$, i.e. an object

$$id_C \in \mathrm{Hom}_{\mathscr{C}}(C, C)_0.$$

It satisfies the following:

Suppose that $f : C \to D$ is a 1-morphism. Then

$$f \circ id_C = f \tag{I.10}$$

and

$$id_D \circ f = f. \tag{I.11}$$

Example 8. The category of small categories, $\mathcal{CAT}$, can be promoted to a 2-category in a natural way. For any two categories $\mathscr{C}$ and $\mathscr{D}$, let $\mathrm{Hom}_{\mathcal{CAT}}(\mathscr{C}, \mathscr{D})$ denote the functor category, whose objects are functors from $\mathscr{C}$ to $\mathscr{D}$ and arrows are natural transformations.

There is also a weakened version of a 2-category called a **bicategory**. Roughly speaking, a bicategory is a category which is weakly enriched in categories. More explicitly, if $\mathscr{C}$ is a bicategory, then any two objects C and D have a category of morphisms $\mathrm{Hom}_{\mathscr{C}}(C, D)$, so hence a notion of (associative) vertical composition of 2-cells, just as the case for 2-categories, and for each triple of objects C, D, and E of $\mathscr{C}$, there exists a composition functor

$$c_{C,D,E} : \mathrm{Hom}_{\mathscr{C}}(C, D) \times \mathrm{Hom}_{\mathscr{C}}(D, E) \to \mathrm{Hom}_{\mathscr{C}}(C, E).$$

However, these composition functors no longer need to encode an *associative* composition; these compositions need only be associative up to isomorphism. Explicitly, for any composable 1-morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D,$$

there is a distinguished 2-morphism

$$a_{(f,g,h)} : h \circ (g \circ f) \Rightarrow (h \circ g) \circ f,$$

which is an isomorphism in $\mathrm{Hom}_{\mathscr{C}}(A, D)$. In particular, $h \circ (g \circ f)$ and $(h \circ g) \circ f$ may not be equal, but are at least isomorphic.

Moreover, bicategories come equipped with identities, but these are also weak. For each object C there is a 1-morphism $id_C : C \to C$ but equations (I.10) and (I.11) need only hold up to invertible 2-morphisms, just as for the case of the associativity of composition of 1-morphisms. These 2-morphisms, as well as the associator 2-morphisms $a_{(f,g,h)}$, must satisfy the obvious coherency conditions. We invite the reader to write them down themselves, or to see [10] for details.

Remark. Every 2-category is canonically a bicategory.

Definition I.1.18. A 1-morphism $f : C \to D$ in a bicategory $\mathscr{C}$ is an **equivalence** if there exists another 1-morphism $g : D \to C$ and two invertible 2-morphisms

$$\alpha : g \circ f \Rightarrow id_C$$

and

$$\beta : id_D \Rightarrow f \circ g.$$

In such a situation, C and D are said to be **equivalent**. We denote equivalence by $C \simeq D$.

In a bicategory, equivalences are much more natural than isomorphisms. For example, an equivalence in $\mathscr{C} = \mathcal{CAT}$ is an equivalence of categories, in the ordinary sense.

By a 2-**functor** between two 2-categories $\mathscr{C}$ and $\mathscr{D}$, we mean a $\mathcal{CAT}$-enriched functor. Such a 2-functor $F : \mathscr{C} \to \mathscr{D}$ consists of a map $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ and for each pair of objects A and B, functors

$$F(A,B) : \mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{\mathscr{D}}(FA,FB),$$

which respect composition and identities. In this thesis, we will sometimes refer to a 2-functor simply as a functor, when its domain and codomain are clearly 2-categories. If $\mathscr{C}$ and $\mathscr{D}$ are bicategories, a homomorphism of bicategories

$$F : \mathscr{C} \to \mathscr{D}$$

is essentially the same as a 2-functor, in that it consists of a map $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ and functors

$$F(A,B) : \mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{\mathscr{D}}(FA,FB),$$

but these functors need only respect composition and identities up to isomorphism, and these isomorphisms must satisfy certain coherency relations. We refer the reader to [10].

Definition I.1.19. A 2-functor (or in the setting of bicategories, homomorphism) $F : \mathscr{C} \to \mathscr{D}$ is an **equivalence of** 2-**categories** (respectively, an equivalence of bicategories), if it satisfies the following two properties:

i) (Essentially surjective): For every object D of $\mathscr{D}$, there exists an object C of $\mathscr{C}$ such that $F(C) \simeq D$.

ii) (Fully-faithful): For every two objets A and B of $\mathscr{C}$, the functor

$$F(A,B) : \mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{\mathscr{D}}(FA,FB)$$

is an equivalence of categories.

Remark. For strict 2-categories, this definition of equivalence agrees with the notion of an equivalence of $\mathcal{CAT}$-enriched categories. This philosophy carries over to many other concepts. For example, a 2-adjunction is the same as $\mathcal{CAT}$-enriched adjunction as in [31].

In this thesis, we will sometimes use the following standard terminology:

Definition I.1.20. A $(2,1)$-category is a bicategory in which every 2-morphism is invertible.

Every bicategory discussed in this thesis will in fact be a $(2,1)$-category. A standard example is the 2-category of small groupoids, which is a full sub-2-category of $\mathscr{CAT}$; any natural transformation between two functors with codomain a groupoid, is automatically invertible.

Some notation:

In this thesis, we will use $\underleftarrow{\operatorname{holim}}$ and $\underrightarrow{\operatorname{holim}}$ to denote weak limits and weak colimits respectively. See Section A.1 for more information.

I.1.6 Grothendieck topoi

A concise definition of a Grothendieck topos is as follows:

Definition I.1.21. A category $\mathscr{E}$ is a Grothendieck topos if it is a reflective subcategory of a presheaf category $Set^{\mathscr{C}^{op}}$ for some small category $\mathscr{C}$,

$$\mathscr{E} \underset{j_*}{\overset{j^*}{\rightleftarrows}} Set^{\mathscr{C}^{op}}, \tag{I.12}$$

with $j^* \dashv j_*$, such that the left adjoint j^* preserves finite limits (i.e. is left-exact). From here on in, topos will mean Grothendieck topos.

Remark. It is standard that this is the same as saying that $\mathscr{E}$ is equivalent to $Sh_J(\mathscr{C})$ for some Grothendieck topology J on $\mathscr{C}$. From the closing remarks Section I.1.3, one direction is clear, namely, since the sheafification functor a_J is left-exact, any category of the form $Sh_J(\mathscr{C})$ is a Grothendieck topos. Conversely, given a left-exact reflective subcategory $\mathscr{E}$ of $\mathrm{Set}^{\mathscr{C}^{op}}$,

$$\mathscr{E} \underset{j_*}{\overset{j^*}{\rightleftarrows}} Set^{\mathscr{C}^{op}},$$

one may declare a sieve on an object C, represented by a monomorphism

$$m : R \hookrightarrow y(C),$$

to be a covering sieve if and only if $j^*(m)$ is an isomorphism. For details, see for example [37].

Definition I.1.22. A **geometric morphism** from a topos $\mathscr{E}$ to a topos $\mathcal{F}$ is an adjoint-pair

$$\mathscr{E} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{F},$$

with $f^* \dashv f_*$, such that f^* preserve finite limits. The functor f_* is called the **direct image** functor, whereas the functor f^* is called the **inverse image** functor.

In particular, this implies, somewhat circularly, that equation (I.12) is an example of a geometric morphism.

Topoi form a 2-category. Their arrows are geometric morphisms. If f and g are geometric morphisms from $\mathscr{E}$ to $\mathcal{F}$, a 2-cell

$$\alpha : f \Rightarrow g$$

is given by a natural transformation

$$\alpha : f^* \Rightarrow g^*.$$

In this thesis, we will simply ignore all non-invertible 2-cells to arrive at a $(2,1)$-category of topoi, $\mathfrak{Top}$.

I.1.7 Stacks

Weak presheaves of groupoids

Let *Gpd* denote the 2-category of groupoids[3], functors, and natural transformations. Note that this is in fact a $(2,1)$-category as every natural transformation of groupoid functors is automatically a natural isomorphism.

Definition I.1.23. Let $\mathscr{C}$ be a small category. A weak presheaf of groupoids $\mathscr{X}$ on $\mathscr{C}$ is a weak 2-functor

$$\mathscr{X} : \mathscr{C}^{op} \to Gpd,$$

that is a contravariant homomorphism of bicategories. Explicitly, it assigns to each object C of $\mathscr{C}$ a groupoid $\mathscr{X}(C)$, to each arrow $f : C \to D$ of $\mathscr{C}$ a functor of groupoids $\mathscr{X}(f) : \mathscr{X}(D) \to \mathscr{X}(C)$, and to each pair $g : B \to C$ and $f : C \to D$ of composable arrows of $\mathscr{C}$, a natural transformation

$$\begin{tikzcd} & \mathscr{X}(C) \arrow[dr, "\mathscr{X}(g)"] \arrow[d, phantom, "\Downarrow \mathscr{X}(f{,}g)"] & \\ \mathscr{X}(D) \arrow[ur, "\mathscr{X}(f)"] \arrow[rr, "\mathscr{X}(fg)"'] & {} & \mathscr{X}(B) \end{tikzcd} \tag{I.13}$$

such that for composable triples

$$A \xrightarrow{h} B \xrightarrow{g} C \xrightarrow{f} D$$

of arrows of $\mathscr{C}$, the pentagon

$$\begin{tikzcd} & \big(\mathscr{X}(h)\mathscr{X}(g)\big)\mathscr{X}(f) \arrow[r, equal] \arrow[dl, Rightarrow, "\mathscr{X}(g{,}h)\mathscr{X}(f)"'] & \mathscr{X}(h)\big(\mathscr{X}(g)\mathscr{X}(f)\big) \arrow[dr, Rightarrow, "\mathscr{X}(h)\mathscr{X}(f{,}g)"] & \\ \mathscr{X}(gh)\mathscr{X}(f) \arrow[drr, Rightarrow, "\mathscr{X}(f{,}gh)"'] & & & \mathscr{X}(h)\mathscr{X}(fg) \arrow[dl, Rightarrow, "\mathscr{X}(fg{,}h)"] \\ & & \mathscr{X}(fgh) & \end{tikzcd}$$

[3]Technically, those groupoids which are equivalent to a small category.

commutes.

Remark. What we have described is technically a *strict* homomorphism of bicategories. In general, a homomorphism need only respect identities up to isomorphism. However, every homomorphism $\mathscr{C}^{op} \to Gpd$ of bicategories is equivalent to one of the form just described.

Example 9. Let G be a topological group. Consider the assignment to each topological space X the category of principal G-bundles over X, $Bun_G(X)$. Since any morphism of principal bundles over a fixed base must be an isomorphism, it follows that $Bun_G(X)$ is a groupoid. If $f : X \to Y$ is a continuous map, then there is a canonically induced functor

$$f^* : Bun_G(Y) \to Bun_G(X)$$

which sends a principal G-bundle P over Y to its pullback-bundle $f^*(P)$ over X. If $g : Y \to Z$ is another continuous functor and Q is a principal G-bundle over Z, then although the two pullback-bundles $f^*(g^*(Q))$ and $(gf)^*(Q)$ are *isomorphic*, they are not *equal*, hence Bun_G fails to be a functor from $\mathbb{TOP}$ to the 1-category of groupoids. However, the reader is encouraged to verify that it is indeed a weak presheaf of groupoids.

Definition I.1.24. A **weak natural transformation** of $\varphi : \mathscr{Y} \to \mathscr{X}$ of weak presheaves of groupoids on $\mathscr{C}$ is an assignment to each object D of $\mathscr{C}$ a functor

$$\varphi(D) : \mathscr{Y}(D) \to \mathscr{X}(D)$$

and to each arrow $f : C \to D$ of $\mathscr{C}$ a natural transformation

$$\begin{tikzcd} \mathscr{Y}(D) \arrow[r, "\varphi(D)"] \arrow[d, "\mathscr{Y}(f)"'] & \mathscr{X}(D) \arrow[d, "\mathscr{X}(f)"] \\ \mathscr{Y}(C) \arrow[r, "\varphi(C)"'] \arrow[ur, Rightarrow, "\varphi(f)"] & \mathscr{X}(C) \end{tikzcd}$$

such that for each pair of composable arrows $g : B \to C$ and $f : C \to D$ of $\mathscr{C}$, the pentagon

$$\begin{tikzcd} & \mathscr{X}(g)\varphi(C)\mathscr{Y}(f) \arrow[rr, Rightarrow, "\mathscr{X}(g)\varphi(f)"] & & \mathscr{X}(g)\mathscr{X}(f)\varphi(D) \arrow[dr, Rightarrow, "\mathscr{X}(f{,}g)\varphi(D)"] & \\ \varphi(B)\mathscr{Y}(g)\mathscr{Y}(f) \arrow[ur, Rightarrow, "\varphi(g)\mathscr{Y}(f)"] \arrow[drr, Rightarrow, "\varphi(B)\mathscr{Y}(f{,}g)"'] & & & & \mathscr{X}(fg)\varphi(D) \\ & & \varphi(B)\mathscr{Y}(fg) \arrow[urr, Rightarrow, "\varphi(fg)"'] & & \end{tikzcd}$$

commutes.

These weak natural transformations can be composed in the obvious way. They form the 1-morphisms for the 2-category of weak presheaves of groupoids, which we will denote by $Gpd^{\mathscr{C}^{op}}$.

Definition I.1.25. The 2-morphisms $\alpha : \psi \Rightarrow \varphi$ between a pair of arrows

$$\varphi : \mathscr{Y} \to \mathscr{X}$$

and

$$\psi : \mathscr{Y} \to \mathscr{X},$$

are called **modifications** and assign to each object D of $\mathscr{C}$ a natural transformation $\alpha(D) : \psi(D) \Rightarrow \varphi(D)$ such that for each arrow $f : C \to D$ of $\mathscr{C}$ the square

$$\begin{tikzcd}
\psi(C)\mathscr{Y}(f) \arrow[r, Rightarrow, "\alpha(C)\mathscr{Y}(f)"] \arrow[d, Rightarrow, "\psi(f)"'] & \varphi(C)\mathscr{Y}(f) \arrow[d, Rightarrow, "\varphi(f)"] \\
\mathscr{X}(f)\psi(D) \arrow[r, Rightarrow, "\mathscr{X}(f)\alpha(D)"] & \mathscr{X}(f)\varphi(D)
\end{tikzcd} \tag{I.14}$$

commutes.

There is an obvious notion of horizontal and vertical composition which give $Gpd^{\mathscr{C}^{op}}$ the structure of a strict 2-category. Furthermore, it follows directly that every modification has an inverse, hence $Gpd^{\mathscr{C}^{op}}$ is in fact a category enriched in groupoids, i.e. a strict $(2,1)$-category.

There exists a canonical inclusion

$$(\,\cdot\,)^{id} : \mathrm{Set}^{\mathscr{C}^{op}} \to Gpd^{\mathscr{C}^{op}},$$

where each presheaf F is sent to the weak presheaf which assigns to each object C the category $(F(C))^{id}$ whose objects are $F(C)$ and whose arrows are all identities. If C is an object of $\mathscr{C}$, we usually denote $(y(C))^{id}$ simply by C.

Remark. The 2-category $Gpd^{\mathscr{C}^{op}}$ is both complete and co-complete; weak limits (See Appendix A.1) are computed "point-wise" :

$$\left(\underleftarrow{\mathrm{holim}}\ \mathscr{X}_i\right)(X) = \underleftarrow{\mathrm{holim}}\ \mathscr{X}_i(X),$$

where the weak limit to the right is computed in Gpd. Similarly for weak colimits.

We end this subsection by stating a direct analogue of the Yoneda lemma.

Lemma I.1.5. *[23]* ***The 2-Yoneda Lemma:*** *If C is an object of $\mathscr{C}$ and $\mathscr{X}$ a weak presheaf, then there is a natural equivalence of groupoids*

$$\mathrm{Hom}_{Gpd^{\mathscr{C}^{op}}}(C, \mathscr{X}) \simeq \mathscr{X}(C).$$

Grothendieck fibrations in groupoids

There is yet another way to describe the data of a weak presheaf of groupoids. This is the notion of a Grothendieck fibration in groupoids.

Definition I.1.26. Let $p : \mathcal{F} \to \mathscr{C}$ be a functor between small categories. An arrow $\eta : e \to e'$ in $\mathscr{C}$ is called **Cartesian** (with respect to p) if for every $f : e'' \to e$ such that there exists $g : p\,(e'') \to p\,(e')$ such that $p\,(f) = p\,(\eta) \circ g$, there exists a unique $\theta : e'' \to e'$ such that $p\,(\theta) = g$ and $\eta \circ \theta = f$.

$$\begin{tikzcd}
e'' \arrow[dr, dashed, "\exists!\theta"'] \arrow[drr, bend left, "\forall f"] \arrow[dd, mapsto] & & \\
& e' \arrow[r, "\eta"'] \arrow[dd, mapsto] & e \arrow[dd, mapsto] \\
p\,(e'') \arrow[dr, "\forall g"'] \arrow[drr, bend left] & & \\
& p(e') \arrow[r] & p\,(e)\,.
\end{tikzcd}$$

Definition I.1.27. A functor $p : \mathcal{F} \to \mathscr{C}$ is a Grothendieck fibration in groupoids over $\mathscr{C}$ if

i) every arrow in $\mathcal{F}$ is Cartesian with respect to p, and

ii) for every $\eta : c \to p\,(e)$ in $\mathscr{C}$, there exists an $\tilde{\eta} : \tilde{e} \to e$ such that $p\,(\tilde{\eta}) = \eta$.

Such a $\tilde{\eta}$ is called a **Cartesian lift** of η.

Definition I.1.28. Let C be an object of $\mathscr{C}$ and $p : \mathcal{F} \to \mathscr{C}$ a Grothendieck fibration in groupoids. The **fiber** over C is defined to be the category $\mathcal{F}_C$ where

$$\mathcal{F}_{C0} := \{e \in \mathcal{F}_0 \text{ such that } p\,(e) = C\}$$

$$\mathcal{F}_{C1} := \left\{e \xrightarrow{f} e' \in \mathcal{F}_1 \text{ such that } p\,(f) = id_C\right\}$$

Remark. It is easy to see that for $p : \mathcal{F} \to \mathscr{C}$ a Grothendieck fibration in groupoids, each fiber $\mathcal{F}_C$ must be a groupoid (hence justifying the terminology).

Example 10. Let $\mathscr{C} = \mathbb{TOP}$, let $\mathcal{F} = Bun_G$ be the category of principal G-bundles (over an arbitrary base), and let p be functor which picks out the base space of a principal bundle. It is easy to check that this is indeed a Grothendieck fibration in groupoids. Moreover, for a space X, the fiber $(Bun_G)_X$ is the same as $Bun_G\,(X)$, the category of principal G-bundles *over* X.

Categories fibered in groupoids over $\mathscr{C}$ form a 2-category:

An arrow from one such fibration $p : \mathcal{F} \to \mathscr{C}$ to another $p' : \mathcal{F}' \to \mathscr{C}$ is a functor $G : \mathcal{F} \to \mathcal{F}'$ which makes

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "G"] \arrow[dr, "p"'] & & \mathcal{F}' \arrow[dl, "p'"] \\ & \mathscr{C} & \end{tikzcd}$$

into a strictly commuting triangle.

A 2-morphism between two arrows $G, G' : p \to p'$, is a natural transformation $\mu : G \Rightarrow G'$ such that $p'\mu = id_p$.

The Grothendieck construction

As we shall see, the 2-category of weak presheaves of groupoids over $\mathscr{C}$ is equivalent to the 2-category of categories fibered in groupoids over $\mathscr{C}$. The construction which takes a weak presheaf and produces a Grothendieck fibration is called the **Grothendieck construction**.

Definition I.1.29. Suppose that $\mathscr{X} : \mathscr{C}^{op} \to Gpd$ is a weak presheaf. Let $\int \mathscr{X}$ denote the following category:

objects: An object is a pair (C, x) with $C \in \mathscr{C}_0$ and $x \in \mathscr{X}(C)_0$.
arrows: An arrow $(C, x) \to (D, y)$ is a pair (f, α) such that

$$f : C \to D$$

and

$$\alpha : x \xrightarrow{\sim} \mathscr{X}(f)(y).$$

Composition of

$$(C, x) \xrightarrow{(f,\alpha)} (D, y) \xrightarrow{(g,\beta)} (E, z)$$

is defined as (gf, h) where h is the composite:

$$x \xrightarrow{\alpha} \mathscr{X}(f)(y) \xrightarrow{\mathscr{X}(f)(\beta)} \mathscr{X}(f)(\mathscr{X}(g)(z)) \xrightarrow{\mathscr{X}(f,g)} \mathscr{X}(gf)(z).$$

There is a canonical functor $\int \mathscr{X} \to \mathscr{C}$ given by sending a pair (C, x) to C and similarly on arrows. This makes $\int \mathscr{X}$ into a category fibered in groupoids over $\mathscr{C}$.

This construction extends in a natural way to a 2-functor

$$\int : Gpd^{\mathscr{C}^{op}} \to Fib_{Gpd}(\mathscr{C})$$

from the 2-category of weak presheaves of groupoids over $\mathscr{C}$ and the 2-category of categories fibered in groupoids over $\mathscr{C}$.

Define a 2-functor

$$St : Fib_{Gpd}(\mathscr{C}) \to Gpd^{\mathscr{C}^{op}}$$

by having it assign a Grothendieck fibration in groupoids $p : \mathcal{F} \to \mathscr{C}$ the strict 2-functor St_p which assigns an object C of $\mathscr{C}$ the groupoid

$$\mathrm{Hom}_{Fib_{Gpd}(\mathscr{C})}\left(\int y(C), p\right).$$

In particular, St_p may be regarded a weak presheaf, even though it happens to be a strict 2-functor. It follows from the 2-Yoneda lemma that St is left 2-adjoint to $\int$. In fact, even more is true:

Theorem I.1.6. *[29] The adjoint pair*

$$Fib_{Gpd}(\mathscr{C}) \underset{St}{\overset{\int}{\leftrightarrows}} Gpd^{\mathscr{C}^{op}},$$

is an equivalence of 2-categories.

Remark. In particular, this implies that any weak presheaf of groupoids $\mathscr{X}$ is equivalent to a strict presheaf of groupoids, i.e. one which is a 2-functor.

Proposition I.1.5. *For any presheaf $\mathscr{X}$, $\mathscr{X}$ is the weak colimit of the functor $\int \mathscr{X} \to \mathscr{C}$. (See Section A.1.)*

Proof. This is a direct consequence of the 2-Yoneda Lemma. □

Remark. Often this proposition is stated by saying "$\mathscr{X}$ is a weak colimit of representables," and is denoted by

$$F \simeq \underset{C \to F}{\underrightarrow{\mathrm{holim}}}\, C.$$

Corollary I.1.1. *The 2-functor St can also be described by*

$$St(p : \mathcal{F} \to \mathscr{C}) \simeq \underrightarrow{\mathrm{holim}}\,(y \circ p),$$

where $y : \mathscr{C} \to Gpd^{\mathscr{C}^{op}}$ is given by the Yoneda embedding.

Now suppose that $\mathscr{D}$ is any co-complete bicategory, i.e. one which has all small weak colimits, and $f : \mathscr{C} \to \mathscr{D}$ is any weak functor. Then f induces a pair of adjoint functors

$$Gpd^{\mathscr{C}^{op}} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathscr{D},$$

with $f^* \dashv f_*$. Explicitly

$$f_*(D)(C) = \mathrm{Hom}_{\mathscr{D}}(f(C), D),$$

and f is uniquely determined by the fact that it is weak colimit preserving and

$$f^*(y(C)) = f(C).$$

It follows that f^* is given explicitly by

$$f^*(\mathscr{X}) = \underset{C \to \mathscr{X}}{\underrightarrow{\mathrm{holim}}}\, f(C).$$

The functor f^* is the **weak left Kan extension** of f along the Yoneda embedding.

Stacks

As we have seen, there are two equivalent ways of viewing weak presheaves of groupoids, namely as weak 2-functors, or as Grothendieck fibrations. Throughout most of this thesis, we will view them as weak functors, since it is this point of view that highlights the fact that stacks are merely categorified sheaves.

Definition I.1.30. Let $(\mathscr{C}, J)$ be a Grothendieck site. A weak presheaf of groupoids $\mathscr{X}$ over $\mathscr{C}$ is called a **stack** if for every object C and covering sieve S, the natural map

$$\mathrm{Hom}_{Gpd^{\mathscr{C}^{op}}}(C, \mathscr{X}) \to \mathrm{Hom}_{Gpd^{\mathscr{C}^{op}}}(S, \mathscr{X})$$

is an equivalence of groupoids.

If this map is fully faithful, $\mathscr{X}$ is called **separated** (or a **prestack**). Although it is not standard, if this map is faithful, we will call it **weakly separated**.

We denote the full sub-2-category of $Gpd^{\mathscr{C}^{op}}$ consisting of those weak presheaves that are stacks by $St_J(\mathscr{C})$. Any 2-category arising this way is an example of a **2-topos**. (A general 2-topos is of this form, where $\mathscr{C}$ is a bicategory)

It is immediate from the definition that the weak limit (See Appendix A.1) of any small diagram of stacks is again a stack.

If $\mathcal{B}$ is a basis for the topology J, then it suffices to check this condition for every sieve of the form $S_{\mathcal{U}}$, where $\mathcal{U}$ is a covering family. Namely, a weak presheaf is a stack if and only if for every covering family $\mathcal{U} = (U_i \to C)_i$ the induced map

$$\mathscr{X}(C) \to \underleftarrow{\mathrm{holim}} \left[\prod \mathscr{X}(U_i) \rightrightarrows \prod \mathscr{X}(U_{ij}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \prod \mathscr{X}(U_{ijk})\right]$$

is an equivalence of groupoids. A weak presheaf $\mathscr{X}$ is separated if and only if this map is fully faithful, and weakly separated if and only if it is faithful.

The associated groupoid

$$\mathfrak{Des}\,(\mathscr{X},\mathcal{U}) := \underleftarrow{\mathrm{holim}}\left[\prod \mathscr{X}(U_i) \rightrightarrows \prod \mathscr{X}(U_{ij}) \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \prod \mathscr{X}(U_{ijk})\right],$$

obtained as weak limit (See Appendix A.1) of the above diagram of groupoids, is called the category of **descent data** for $\mathscr{X}$ at $\mathcal{U}$.

A concrete model for this groupoid can be given as follows:

Its objects are pairs (ϕ_i, α_{ij}) with $\phi_i : U_i \to \mathscr{X}$ and with

$$\begin{tikzcd} U_{ij} \arrow[r, bend left, "\phi_j|_{U_{ij}}"] \arrow[r, bend right, "\phi_i|_{U_{ij}}"'] \arrow[r, phantom, "\Downarrow\alpha_{ij}"] & \mathscr{X}, \end{tikzcd}$$

such that the transition functions α_{ij} satisfy the cocycle condition

$$\alpha_{ij}|_{U_{ijk}} \circ \alpha_{jk}|_{U_{ijk}} = \alpha_{ik}|_{U_{ijk}}. \tag{I.15}$$

An arrow in $\mathfrak{Des}\,(\mathscr{X},\mathcal{U})$ from (ϕ_i, α_{ij}) to (ψ_i, β_{ij}) is a collection of 2-morphisms θ_i

$$\begin{tikzcd} U_i \arrow[r, bend left, "\phi_i"] \arrow[r, bend right, "\psi_i"'] \arrow[r, phantom, "\Downarrow\theta_i"] & \mathscr{X}, \end{tikzcd}$$

such that for each i and j the following diagram of 2-morphisms commutes

$$\begin{tikzcd} \phi_j|_{U_{ij}} \arrow[r, Rightarrow, "\theta_j"] \arrow[d, Rightarrow, "\alpha_{ij}"'] & \psi_j|_{U_{ij}} \arrow[d, Rightarrow, "\beta_{ij}"] \\ \phi_i|_{U_{ij}} \arrow[r, Rightarrow, "\theta_i"] & \psi_i|_{U_{ij}}. \end{tikzcd}$$

Proposition I.1.6. *If J is subcanonical and $(U_i \to C)_i$ is a covering family for an object C, then,* ***in the 2-category of stacks***

$$C \simeq \underrightarrow{\mathrm{holim}}\left[\coprod U_{ijk} \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \coprod U_{ij} \rightrightarrows \coprod U_i\right]$$

We will often simply write

$$C \simeq \underrightarrow{\mathrm{holim}}_{U_i \to C} U_i.$$

Definition I.1.31. A morphism $\varphi : \mathscr{X} \to \mathscr{Y}$ of stacks is called **representable** if for any object $C \in \mathscr{C}$ and any morphism $C \to \mathscr{Y}$, the weak pullback $C \times_{\mathscr{Y}} \mathscr{X}$ in the category of stacks is (equivalent to) an object D of $\mathscr{C}$. Weak pullbacks are explained in Section I.2.2.

We can now define a 2-functor

$$(\cdot)^+ : Gpd^{\mathscr{C}^{op}} \to Gpd^{\mathscr{C}^{op}}$$

by

$$\mathscr{X}^+(C) := \underset{S \in Cov(C)}{\underrightarrow{\text{holim}}}\ \text{Hom}_{Gpd^{\mathscr{C}^{op}}}(S, F).$$

Just as in the 1-categorical case, we call this 2-functor the **plus-construction.** We can alternatively define $(\cdot)^+$ by the equation

$$\mathscr{X}^+(C) := \underset{\mathcal{U} \in cov(C)}{\underrightarrow{\text{holim}}}\ \mathfrak{Des}(\mathscr{X}, \mathcal{U})$$

and obtain a naturally equivalent 2-functor.

Remark. The weak colimit (See Appendix A.1) in either definition must be indexed over a suitable 2-category of covers.

A weak presheaf $\mathscr{X}$ is separated if and only if the canonical map

$$\mathscr{X} \to \mathscr{X}^+$$

is a fully faithful, and weakly separated if and only if it is faithful. It is a stack if and only if this map is an equivalence.

If $\mathscr{X}$ is separated, then $\mathscr{X}^+$ is a stack, and if $\mathscr{X}$ is only weakly separated, then $\mathscr{X}^+$ is separated. Furthermore, $\mathscr{X}^+$ is always weakly separated, for any $\mathscr{X}$. Hence, $\mathscr{X}^{+++}$ is always a stack.

Definition I.1.32. We denote by a_J the 2-functor $\mathscr{X} \mapsto \mathscr{X}^{+++}$. It is called the **stackification** 2-functor .

If $\mathscr{X}$ is separated, $a_J(\mathscr{X}) \simeq \mathscr{X}^+$, and if $\mathscr{X}$ is a stack, $a_J(\mathscr{X}) \simeq \mathscr{X}$. The 2-functor a_J is left-2-adjoint to the inclusion

$$i : St_J(\mathscr{C}) \hookrightarrow Gpd^{\mathscr{C}^{op}}$$

and preserves finite weak limits.

Remark. The 2-category $St_J(\mathscr{C})$ is both complete and co-complete. Since i is a right adjoint, it follows that the computation of weak limits of stacks can be done in the category $Gpd^{\mathscr{C}^{op}}$, hence can be done "point-wise". To compute the weak colimit of a diagram of stacks, one must first compute it in $Gpd^{\mathscr{C}^{op}}$ and then apply the stackification functor a_J.

We will now give an explicit description of the epimorphisms and monomorphisms in the $(2,1)$-category of stacks.

Definition I.1.33. [38],[49] A morphism $\varphi : \mathscr{X} \to \mathscr{Y}$ between two stacks over $(\mathscr{C}, J)$ is said to be an **epimorphism** (or epi) if for every object C and every object $x \in \mathscr{Y}(C)_0$, there exists a J-covering family $\mathcal{U} = (f_i : U_i \to C)_i$ of C such that for each i there exists an object $x_i \in \mathscr{X}(U_i)_0$ and a(n) (iso)morphism

$$\alpha_i : \varphi\left(\mathscr{X}\left(f_i\right)\left(x_i\right)\right) \to \mathscr{Y}\left(f_i\right)(x).$$

In view of the 2-Yoneda lemma, this just says that every map $C \to \mathscr{Y}$ from a representable C locally factors through φ up to isomorphism.

It is sometimes useful to extend this definition for maps between arbitrary weak presheaves of groupoids:

Definition I.1.34. A morphism

$$\varphi : \mathscr{X} \to \mathscr{Y}$$

is a J-**covering morphism** if it satisfies the properties of being an epimorphism, except for the fact that its source and target need not be stacks.

Remark. A morphism

$$\varphi : \mathscr{X} \to \mathscr{Y}$$

is a J-covering morphism if and only if $a_J(\varphi)$ is a J-epimorphism, where a_J denotes stackification.

Definition I.1.35. A morphism $\varphi : \mathscr{X} \to \mathscr{Y}$ between two stacks over $(\mathscr{C}, J)$ is said to be a **monomorphism** (or mono) if for each object C,

$$\varphi(C) : \mathscr{X}(C) \to \mathscr{Y}(C)$$

is a full and faithful functor of groupoids.

Remark. In any $(2,1)$-category, a morphism $f : C \to C'$ is a monomorphism if and only if the diagonal map

$$C \to C \times_{C'} C$$

is an equivalence, where $C \times_{C'} C$ denotes the weak 2-pullback (explained in Section I.2.2). See for instance [35]. Note that this immediately implies that any finite weak limit preserving functor between $(2,1)$-categories preserves monos, so the stackification of a monomorphism is again a monomorphism.

Dually, a morphism $f : C \to C'$ is an epimorphism if and only if the co-diagonal

$$C' \coprod_{C} C' \to C'$$

(the left-hand side here is a weak colimit, see Appendix A.1) is an equivalence.

Remark. A morphism of stacks which is both a monomorphism and an epimorphism is an equivalence. (This is true in any n-topos in the sense of [35]).

We end by remarking that the 2-category $St_J(\mathscr{C})$ is Cartesian closed:

Definition I.1.36. Let $\mathscr{C}$ be a bicategory. An object X is said to be **exponentiable** if the functor

$$\begin{array}{rcl} (\,\cdot\,)\times X:\mathscr{C} & \to & \mathscr{C} \\ Y & \mapsto & Y\times X \end{array}$$

has a right 2-adjoint

$$\begin{array}{rcl} (\,\cdot\,)^X:\mathscr{C} & \to & \mathscr{C} \\ Y & \mapsto & Y^X. \end{array}$$

In other words, for every object Y of $\mathscr{C}$, there exists an "object of maps" from Y to X, denoted by Y^X such that for any object Z, there is a natural equivalence of groupoids

$$\operatorname{Hom}\left(Z, Y^X\right) \simeq \operatorname{Hom}\left(Z\times X, Y\right).$$

Definition I.1.37. A bicategory $\mathscr{C}$ is **Cartesian closed** if it has binary products, and every object X of $\mathscr{C}$ is exponentiable.

Theorem I.1.7. *Let $(\mathscr{C}, J)$ be any Grothendieck site. Then* $\mathrm{St}_J(\mathscr{C})$ *is Cartesian closed.*

The proof of this theorem is completely analogous to the proof given in Section I.1.4.

The exponent $\mathscr{X}^{\mathscr{Y}}$ of two stacks is given by

$$\mathscr{X}^{\mathscr{Y}}(C) = \operatorname{Hom}_{Gpd^{\mathscr{C}^{op}}}\left(\mathscr{Y}\times C, \mathscr{X}\right),$$

and satisfies

$$\operatorname{Hom}_{Gpd^{\mathscr{C}^{op}}}\left(\mathscr{Z}, \mathscr{X}^{\mathscr{Y}}\right) \simeq \operatorname{Hom}_{Gpd^{\mathscr{C}^{op}}}\left(\mathscr{Y}\times\mathscr{Z}, \mathscr{X}\right)$$

for all stacks $\mathscr{Z}$.

I.1.8 Sheaves of groupoids vs. Stacks

Definition I.1.38. Let $\mathscr{C}$ be a small category. A **strict presheaf of groupoids** over $\mathscr{C}$ is a strict 2-functor $F:\mathscr{C}^{op}\to Gpd$ to the 2-category of (small) groupoids. Notice that this is the same as a 1-functor

$$\mathscr{C}^{op}\to\tau_1\left(Gpd\right),$$

where the target is the 1-category of groupoids. A morphism of strict presheaves is a strict natural transformation (i.e. a natural transformation between their corresponding 1-functors into $\tau_1(Gpd)$). A 2-morphism between two natural transformations $\alpha_i : F \Rightarrow G$, $i = 1, 2$, is an assignment to each object C of $\mathscr{C}$ a natural transformation

$$w(C) : \alpha_1(C) \Rightarrow \alpha_2(C)$$

subject to the following condition:

For all $f : D \to C$, we have two functors from $F(C)$ to $G(D)$, namely

$$G(f)\alpha_1(C) = \alpha_1(D)F(f)$$

and

$$G(f)\alpha_2(C) = \alpha_2(D)F(f).$$

Given our assignment $C \mapsto w(C)$, we have two different natural transformations between these functors: $G(f)w(C)$ and $w(D)F(f)$. Our assignment w is called a **modification** if these two natural transformations are equal. Modifications are the 2-cells of strict presheaves. This yields a strict 2-category of strict presheaves of groupoids $Psh(\mathscr{C}, Gpd)$.

Proposition I.1.7. *The 2-category $Psh(\mathscr{C}, Gpd)$ is equivalent to the 2-category of groupoid objects in $\mathrm{Set}^{\mathscr{C}^{op}}$.*

Proof. Let $(\cdot)_i : \tau_1(Gpd) \to \mathrm{Set}$, $i = 0, 1, 2$ be the functors which associate to a groupoid $\mathcal{G}$ its set of objects $\mathcal{G}_0$, its set of arrows $\mathcal{G}_1$, and its set $\mathcal{G}_2$ of composable arrows respectively. Let

$$F : \mathscr{C}^{op} \to \tau_1(Gpd)$$

be a strict presheaf of groupoids. Then each F_i is an ordinary presheaf of sets. Moreover, for each C, $F(C)$ is a groupoid, which we may write as demanding certain diagram involving each $F(C)_i$ to commute. These assemble to a corresponding diagram for the global F_i's, showing they form a groupoid object in $\mathrm{Set}^{\mathscr{C}^{op}}$, $Q(F)$. Given a 1-morphism

$$\alpha : F \Rightarrow G$$

in $Psh(\mathscr{C}, Gpd)$, let $Q(\alpha) : Q(F) \to Q(F)$ be the internal functor with components

$$Q(\alpha)_i(C) = \alpha(C)_i$$

for $i = 0, 1$. Finally, let w be a modification from α to β. Then, in particular, for each C, $w(C) : \alpha(C) \Rightarrow \beta(C)$ is a natural transformation, so is a map $w(C) : F(C)_0 \to G(C)_1$ satisfying the obvious properties. It is easy to check that the conditions for w to be a modification are precisely those for the family

$$(w(C) : F(C)_0 \to G(D)_1)$$

to assemble into a natural transformation

$$Q(w) : F_0 \Rightarrow G_1.$$

Since w is point-wise a natural transformation, $Q(w)$ is an internal natural transformation. It is easy to check that this is indeed an equivalence of 2-categories with an explicit inverse on objects given by

$$\tilde{y} : \mathbb{G} \mapsto \operatorname{Hom}(\cdot, \mathbb{G}).$$

$\square$

Definition I.1.39. Let $(\mathscr{C}, J)$ be a Grothendieck site. Then a **sheaf** of groupoids is a strict presheaf $F : \mathscr{C}^{op} \to \tau_1(Gpd)$ such that for any covering family $(C_i \to C)_i$, the induced morphism

$$F(C) \to \varprojlim \left[\prod_i F(C_i) \rightrightarrows \prod_{i,j} F(C_{ij})\right]$$

is an isomorphism of groupoids. Sheaves of groupoids form a full sub-2-category $Sh(\mathscr{C}, Gpd)$ of strict presheaves of groupoids.

The following proposition is easily checked:

Proposition I.1.8. *The 2-functor $Q : Psh(\mathscr{C}, Gpd) \to Gpd\left(\mathrm{Set}^{\mathscr{C}^{op}}\right)$ restricts to an equivalence $Q : Sh(\mathscr{C}, Gpd) \to Gpd\left(Sh\left(\mathscr{C}\right)\right)$.*

Analogously to sheaves of sets, there is a 2-adjunction

$$Sh(\mathscr{C}, Gpd) \underset{i}{\overset{sh}{\rightleftarrows}} Psh(\mathscr{C}, Gpd),$$

where sh denotes sheafification.

Denote by $j : Psh(\mathscr{C}, Gpd) \to Gpd^{\mathscr{C}^{op}}$ the "inclusion" of strict presheaves into weak presheaves. We use quotations since this functor is not full. The following proposition is standard:

Proposition I.1.9. *Let $\mathscr{X}$ be a strict presheaf of groupoids. Then*

$$a \circ j(\mathscr{X}) \simeq a \circ j \circ i \circ sh(\mathscr{X}),$$

where a denotes stackification.

In other words, if you start with a strict presheaf of groupoids, sheafify it to a sheaf of groupoids, and then stackify the result, this is equivalent to stackifying the original presheaf.

Corollary I.1.2. *Every stack is equivalent to $a \circ j \circ i(\mathscr{W})$ for some sheaf of groupoids $\mathscr{W}$.*

The category of descent in terms of groupoid objects

Let $\mathscr{X}$ be a weak presheaf of groupoids. Without loss of generality, we may assume that $\mathscr{X} \simeq \tilde{y}(\mathbb{G})$, for some groupoid object $\mathbb{G}$ in presheaves, where

$$\tilde{y} : Gpd\left(\mathrm{Set}^{\mathscr{C}^{op}}\right) \to Psh(\mathscr{C}, Gpd)$$

is as in the proof of Proposition I.1.7. Let us try to interpret the category of descent in terms of groupoid objects. We assume that our category C has arbitrary co-products.

For each cover $\mathcal{U} = (\eta_i : C_i \to C)_{i \in I}$, we can construct a groupoid object (in $\mathscr{C}$) $C_{\mathcal{U}}$ as follows. To fix notation, consider each Cartesian square

$$\begin{tikzcd}
C_{ij} \arrow[r, "\pi^i_{ij}"] \arrow[d, "\pi^j_{ij}"'] & C_j \arrow[d, "\eta_j"] \\
C_i \arrow[r, "\eta_i"'] & C
\end{tikzcd}$$

and each Cartesian cube:

$$\begin{tikzcd}
& C_{ijk} \arrow[rr, "\pi^i_{ijk}"] \arrow[dl, "\pi^k_{ijk}"'] \arrow[dd, "\pi^j_{ijk}"] & & C_{jk} \arrow[dl] \arrow[dd] \\
C_{ij} \arrow[rr] \arrow[dd] & & C_j \arrow[dd] & \\
& C_{ik} \arrow[rr] \arrow[dl] & & C_k \arrow[dl] \\
C_i \arrow[rr] & & C &
\end{tikzcd}$$

As objects, $C_{\mathcal{U}}$ has

$$(C_{\mathcal{U}})_0 = \coprod_i C_i$$

and for arrows it has

$$(C_{\mathcal{U}}) = \coprod_{i,j} C_{ij}.$$

The source map is given by

$$s = \coprod_{i,j} \pi^j_{i,j}$$

and the target map is given by

$$t = \coprod_{i,j} \pi^i_{i,j}.$$

The unit map is given by
$$u = \coprod_i \Delta_i,$$
where Δ_i is the diagonal map. Finally, the fibered product

$$(C_{\mathcal{U}})_1 \times_{(C_{\mathcal{U}})_0} (C_{\mathcal{U}})_1$$

can be identified with
$$\coprod_{i,j,k} C_{ijk}$$
and the multiplication map is then given by

$$m = \coprod_{i,j,k} \pi^j_{ijk}.$$

This internal groupoid is called the **Čech groupoid** of the cover $\mathcal{U}$. By using the Yoneda-embedding on objects and arrows, we may consider this to be a groupoid object in presheaves.

An object (f_i, α_{ij}) of $\mathfrak{Des}\left(\tilde{y}\left(\mathbb{G}\right), \mathcal{U}\right)$ produces homomorphism

$$\phi : C_{\mathcal{U}} \to \mathbb{G}$$

of groupoid objects in presheaves:

$$\begin{aligned} \phi_0 &= \coprod_i (f_i)_0 \\ \phi_1 &= \coprod_{i,j} \alpha_{ji} \end{aligned}$$

viewing each α_{ij} as a map $C_{ij} \to \mathcal{G}_1$. It can be checked that this produces an equivalence of categories

$$\mathfrak{Des}\left(\tilde{y}\left(\mathbb{G}\right), \mathcal{U}\right) \simeq \mathrm{Hom}_{Gpd\left(\mathrm{Set}^{\mathscr{C}^{op}}\right)}\left(C_{\mathcal{U}}, \mathcal{G}\right)$$

which is functorial in $\mathcal{U}$.

We we state this as proposition for reference later:

Proposition I.1.10. *Let $C \in \mathscr{C}_0$ be an object of $\mathscr{C}$ and $\mathscr{X}$ a weak presheaf of groupoids such that $\mathscr{X} \simeq \tilde{y}\left(\mathbb{G}\right)$ for a groupoid object in presheaves $\mathbb{G}$. Then there is a functorial equivalence of groupoids*

$$\mathfrak{Des}\left(\tilde{y}\left(\mathbb{G}\right), \mathcal{U}\right) \simeq \mathrm{Hom}_{Gpd\left(\mathrm{Set}^{\mathscr{C}^{op}}\right)}\left(C_{\mathcal{U}}, \mathbb{G}\right)$$

for each covering family $\mathcal{U}$ of C.

I.2 Topological and Differentiable Stacks

I.2.1 Topological Groupoids and Lie groupoids

Definition I.2.1. A **topological groupoid** is a groupoid object in $\mathbb{TOP}$. Explicitly, it is a diagram

$$\begin{tikzcd} \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \arrow[r, "m"] & \mathcal{G}_1 \arrow[loop above, "i"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathcal{G}_0 \arrow[l, bend left, "\mathbb{1}"] \end{tikzcd}$$

of topological spaces and continuous maps satisfying the usual axioms. Forgetting the topological structure (i.e. applying the forgetful functor from $\mathbb{TOP}$ to Set), one obtains an ordinary (small) groupoid.

Similarly, a Lie groupoid is a groupoid object in the category of smooth manifolds *Mfd*, where we allow the arrow space $\mathcal{G}_1$ to possibly be a non-Hausdorff manifold, and where the source and target maps s and t are required to be submersions.

Topological groupoids form a 2-category with continuous functors as 1-morphisms and continuous natural transformations as 2-morphisms. We denote the 2-category of topological groupoids by $\mathbb{TOP}$*Gpd*. Similarly, Lie groupoids form a 2-category where the analogous maps are required to be smooth. We denote this 2-category by *LieGpd*.

Example 11. Any Lie group G can be viewed as a Lie groupoid $G \rightrightarrows *$ with one object.

Definition I.2.2. Given a topological space X, we denote by $(X)^{id}$ the topological groupoid whose object and arrow space are both X and all of whose structure maps are the identity morphism of X. The arrow space is the collection of all the identity arrows for the objects X. Similarly we may view any manifold M as a Lie groupoid $\left(M^{id}\right)$. In either case, we will often denote this groupoid simply by X or M.

Definition I.2.3. Given a space or manifold X, the **pair groupoid** $Pair\,(X)$ is the (topological or Lie) groupoid whose object space is X and whose arrow space is $X \times X$, where an element

$$(x, y) \in X \times X$$

is viewed as an arrow from y to x, and composition is defined by the rule

$$(x, y) \cdot (y, z) = (x, z)\,.$$

Definition I.2.4. Given a continuous map (respectively submersion)

$$\phi : U \to X,$$

the **relative pair groupoid** $Pair\,(\phi)$ is defined to be the topological (respectively Lie) groupoid whose arrow space is the fibered product $U \times_X U$ and whose object space is U, where an element

$$(x, y) \in U \times_X U \subset U \times U$$

is viewed as an arrow from y to x and composition is defined by the rule

$$(x, y) \cdot (y, z) = (x, z)\,.$$

The pair groupoid of a space X is the relative pair groupoid of the unique map from X to the one-point space.

Definition I.2.5. Given a topological groupoid $\mathcal{G}$ and a continuous map $f : X \to \mathcal{G}_0$, there is an induced **pullback groupoid** $f^*(\mathcal{G})$, which is a topological groupoid, whose object space is X, such that arrows between x and y in $f^*(\mathcal{G})$ are in bijection with arrows between $f(x)$ and $f(y)$ in $\mathcal{G}$. In other words, the arrows fit in the following pullback diagram

$$\begin{tikzcd} f^*(\mathcal{G})_1 \arrow[r] \arrow[d] & \mathcal{G}_1 \arrow[d, "(s,t)"] \\ X \times X \arrow[r, "f \times f"] & \mathcal{G}_0 \times \mathcal{G}_0. \end{tikzcd}$$

This construction also works for manifolds provided that this fibered product is a manifold, so in particular, when f is a submersion.

When $X = \coprod_\alpha U_\alpha$ with $\mathcal{U} = (U_\alpha \hookrightarrow X)_\alpha$ an open cover of $\mathcal{G}_0$ and $X \to \mathcal{G}_0$ the canonical map, $f^*(\mathcal{G})$ is denoted by $\mathcal{G}_\mathcal{U}$. If in addition to this,

$$\mathcal{G} = (T)^{id}$$

for some topological space T or manifold, then this is called the **Čech groupoid** associated to the cover $\mathcal{U}$ of T and is denoted by $T_\mathcal{U}$.

Remark. If the open cover $\mathcal{U}$ is instead a cover for a different Grothendieck topology, the above still makes sense. This will be important later.

I.2.2 Weak 2-pullbacks

We will take this section to explain a very important construction which we will use frequently in this thesis. Suppose that we are given a diagram $\mathscr{D}$ of groupoids:

$$\begin{tikzcd} & \mathcal{H} \arrow[d, "\psi"] \\ \mathcal{G} \arrow[r, "\varphi"] & \mathcal{K}. \end{tikzcd}$$

We will construct a groupoid $\mathcal{G}\times_{\mathcal{K}}\mathcal{H}$ which is a model for the weak 2-pullback. The objects of $\mathcal{G}\times_{\mathcal{K}}\mathcal{H}$ consist of triples

$$(x, y, k) \in \mathcal{G}_0 \times \mathcal{H}_0 \times \mathcal{K}_1$$

such that

$$k : \varphi(x) \to \psi(y).$$

The arrows of $\mathcal{G}\times_{\mathcal{K}}\mathcal{H}$ between two objects

$$(x, y, k)$$

and

$$(x', y', k')$$

are pairs

$$(g, h) \in \mathcal{G}_1 \times \mathcal{H}_1$$

such that

$$g : x \to x',$$
$$h : y \to y',$$

and such that the following diagram commutes:

$$\begin{tikzcd}
\varphi(x) \arrow[r, "k"] \arrow[d, "\varphi(g)"'] & \psi(y) \arrow[d, "\psi(h)"] \\
\varphi(x') \arrow[r, "k'"] & \psi(y').
\end{tikzcd}$$

Notice that $\mathcal{G}\times_{\mathcal{K}}\mathcal{H}$ comes equipped with two canonical homomorphisms:

$$\begin{aligned} pr_1 : \mathcal{G}\times_{\mathcal{K}}\mathcal{H} &\to \mathcal{G} \\ (x, y, k) &\mapsto x \end{aligned}$$

and

$$\begin{aligned} pr_2 : \mathcal{G}\times_{\mathcal{K}}\mathcal{H} &\to \mathcal{H} \\ (x, y, k) &\mapsto y. \end{aligned}$$

Unlike the ordinary fibered product, we do not have

$$\psi \circ pr_2 = \varphi \circ pr_1,$$

however there is a canonical natural isomorphism

$$\begin{aligned} \alpha : \psi \circ pr_2 &\Rightarrow \varphi \circ pr_1 \\ \alpha(x, y, k) = k : \varphi(x) &\to \psi(y). \end{aligned}$$

Therefore, the following diagram 2-commutes:

$$\begin{tikzcd}
\mathcal{G} \times_{\mathcal{K}} \mathcal{H} \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{H} \arrow[d, "\psi"] \\
\mathcal{G} \arrow[r, "\varphi"'] \arrow[ur, Rightarrow, "\alpha"] & \mathcal{K}.
\end{tikzcd}$$

Notice that if $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{K}$ are topological groupoids, there is a canonical topological structure on the groupoid $\mathcal{G} \times_{\mathcal{K}} \mathcal{H}$. The corresponding statement is not always true in the smooth setting. However, the objects and arrows of the groupoid $\mathcal{G} \times_{\mathcal{K}} \mathcal{H}$ are constructing by forming ordinary pullbacks, hence if the maps involved are transverse, $\mathcal{G} \times_{\mathcal{K}} \mathcal{H}$ will have the structure of a Lie groupoid. This is in particular the case if one of φ or ψ has both its object and arrow maps submersions.

We will now briefly describe the universal property of this weak 2-pullback.

Definition I.2.6. Suppose we are given a diagram $\mathscr{D}$ in a bicategory $\mathscr{C}$ of the form

$$\begin{tikzcd}
 & D \arrow[d, "\psi"] \\
C \arrow[r, "\varphi"] & E.
\end{tikzcd}$$

For F an object of $\mathscr{C}$, a **cone** on $\mathscr{D}$ with vertex F is two maps

$$f : F \to C$$

and

$$g : F \to D,$$

and a two-cell α, such that the following diagram 2-commutes:

$$\begin{tikzcd}
F \arrow[r, "g"] \arrow[d, "f"'] & D \arrow[d, "\psi"] \\
C \arrow[r, "\varphi"'] \arrow[ur, Rightarrow, "\alpha"] & E.
\end{tikzcd}$$

A map of cones with vertex F from (f, g, α) to (f', g', α') is a pair of two-cells

$$\gamma : f \Rightarrow f'$$

and

$$\delta : g \Rightarrow g',$$

such that the following diagram of two-cells commutes:

$$\begin{tikzcd}
\varphi \circ f \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\gamma"'] & \psi \circ g \arrow[d, Rightarrow, "\delta"] \\
\varphi \circ f' \arrow[r, Rightarrow, "\alpha'"'] & \psi \circ g'.
\end{tikzcd}$$

Cones for $\mathscr{D}$ with vertex F, with this notion of map, form a category $\mathit{Cone}(\mathscr{D}, F)$.

Note that given any cone $\sigma = (f, g, \alpha)$ for $\mathscr{D}$ with vertex F, and any object A of $\mathscr{C}$, there a canonical functor

$$\begin{aligned} \hat{\sigma}_A : \operatorname{Hom}(A, F) &\rightarrow \mathcal{C}one(\mathscr{D}, A) \\ A \xrightarrow{\theta} F &\mapsto (f\theta, g\theta, \alpha\theta) \end{aligned}$$

(and similarly on arrows).

Definition I.2.7. A cone $\sigma = (f, g, \alpha)$ for $\mathscr{D}$ with vertex F is said to be **limiting** if for each object A of $\mathscr{C}$, $\hat{\sigma}_A$ is an equivalence of categories. In such a situation, one often says F is a weak 2-pullback of the diagram $\mathscr{D}$ (or the weak fibered product).

Remark. Note that if F' is equivalent to F, and F is a weak 2-pullback of the diagram $\mathscr{D}$, then so is F'. It is easy to check that the for $\mathscr{D}$ the diagram of groupoids

$$\begin{tikzcd} & \mathcal{H} \arrow[d, "\psi"] \\ \mathcal{G} \arrow[r, "\varphi"] & \mathcal{K}, \end{tikzcd}$$

(pr_1, pr_2, α) is a limiting cone for $\mathscr{D}$ with vertex $\mathcal{G} \times_{\mathcal{K}} \mathcal{H}$. This holds just as well in the topological setting, and in the smooth setting provided that $\mathcal{G} \times_{\mathcal{K}} \mathcal{H}$ exists as a Lie groupoid.

Example 12. Let X be a set and $\mathcal{G}$ a groupoid. Suppose that $f : X \to \mathcal{G}$. This is the same data as a map from X to $\mathcal{G}_0$, but we choose to view it as a map of groupoids. We can then form the weak 2-pullback $X \times_{\mathcal{G}} X$ using the model described in this section. Direct inspection shows that this groupoid is actually a set, since X as a groupoid has no arrows. We can therefore identify $X \times_{\mathcal{G}} X$ with the set

$$\begin{tikzcd} X \times_{\mathcal{G}} X \arrow[r] \arrow[d] & \mathcal{G}_1 \arrow[d, "(s,t)"] \\ X \times X \arrow[r, "f \times f"] & \mathcal{G}_0 \times \mathcal{G}_0. \end{tikzcd}$$

However, from Section I.2.1, we know that this is the arrow space for the induced pullback groupoid $f^*(\mathcal{G})$. This gives

$$X \times_{\mathcal{G}} X \rightrightarrows X$$

the structure of a groupoid, whose source and target map are given by pr_1 and pr_2 respectively. This groupoid is canonically isomorphic to $f^*(\mathcal{G})$.

Since $X \times_{\mathcal{G}} X$ is the vertex for a cone, we know we have a two-cell

$$\alpha : f \circ pr_1 \Rightarrow f \circ pr_2$$

explicitly given by

$$\alpha(x, y, g) = g.$$

If we apply this to the particular case that $f = id_{\mathcal{G}_0}$ we get that for every groupoid $\mathcal{G}$, the following diagram is a weak 2-pullback:

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\alpha"] & \mathcal{G}_0 \arrow[d] \\ \mathcal{G}_0 \arrow[r] & \mathcal{G}, \end{tikzcd}$$

where the map $\mathcal{G}_0 \to \mathcal{G}$ corresponds to $id_{\mathcal{G}_0}$ and where $\alpha(g) = g$. More generally, it is an easy exercise to show that $X \times_{\mathcal{G}} X \rightrightarrows X$ is equivalent to $\mathcal{G}$ if and only if

$$f : X \to \mathcal{G}$$

is essentially surjective.

Example 13. Since weak limits in the 2-category $\mathrm{St}(\mathbb{TOP})$ of stacks on topological spaces are computed "point-wise", if $\mathscr{D}$ is a diagram of stacks

$$\begin{tikzcd} & \mathscr{Y} \arrow[d, "\psi"] \\ \mathscr{X} \arrow[r, "\varphi"] & \mathscr{Z}, \end{tikzcd}$$

then we can model the weak 2-pullback $\mathscr{X} \times_{\mathscr{Z}} \mathscr{Y}$ as the stack which assigns to each space T the weak 2-pullback of groupoids $\mathscr{X}(T) \times_{\mathscr{Z}(T)} \mathscr{Y}(T)$. This works equally as well if we replace topological spaces with the category of smooth manifolds.

I.2.3 The stacky quotient of a manifold by a Lie group action

One way in which topological and differentiable stacks naturally arise is by considering quotients of spaces or manifolds by the action of a (topological or Lie) group. In all but the most special examples, such quotients when computed in the category of spaces or manifolds will be very ill behaved. For example, if G is a Lie group acting on a manifold, the quotient space M/G is rarely a manifold, and even as a topological space may be quite pathological. It is standard however that if the action of the Lie group is both free and proper, then the there is a canonical differentiable structure on M/G and the projection map $M \to M/G$ inherits the structure of a principal G-bundle. However, even without the assumptions of the action being free and proper, there always exists a "stacky quotient" $M//G$ which makes the projection map $M \to M//G$ into a principal G-bundle over $M//G$. In the case of a free and proper action, this stacky quotient is nothing more than the canonical manifold structure on M/G. In this subsection, we will explain how to construct such a quotient. For simplicity of exhibition, we will work in the category of smooth manifolds, however, everything holds equally well for the category of topological spaces.

Let be G be a Lie group acting on a manifold M. Recall that the action is free if and only if for each point $x \in M$, the stabilizer group

$$G_x := \{g \in G \,|\, g \cdot x = x\}$$

is trivial . This may be rephrased in a more abstract way by saying that the diagram

$$\begin{tikzcd} G \times M \arrow[r, "\rho"] \arrow[d, "pr_2"'] & M \arrow[d, "\pi"] \\ M \arrow[r, "\pi"] & M/G, \end{tikzcd}$$

in which ρ is the map encoding the action of the Lie group and π is the quotient map, is a pullback diagram in topological spaces (or in manifolds, provided that we know M/G is smooth).

Given our action $\rho : G \times M \to M$, we can construct a groupoid $G \ltimes M$, called the action groupoid whose objects are M and whose arrows are $G \times M$, where a pair (g, m) is seen as an arrow from m to $g \cdot m$.

Remark. Both the objects M and the arrows $G \times M$ are in fact manifolds and the structure maps of this groupoid are smooth maps of manifolds (and the source and target maps are submersions) hence the groupoid we have constructed is a Lie groupoid.

We know from Section I.2.2 that given a groupoid $\mathcal{G}$, the following is a weak 2-pullback diagram in the 2-category of groupoids

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, "t"] \arrow[d, "s"'] & \mathcal{G}_0 \arrow[d] \\ \mathcal{G}_0 \arrow[r] & \mathcal{G}, \end{tikzcd}$$

where we view the sets $\mathcal{G}_1$ and $\mathcal{G}_0$ as groupoids with only identity morphisms, and where the map $\mathcal{G}_0 \to \mathcal{G}$ is induced by the unit map of the groupoid. We are suppressing the canonical two-cell from our notation for simplicity.

Applying this to our action groupoid, it means that

$$\begin{tikzcd} \mathcal{G} \times M \arrow[r, "\rho"] \arrow[d, "pr_2"'] & M \arrow[d] \\ M \arrow[r] & \mathcal{G} \ltimes M \end{tikzcd}$$

is a weak 2-pullback diagram of groupoids.

Heuristically, this means that if we could somehow replace M/G with the action groupoid $\mathcal{G} \ltimes M$, then the action would "behave like it was free".

However, there is another problem lurking in the background. Consider the obvious fact that, given two manifolds N and M, for every open cover $\mathcal{U} = (U_\alpha)$ of N, there is a bijection between families of smooth maps

$$f_\alpha : U_\alpha \to M$$

such that for all α and β,

$$f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta},$$

and smooth maps $f : N \to M$.

This fact can be expressed abstractly by saying that for every open cover $\mathcal{U} = (U_\alpha)$ of N, the natural map

$$\text{(I.16)} \qquad \mathrm{Hom}\,(X, Y) \to \varprojlim \left[\prod \mathrm{Hom}\,(U_\alpha, M) \rightrightarrows \prod \mathrm{Hom}\,(U_\alpha \cap U_\beta, M)\right]$$

is a bijection.

In view of the Yoneda Lemma, this is just the statement that $y\,(N)$ is a sheaf with respect to the open cover Grothendieck topology.

This suggests that in order for $\mathcal{G} \ltimes M$ to "behave like a space," in the sense that having compatible maps out of each element of an open covering is the same as having a well defined global map, maps into $\mathcal{G} \ltimes M$ should form a sheaf. However, for a given manifold N, the morphisms $\mathrm{Hom}\,(N, \mathcal{G} \ltimes M)$ are not a set but a groupoid. This is not something that we want to destroy, for otherwise we would lose the fact that

$$\begin{tikzcd}
\mathcal{G} \times M \arrow[r, "\rho"] \arrow[d, "pr_2"'] & M \arrow[d] \\
M \arrow[r] & \mathcal{G} \ltimes M
\end{tikzcd}$$

is a weak 2-pullback diagram. Therefore, instead of sheaf, we should want

$$N \mapsto \mathrm{Hom}\,(N, \mathcal{G} \ltimes M)$$

to be a stack. Unfortunately, this assignment is not a stack unless the action is trivial, however, it can be *turned into* a stack. Explicitly, we define the stacky-quotient $M//G$ to be the stackification of the weak presheaf of groupoids

$$N \mapsto \mathrm{Hom}\,(N, \mathcal{G} \ltimes M)\,.$$

Denote this un-stackified weak presheaf by $\tilde{y}\,(\mathcal{G} \ltimes M)$. It is easy to check that $\tilde{y}$ is actually a fully faithful 2-functor from the 2-category of Lie groupoids to the 2-category of weak presheaves of groupoids over the category of manifolds and it preserves all weak 2-limits. In particular, this implies that in the 2-category $Gpd^{Mfd^{op}}$, the following is a weak 2-pullback diagram:

$$\begin{tikzcd} \mathcal{G} \times M \arrow[r, "\rho"] \arrow[d, "pr_2"'] & M \arrow[d] \\ M \arrow[r] & \tilde{y}\left(\mathcal{G} \ltimes M\right). \end{tikzcd}$$

Finally, since the stackification 2-functor preserves finite weak limits, this implies that the following is a weak 2-pullback diagram of stacks:

$$\begin{tikzcd} \mathcal{G} \times M \arrow[r, "\rho"] \arrow[d, "pr_2"'] & M \arrow[d] \\ M \arrow[r] & M//G. \end{tikzcd}$$

Therefore, $M//G$ satisfies our desired property as well, but has the added benefit of being a stack. Before we explain in what sense the projection map $M \to M//G$ is a principal G-bundle, we will look at a very important example of a global quotient, namely a global quotient where the manifold M is the one-point manifold $*$. In this case, the action groupoid $G \ltimes *$ is canonically isomorphism to G when viewed as a Lie groupoid with one-object. It is easy to check that any weak presheaf of the form $\tilde{y}\left(\mathcal{G}\right)$ for some Lie groupoid $\mathcal{G}$ is automatically a prestack for the open cover topology. Hence,

$$*//G \simeq \tilde{y}\left(G\right)^{+}.$$

It follows that

$$\operatorname{Hom}\left(N, *//G\right) := \underset{\mathcal{U} \in cov(N)}{\underrightarrow{\operatorname{holim}}} \mathfrak{Des}\left(\tilde{y}\left(G\right), \mathcal{U}\right),$$

where $cov(N)$ is a suitable 2-category of open covers of N.

By Proposition I.1.10,

$$\mathfrak{Des}\left(\tilde{y}\left(G\right), \mathcal{U}\right) \simeq \operatorname{Hom}_{LieGpd}\left(N_{\mathcal{U}}, G\right).$$

Since the object space of G is the one-point space, an object of this category reduces to the data of a collection of maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ which satisfy the cocycle condition

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma},$$

when restricted to triple intersections. The geometer may recognize that this data is precisely cocycle data for a principal G-bundle over N which trivializes over the cover $\mathcal{U}$. The effect of taking the weak colimit over all open covers is to glue together the groupoids of principal G-bundles over N which trivialize over various open covers to obtain the groupoid of *all* principal G-bundles over N. Hence, $*//G \simeq Bun_G$.

To illustrate the drastic distance difference between the stacky quotient

$$*//G = Bun_G$$

and the coarse quotient $*/G = *$, we note the following: for any manifold M,

$$\mathrm{Hom}\,(M, *) \cong *,$$

that is, there is only one map from M to $*$. However, in general there will be several maps to $*//G$ since by the 2-Yoneda lemma,

$$\mathrm{Hom}\,(M, *//G) \simeq Bun_G\,(M)\,.$$

So, whereas there is only one map to the coarse quotient, a map to the *stacky* quotient is the same thing as a principal G-bundle.

We note that the stack Bun_G classifies principal G-bundles:

Proposition I.2.1. *For any principal G-bundle $\pi : P \to N$, the following is a weak 2-pullback diagram in stacks:*

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\pi"'] & * \arrow[d] \\ N \arrow[r, "\tilde{P}"] & Bun_G, \end{tikzcd}$$

where $* \to Bun_G \simeq G \ltimes *$ is induced by the unit of the group, and $\tilde{P}$ is a map which under the 2-Yoneda lemma corresponds to a principal G-bundle over N isomorphic to P.

In this sense, $* \to Bun_G$ plays the role of a universal principal G-bundle.

Proof. It suffices to show that for every manifold X, there is a functorial equivalence

$$\mathrm{Hom}\,(X, P) \simeq \mathrm{Hom}\,(X, N) \times_{Bun_G(X)} *,$$

where the latter is the weak 2-pullback

$$\begin{tikzcd} \mathrm{Hom}\,(X, N) \times_{Bun_G(X)} * \arrow[r] \arrow[d] & * \arrow[d] \\ \mathrm{Hom}\,(X, N) \arrow[r] & Bun_G\,(X)\,, \end{tikzcd}$$

where the map $\mathrm{Hom}\,(X, N) \to Bun_G\,(X)$ corresponds to the map

$$\begin{array}{rcl} \mathrm{Hom}\,(X, N) & \to & Bun_G\,(X)_0 \\ f : X \to N & \mapsto & f^*\,(P)\,, \end{array}$$

and the map $* \to Bun_G(X)$ corresponds to the map $* \to Bun_G(X)_0$ picking out the trivial principal G-bundle $G \times X \to X$. Since both $\mathrm{Hom}(X, N)$ and $*$ are sets, it follows that the weak 2-pullback is also a set. Explicitly, it can be described as the set of pairs (f, α) with

$$f : X \to N$$

a smooth map, and

$$\alpha : f^*(P) \to G \times X$$

an isomorphism of principal G-bundles. Given such a pair, we can consider the composite

$$N \to G \times N \xrightarrow{\alpha} f^*(P) \xrightarrow{\hat{f}} P,$$

where $N \to G \times N$ is the canonical global section $n \mapsto (e, n)$, and with

$$f^*(P) = X \times_N P,$$

$\hat{f}$ is the projection $pr_2 : X \times_N P \to P$. This gives a map

$$\theta : \mathrm{Hom}(X, N) \times_{Bun_G(X)} * \to \mathrm{Hom}(X, P).$$

Conversely, given a map $\varphi : X \to P$, consider the map

$$\pi \circ \varphi : X \to N,$$

where $\pi : P \to N$ is the projection map of the principal G-bundle P. Explicitly, $(\pi \circ \varphi)^*(P)$ consists of pairs (x, p) in $X \times P$ such that

$$\pi(\varphi(x)) = \pi(p).$$

Since P is a principal bundle, for each such pair, there exists a unique element of g of G such that

$$g\varphi(x) = p,$$

and this choice assembles into a smooth map

$$\sigma : (\pi \circ \varphi)^*(P) \to G.$$

From this data, we define an isomorphism of principal G-bundles over X:

$$\begin{array}{rcl} \alpha : (\pi \circ \varphi)^*(P) & \to & G \times X \\ (x, p) & \mapsto & (\sigma(x, p), x). \end{array}$$

This assembles into a map

$$\omega : \mathrm{Hom}(X, P) \to \mathrm{Hom}(X, N) \times_{Bun_G(X)} *.$$

It is easy to check that θ and ω are mutually inverse to each other. $\square$

This enables us to extend the definition of a principal G-bundle to allow for the base to be any stack $\mathscr{X}$. We may simply define it as a map

$$\pi : \mathscr{P} \to \mathscr{X}$$

which fits into a weak 2-pullback diagram of the form

$$\begin{tikzcd} \mathscr{P} \arrow[r] \arrow[d, "\pi"'] & * \arrow[d] \\ \mathscr{X} \arrow[r] & Bun_G. \end{tikzcd}$$

Let us return now to the stacky quotient $M//G$, where M is no longer required to be the one-point space. Note that the action groupoid $G \ltimes M$ comes equipped with a canonical smooth groupoid homomorphism

$$pr_1 : G \ltimes M \to G.$$

Proposition I.2.2. *In the* 2*-category of Lie groupoids and smooth functors, the following is a weak* 2*-pullback diagram:*

$$\begin{tikzcd} M \arrow[r] \arrow[d] & * \arrow[d] \\ G \ltimes M \arrow[r, "pr_1"] & G. \end{tikzcd}$$

Proof. Using the model for the weak pullback in Section I.2.2,

$$(G \ltimes M) \times_G *$$

has objects pairs (x, g) with $x \in M$ and $g \in G$, and an arrow from (x, g) to (x', g') is a group element $h \in G$ such that

$$hx = x'$$

and

$$g = g'h.$$

It follows that there is an arrow from (x, g) to (x', g') if and only if $gx = g'x'$, and in this case, the arrow is unique (given by $g'^{-1}g$). There is a canonical smooth functor

$$\psi : M \to (G \ltimes M) \times_G *$$

which sends x to (x, e), where $e \in G$ is the identity element. There is also a canonical functor

$$\varphi : (G \ltimes M) \times_G * \to M$$

which sends (g, x) to gx, and sends the unique arrow from (x, g) to (x', g') (if it exists) to $id_{gx} = id_{g'x'}$. By direct inspection, we see that

$$\varphi \circ \psi = id_M.$$

Finally, notice that

$$egx = gx$$

so, for each (g, x), there is a unique map from

$$\psi \circ \varphi(x, g) = (gx, e)$$

to

$$(x, g)$$

comprising a canonical smooth natural isomorphism

$$\alpha : \psi \circ \phi \Rightarrow id_{(G \ltimes M) \times_G *}.$$

So M and $(G \ltimes M) \times_G *$ are equivalent, hence M is a model for the weak 2-pullback. $\square$

Since $\tilde{y}$ preserves all weak 2-limits and stackification preserves finite ones, it follows that the induced diagram in stacks

$$\begin{tikzcd} M \arrow[r] \arrow[d] & * \arrow[d] \\ M//G \arrow[r, "pr_1"] & *//G, \end{tikzcd}$$

is also a pullback diagram. Hence, the canonical map $M \to M//G$ is a principal G-bundle over $M//G$ in the sense we just defined.

I.2.4 Definition of topological and differentiable stacks

In the last subsection, we saw how stacks can be used to form the stacky quotient of any Lie group action on a manifold. These quotient stacks are first examples of differentiable stacks. Recall that such a stacky quotient is the stackification of the prestack of maps into the corresponding action Lie groupoid. More generally, let $\mathcal{G}$ be any Lie groupoid. Then $\mathcal{G}$ determines a weak presheaf on *Mfd* by the rule

$$X \mapsto \mathrm{Hom}_{LieGpd}\left((X)^{(id)}, \mathcal{G}\right).$$

This defines an extended Yoneda 2-functor $\tilde{y} : LieGpd \to Gpd^{Mfd^{op}}$ and we have the obvious commutative diagram

$$\begin{tikzcd} Mfd \arrow[r, "y"] \arrow[d, "(\cdot)^{(id)}"'] & \mathrm{Set}^{Mfd^{op}} \arrow[d, "(\cdot)^{(id)}"] \\ LieGpd \arrow[r, "\tilde{y}"'] & Gpd^{Mfd^{op}}. \end{tikzcd}$$

Remark. $\tilde{y}$ preserves all weak-limits.

Remark. There is an analogously defined 2-functor $\tilde{y} : \mathbb{TOP}Gpd \to Gpd^{\mathbb{TOP}^{op}}$ in the topological setting.

Given a Lie groupoid or topological groupoid $\mathcal{G}$, we denote by $[\mathcal{G}]$ the stackification of the prestack $\tilde{y}(\mathcal{G})$. This is a stack over manifolds or topological spaces, respectively.

Definition I.2.8. A **differentiable stack** is a stack over manifolds which is equivalent to one of the form $[\mathcal{G}]$ for some Lie groupoid $\mathcal{G}$. A **topological stack** is a stack over topological spaces which is equivalent to $[\mathcal{G}]$ for some topological groupoid $\mathcal{G}$.

We note that, so far, everything would work for more general Grothendieck topologies than the open cover topology. This will be very important later, in the topological setting, in Chapter II.

Given a subcanonical Grothendieck topology J on $\mathbb{TOP}$, we denote by $[\mathcal{G}]_J$ the associated stack on $(\mathbb{TOP}, J)$, $a_J \circ \tilde{y}(\mathcal{G})$, where a_J is the stackification 2-functor (Definition I.1.32).

Definition I.2.9. A stack $\mathscr{X}$ on $(\mathbb{TOP}, J)$ is **presentable** if it is equivalent to $[\mathcal{G}]_J$ for some topological groupoid $\mathcal{G}$. In this case, $\mathcal{G}$ is said to be a **presentation** of $\mathscr{X}$.

We denote the full sub-2-category of $St_J(\mathbb{TOP})$ consisting of presentable stacks by $\mathfrak{Pres}St_J(\mathbb{TOP})$.

I.2.5 Principal Bundles

Principal bundles for Lie groups and topological groups (and more generally for groupoids) are classical objects of study. However, principal bundles (and many other objects involving a local triviality condition) should not be thought of as objects associated to the *category* $\mathbb{TOP}$ (or manifolds), but rather as objects associated to the *Grothendieck site* $(\mathbb{TOP}, \mathcal{O})$, where $\mathcal{O}$ is the **open cover topology** on $\mathbb{TOP}$. This Grothendieck topology is defined by declaring a family of maps $(O_\alpha \to X)_\alpha$ to be a covering family if and only if it constitutes an open cover of X. The concept of principal bundles generalizes to other Grothendieck topologies and we will need this generality later when we introduce the compactly generated Grothendieck topology on compactly generated Hausdorff spaces. For the remainder of this subsection, in the topological setting, let J be an arbitrary subcanonical Grothendieck topology on $\mathbb{TOP}$. When working in the smooth setting, we will still assume we are working with open covers.

Definition I.2.10. Given a topological groupoid (or Lie groupoid), $\mathcal{G}$, a (left) **$\mathcal{G}$-space** is a space (or manifold) E equipped with a **moment map**

$\mu : E \to \mathcal{G}_0$ and an **action map** $\rho : \mathcal{G}_1 \times_{\mathcal{G}_0} E \to E$, where

$$\begin{tikzcd} \mathcal{G}_1 \times_{\mathcal{G}_0} E \arrow[r] \arrow[d] & E \arrow[d, "\mu"] \\ \mathcal{G}_1 \arrow[r, "s"] & \mathcal{G}_0 \end{tikzcd}$$

is the fibered product, such that the following conditions hold:

i) $(gh) \cdot e = g \cdot (h \cdot e)$ whenever e is an element of E and g and h elements of $\mathcal{G}_1$ with domains such that the composition makes sense,

ii) $\mathbb{1}_{\mu(e)} \cdot e = e$ for all $e \in E$, and

iii) $\mu(g \cdot e) = t(g)$ for all $g \in \mathcal{G}_1$ and $e \in E$.

There is an obvious dual notion of a right action.

Definition I.2.11. Suppose that $\mathcal{G} \circlearrowright E$ is a $\mathcal{G}$-space. Then the **action groupoid** $\mathcal{G} \ltimes E$ is defined to be the topological or Lie groupoid whose arrow space is $\mathcal{G} \times_{\mathcal{G}_0} E$ and whose object space is E. An element

$$(g, e) \in \mathcal{G} \times_{\mathcal{G}_0} E \subset \mathcal{G} \times E$$

is viewed as an arrow from e to $g \cdot e$. Composition is defined by the rule

$$(h, g \cdot e) \cdot (g, e) = (hg, e).$$

Definition I.2.12. Given a topological or Lie groupoid $\mathcal{G}$, the **translation groupoid** $\mathbb{E}\mathcal{G}$ is defined to be the action groupoid $\mathcal{G} \ltimes \mathcal{G}_1$ with respect to the left action of $\mathcal{G}$ on $\mathcal{G}_1$ by composition.

Definition I.2.13. For a topological groupoid $\mathcal{G}$, a (left) $\mathcal{G}$**-bundle** over a space X (with respect to J) is a (left) $\mathcal{G}$-space P equipped with $\mathcal{G}$-invariant **projection map**

$$\pi : P \to X$$

which **admits local sections** with respect to the Grothendieck topology J. Recall that this last condition means that there exists a covering family $\mathcal{U} = (U_i \to X)_i$ in J and morphisms $\sigma_i : U_i \to P$ called **local sections** such that the following diagram commutes for all i:

$$\begin{tikzcd} & P \arrow[dr, "\pi"] & \\ U_i \arrow[ur, "\sigma_i"] \arrow[rr, "f_i"] & & X. \end{tikzcd}$$

This condition is equivalent to requiring that the projection map is an epimorphism of (representable) J-sheaves.

In the smooth setting, we simply demand that the map π is a surjective submersion.

Such a $\mathcal{G}$-bundle is called (J)-**principal** if the induced map,

$$\mathcal{G}_1 \times_{\mathcal{G}_0} P \to P \times_X P$$

is a homeomorphism (or diffeomorphism in the smooth setting).

We typically denote such a principal bundle by

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[d, shift right] \arrow[d, shift left] & \circlearrowleft & P \arrow[dll, "\mu"'] \arrow[d, "\pi"] \\
\mathcal{G}_0 & & X.
\end{tikzcd}$$

Remark. To ease terminology, for the rest of this section, the term principal bundle, will refer to a J-principal bundle for our fixed topology J. When we are dealing with smooth manifolds, we will always assume that J is the open cover topology.

Definition I.2.14. Any topological or Lie groupoid $\mathcal{G}$ determines a principal $\mathcal{G}$-bundle over $\mathcal{G}_0$, by

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[d, shift right] \arrow[d, shift left] & \circlearrowleft & \mathcal{G}_1 \arrow[dll, "t"'] \arrow[d, "s"] \\
\mathcal{G}_0 & & \mathcal{G}_0,
\end{tikzcd}$$

called the **unit bundle**, $1_{\mathcal{G}}$, where the action is given by composition.

Definition I.2.15. Let $f : Y \to X$ be a map and suppose that P is a principal $\mathcal{G}$-bundle over X. Then we can give $P \times_X Y \to Y$ the structure of a principal $\mathcal{G}$-bundle $f^*(P)$ over Y, called the **pull-back bundle**. The action is given by

$$g \cdot (p, y) := (gp, y),$$

and the moment map is given by the composite

$$P \times_X Y \to P \xrightarrow{\mu} \mathcal{G}_0.$$

Proposition I.2.3. *An equivalent condition for a $\mathcal{G}$-bundle*

$$\pi : P \to X$$

to be a principal $\mathcal{G}$-bundle is for the induced map,

$$\mathcal{G}_1 \times_{\mathcal{G}_0} P \to P \times_X P,$$

to be a homeomorphism (or diffeomorphism in the smooth setting), and for the following local triviality condition to be satisfied:

There exists a covering family $(f_i : U_i \to X)$ such that for each i, there is a map $\xi_i : U_i \to \mathcal{G}_0$ such that the pullback of the unit bundle

$$\begin{tikzcd} (\xi_i)^* (1_{\mathcal{G}}) \arrow[r] \arrow[d] & \mathcal{G}_1 \arrow[d, "s"] \\ U_i \arrow[r, "\xi_i"] & \mathcal{G}_0 \end{tikzcd}$$

is isomorphic to $P_{|U_i} := (f_i)^* (P)$.

Proof. If

$$\begin{tikzcd} \mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowright"] & P \arrow[dl, "\mu"'] \arrow[d, "\pi"] \\ \mathcal{G}_0 & X \end{tikzcd}$$

is a principal $\mathcal{G}$-bundle over X, let $\mathcal{U} = \left(U_i \xrightarrow{f_i} X\right)_i$ be a J-cover together with local sections $\sigma_i : U_i \to P$. Define for each i

$$\xi_i := \mu \circ \sigma.$$

Then we have

$$(\xi_i)^* (1_{\mathcal{G}}) \simeq \{(u, g) \in U_i \times \mathcal{G}_1 | \xi_i (u) = s (g)\} .$$

Define a map

$$\varphi_i : (\xi_i)^* (1_{\mathcal{G}}) \to P$$

by

$$\varphi (u, g) := g \cdot \sigma (u) .$$

Then φ_i is the desired isomorphism.

Conversely, suppose P is a $\mathcal{G}$-bundle satisfying the local-triviality condition. It suffices to show that the projection map

$$\varphi : (\xi_i)^* (1_{\mathcal{G}}) \to U_i$$

has a section. Such a section is given by the equation

$$\sigma_i (u) := \left(u, \mathbb{1}_{\xi_i(u)}\right) .$$

□

Remark. If J is the open cover topology and $\mathcal{G}$ is topological group G viewed as a groupoid with one object, then the locally triviality-condition of Proposition I.2.3 means that P is a locally-trivial fiber-bundle with fiber homeomorphic to G. Hence, we can see that Definition I.2.13 agrees with the notion of a principal G-bundle of a topological group.

Definition I.2.16. Given P and P', two principal $\mathcal{G}$-bundle over X, a map $f : P \to P'$ is a **map of principal bundles** if it respects the projection maps and is $\mathcal{G}$-equivariant. It is easy to check that any such map must be an isomorphism of principal bundles. For each space X, denote the groupoid of principal $\mathcal{G}$-bundles over X by $Bun_{\mathcal{G}}(X)$.

Definition I.2.17. If $\mathcal{G}$ is topological or Lie groupoid and P is a principal $\mathcal{G}$-bundle over X, then we define the **gauge groupoid** $Gauge(P)$ to be the following topological groupoid:

The fibered product

$$\begin{tikzcd} P \times_{\mathcal{G}_0} P \arrow[r] \arrow[d] & P \arrow[d, "\mu"] \\ P \arrow[r, "\mu"] & \mathcal{G}_0 \end{tikzcd}$$

(if it exists) admits a left-$\mathcal{G}$-action with moment map $\tilde{\mu}((p,q)) = \mu(p)$ via

$$g \cdot (p,q) = (g \cdot p, g \cdot q).$$

The arrow space of $Gauge(P)$ is the quotient

$$P \times_{\mathcal{G}_0} P/\mathcal{G}$$

and the object space is X. An equivalence class $[(p,q)]$ is viewed as an arrow from $\pi(p)$ to $\pi(q)$ (which is well defined as π is $\mathcal{G}$-invariant.) Composition is determined by the rule

$$[(p,q)] \cdot [(q',r)] = [(p, g \cdot r)]$$

where g is the unique element of $\mathcal{G}_1$ such that $g \cdot q' = q$.

Definition I.2.18. Let $\mathcal{G}$ and $\mathcal{H}$ be topological or Lie groupoids. A (left) **principal $\mathcal{G}$-bundle over $\mathcal{H}$** is a left principal $\mathcal{G}$-bundle

$$\begin{tikzcd} \mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & P \arrow[dll, "\mu"'] \arrow[d, "\nu"] \\ \mathcal{G}_0 & & \mathcal{H}_0 \end{tikzcd}$$

over $\mathcal{H}_0$ such that P also has the structure of a right $\mathcal{H}$-bundle with moment map ν, with the $\mathcal{G}$ and $\mathcal{H}$ actions commuting in the obvious sense. We typically denote such a bundle by

$$\begin{tikzcd} \mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & P \arrow[dll, "\mu"'] \arrow[drr, "\nu"] & \circlearrowleft & \mathcal{H}_1 \arrow[d, shift left] \arrow[d, shift right] \\ \mathcal{G}_0 & & & & \mathcal{H}_0. \end{tikzcd}$$

Remark. To a continuous (or smooth) functor $\varphi : \mathcal{H} \to \mathcal{G}$, one can canonically associate a principal $\mathcal{G}$-bundle over $\mathcal{H}$. It is defined by putting the obvious (right) $\mathcal{H}$-bundle structure on the total space of the pullback bundle

$$\varphi_0^* (1_{\mathcal{G}}) = \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{H}_0,$$

induced via φ :

$$(g, x) \cdot h := (g\varphi (h), s (g)).$$

Definition I.2.19. A map of principal $\mathcal{G}$-bundles over $\mathcal{H}$ between P and P' is a continuous (smooth) map which is bi-equivariant, i.e. respects both the left and right actions. In particular, such a map is a map of underlying principal $\mathcal{G}$-bundles over $\mathcal{H}_0$, hence an isomorphism. We denote the corresponding groupoid of principal $\mathcal{G}$-bundles over $\mathcal{H}$ by $Bun_{\mathcal{G}} (\mathcal{H})$.

Proposition I.2.4. *Let $\mathcal{U}$ be J-cover of $\mathcal{H}_0$. Denote by $Bun_{\mathcal{G}} (\mathcal{H},\mathcal{U})$ the full subgroupoid of $Bun_{\mathcal{G}} (\mathcal{H})$ consisting of those principal bundles whose underlying principal $\mathcal{G}$-bundle admits local sections over $\mathcal{U}$. Then there is a natural equivalence of groupoids*

$$Bun_{\mathcal{G}} (\mathcal{H},\mathcal{U}) \simeq \mathrm{Hom} (\mathcal{H}_{\mathcal{U}}, \mathcal{G}).$$

Proof. For simplicity, we will assume that the J is the open cover topology. The proof for general J is completely analogous. Hence, let us assume

$$\mathcal{U} = (U_\alpha)$$

is an open cover over of $\mathcal{H}_0$. Denote

$$\mathcal{H} (U_\alpha, U_\beta) = s^{-1} (U_\alpha) \cap t^{-1} (U_\beta).$$

An object of $\mathrm{Hom} (\mathcal{H}_{\mathcal{U}}, \mathcal{G})$ can be described as a collection of maps

$$(\varphi_\alpha, g_{\alpha\beta})$$

such that

i) $\varphi_\alpha : U_\alpha \to \mathcal{G}_0$,

ii) $g_{\alpha\beta} : \mathcal{H} (U_\alpha, U_\beta) \to \mathcal{G}_1$,

iii) $g_{\alpha\beta} (h) : \varphi_\beta (t (h)) \to \varphi_\alpha (s (h))$, and

iv) $g_{\alpha\beta} (h) g_{\beta\gamma} (h') = g_{\alpha\gamma} (h'h)$.

Let

$$\tilde{Q} := \left(\coprod_\alpha \varphi_\alpha^* (\mathcal{G}) \right) \Big/ \left\{ (g_\alpha, x_\alpha) \sim (g_\beta, x_\beta) \mid x_\alpha = x_\beta \text{ and } g_\beta g_{\beta\alpha} (\mathbb{1}_{x_\alpha}) = g_\alpha \right\},$$

where each $\varphi_\alpha^*(\mathcal{G})$ is the pullback of the unit $\mathcal{G}$-bundle along φ_α :

$$\varphi_\alpha^*(\mathcal{G}) := \{(g_\alpha, x_\alpha) \in \mathcal{G} \times U_\alpha \mid s(g_\alpha) = \varphi_\alpha(x_\alpha)\}.$$

We now make $\tilde{Q}$ into a left principal $\mathcal{G}$-bundle over $\mathcal{H}$ as follows:

Let $\tilde{\nu}$ be defined as

$$\begin{aligned} \tilde{\nu} : \tilde{Q} &\to \mathcal{H}_0 \\ [(g_\alpha, x_\alpha)] &\mapsto x_\alpha, \end{aligned}$$

which is well-defined since equivalent points have the same second projection. Let $\tilde{\mu}$ be defined as

$$\begin{aligned} \tilde{\mu} : \tilde{Q} &\to \mathcal{G}_0 \\ [(g_\alpha, x_\alpha)] &\mapsto t(g_\alpha), \end{aligned}$$

which is well-defined since right-multiplication does not change the target. Define a left $\mathcal{G}$-action on $\tilde{Q}$ along the moment map $\tilde{\mu}$ by

$$g \cdot [(g_\alpha, x_\alpha)] := [(gg_\alpha, x_\alpha)].$$

This action is (clearly) well-defined and it is easy to check that it is a principal action with bundle map $\tilde{\nu}$. Define a right $\mathcal{H}$-action on $\tilde{Q}$ along the moment map $\tilde{\nu}$ by

$$[(g_\beta, y_\beta)] \cdot h := \left[\left(g_\beta g_{\alpha\beta}(h)^{-1}, x_\alpha\right)\right],$$

where $h : x_\alpha \to y_\beta$. It is simple to check that this action is well-defined as a consequence of the generalized cocycle condition iv). It follows that $\tilde{Q}$ is a left principal $\mathcal{G}$-bundle over $\mathcal{H}$. So we have constructed the object part of a functor

$$\mathbb{BUN}_{\mathcal{U}} : \mathrm{Hom}(\mathcal{H}_{\mathcal{U}}, \mathcal{G}) \to Bun_{\mathcal{G}}(\mathcal{H}, \mathcal{U}).$$

Let us now define it on arrows. An arrow in $\mathrm{Hom}(\mathcal{H}_{\mathcal{U}}, \mathcal{G})$ can be described as a collection of maps

$$(\theta_\alpha, g_{\alpha\beta}),$$

such that

i) $\theta_\alpha : U_\alpha \to \mathcal{G}_1$,

ii) $g_{\alpha\beta} : \mathcal{H}(U_\alpha, U_\beta) \to \mathcal{G}_1$,

iii) $g_{\alpha\beta}(h) : s \circ \theta_\beta(t(h)) \to s \circ \theta_\alpha(s(h))$, and

iv) $g_{\alpha\beta}(h)\, g_{\beta\gamma}(h') = g_{\alpha\gamma}(h'h)$.

Such a collection is an arrow

$$(s \circ \theta_\alpha, g_{\alpha\beta}) \to \left(t \circ \theta_\alpha, (\theta_\alpha \circ s) \cdot g_{\alpha\beta} \cdot (\theta_\beta \circ t)^{-1}\right). \tag{I.17}$$

Composition is computed as

$$\left(\theta'_\alpha, g'_{\alpha\beta}\right) \cdot (\theta_\alpha, g_{\alpha\beta}) := (\theta'_\alpha \cdot \theta_\alpha, g_{\alpha\beta}).$$

Now, if

$$(\theta_\alpha, g_{\alpha\beta}) : (\varphi_\alpha, g_{\alpha\beta}) \to \left(\varphi'_\alpha, g'_{\alpha\beta}\right)$$

is an arrow in $\mathrm{Hom}\,(\mathcal{H}_\mathcal{U}, \mathcal{G})$, let

$$\begin{array}{rcl} \mathbb{BUN}_\mathcal{U}\left((\theta_\alpha, g_{\alpha\beta})\right) : \mathbb{BUN}_\mathcal{U}\left((\varphi_\alpha, g_{\alpha\beta})\right) & \to & \mathbb{BUN}_\mathcal{U}\left(\left(\varphi'_\alpha, g'_{\alpha\beta}\right)\right) \\ \left[(g_\alpha, x_\alpha)\right] & \mapsto & \left[\left(g_\alpha \theta_\alpha (x_\alpha)^{-1}, x_\alpha\right)\right]. \end{array}$$

It is an easy consequence of (I.17) that this is a well defined morphism of principal $\mathcal{G}$-bundles over $\mathcal{H}$. It is readily verified that this is indeed a functor. We will show that show that $\mathbb{BUN}_\mathcal{U}$ is full and faithful. To fix notation, consider for each α the pullback diagram

$$\begin{tikzcd} \varphi_\alpha^*(\mathcal{G}) \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{G}_1 \arrow[d, "s"] \\ U_\alpha \arrow[r, "\varphi_\alpha"] & \mathcal{G}_0. \end{tikzcd}$$

For $z \in U_\alpha$, let

$$\sigma_\alpha(z) := \left[\left(\mathbb{1}_{\varphi_\alpha(z)}, z\right)\right]. \tag{I.18}$$

Then

$$\theta_\alpha(z) = \left(pr_2\left(\mathbb{BUN}_\mathcal{U}\left((\theta_\alpha, g_{\alpha\beta})\right)(\sigma_\alpha(z))\right)\right)^{-1}.$$

Moreover, the domain of $\mathbb{BUN}_\mathcal{U}\left((\theta_\alpha, g_{\alpha\beta})\right)$ is $\mathbb{BUN}_\mathcal{U}\left((\varphi_\alpha, g_{\alpha\beta})\right)$, and $g_{\alpha\beta}$ may be recovered from this bundle, since it can be easily checked that given

$$h : x \to y$$

with $x \in U_\alpha$ and $y \in U_\beta$, $g_{\alpha\beta}(h)$ is the unique arrow of $\mathcal{G}$ such that

$$g_{\alpha\beta}(h)\,\sigma_\beta(y)\,h = \sigma_\alpha(x),$$

with σ_α and σ_β defined as in (I.18). Hence, $\mathbb{BUN}_\mathcal{U}$ is faithful. Suppose now that

$$f : \mathbb{BUN}_\mathcal{U}\left((\varphi_\alpha, g_{\alpha\beta})\right) \to \mathbb{BUN}_\mathcal{U}\left(\left(\varphi'_\alpha, g'_{\alpha\beta}\right)\right)$$

is an arrow in $\mathrm{Bun}_{\mathcal{G}}(\mathcal{H},\mathcal{U})$. Then, for each $z \in U_\alpha$, there exists a unique element $\theta_\alpha(z)$ of $\mathcal{G}_1$ such that

$$\theta_\alpha(z) f\left(\left[\left(\mathbb{1}_{\varphi_\alpha(z)}, z\right)\right]\right) = \left[\left(\mathbb{1}_{\varphi'_\alpha(z)}, z\right)\right].$$

Hence,

$$\begin{aligned} f([(g_\alpha, z)]) &= g_\alpha f\left(\left[\left(\mathbb{1}_{\varphi_\alpha(z)}, z\right)\right]\right) \\ &= g_\alpha \left[\left(\theta_\alpha(z)^{-1}, z\right)\right] \\ &= \left[\left(g_\alpha \theta_\alpha(z)^{-1}, z\right)\right]. \end{aligned}$$

It follows that $f = \mathbb{BUN}_{\mathcal{U}}((\theta_\alpha, g_{\alpha\beta}))$. So $\mathbb{BUN}_{\mathcal{U}}$ is also full. It suffices to show that it is essentially surjective.

Let Q be principal $\mathcal{G}$-bundle over $\mathcal{H}$ in $Bun_{\mathcal{G}}(\mathcal{H},\mathcal{U})$. Let μ denote the moment map of the underlying principal $\mathcal{G}$-bundle, and ν its projection onto $\mathcal{H}_0$. Choose for each element of the cover U_α a section of ν,

$$\sigma_\alpha : U_\alpha \to Q.$$

Let $h : x \to y$ be an arrow in $\mathcal{H}$, with $x \in U_\alpha$ and $y \in U_\beta$. Then h can act on $\sigma_\beta(y)$ on the right. Notice that this implies

$$\nu(\sigma_\beta(y) \cdot h) = \nu(\sigma_\alpha(x)).$$

Since the $\mathcal{G}$-action is principal, there exists a unique $g_{\alpha\beta}(h) \in \mathcal{G}_1$ such that

$$g_{\alpha\beta}(h)\, \sigma_\beta(y)\, h = \sigma_\alpha(x). \tag{I.19}$$

Let

$$\varphi_\alpha := \mu \circ \sigma_\alpha$$

and similarly for β. Then we have

$$g_{\alpha\beta}(h) : \varphi_\beta(y) \to \varphi_\alpha(x)$$

in $\mathcal{G}$. Suppose we have two composable arrows

$$x \xrightarrow{h} y \xrightarrow{h'} x,$$

with $x \in U_\alpha$, $y \in U_\beta$, $z \in U_\gamma$. Then we have

$$\begin{aligned} g_{\alpha\beta}(h)\, \sigma_\beta(y)\, h &= \sigma_\alpha(x) \\ g_{\beta\gamma}(h')\, \sigma_\gamma(z)\, h' &= \sigma_\beta(y) \\ g_{\alpha\gamma}(h)\, \sigma_\gamma(z)\, h'h &= \sigma_\alpha(x), \end{aligned}$$

from which it follows that

$$g_{\alpha\beta}(h)\, g_{\beta\gamma}(h')\, \sigma_\alpha(x) = g_{\alpha\gamma}(h)\, \sigma_\alpha(x).$$

Since the $\mathcal{G}$-action is principal, this yields the generalized cocycle condition

$$g_{\alpha\beta}(h)\, g_{\beta\gamma}(h') = g_{\alpha\gamma}(h'h).$$

Hence $(\varphi_\alpha, g_{\alpha,\beta})$ is in $\mathrm{Hom}(\mathcal{H}_{\mathcal{U}}, \mathcal{G})_0$. Define a map

$$\begin{array}{rcl} \sigma : \mathbb{BUN}_{\mathcal{U}}(\varphi_\alpha, g_{\alpha\beta}) & \to & Q \\ [(g_\alpha, x_\alpha)] & \mapsto & g_\alpha \cdot \sigma_\alpha(x_\alpha). \end{array}$$

It is well defined as a consequence of equation (I.19) applied to $h = \mathbb{1}_{x_\alpha}$. The fact that it commutes with the $\mathcal{G}$-action is clear. The fact that it commutes with the $\mathcal{H}$-action is again a direct application of equation (I.19). Since the map is well-defined and $Bun_{\mathcal{G}}(\mathcal{H},\mathcal{U})$ is a groupoid, it is an isomorphism. □

Corollary I.2.1. *Let X be a topological space (or manifold) Let $\mathcal{U}$ be J-cover (or open cover) of X. Denote by $Bun_{\mathcal{G}}(X,\mathcal{U})$ the full subgroupoid of $Bun_{\mathcal{G}}(X)$ consisting of those principal bundles which admit local sections over $\mathcal{U}$. Then*

$$Bun_{\mathcal{G}}(X,\mathcal{U}) \simeq \mathrm{Hom}(X_{\mathcal{U}}, \mathcal{G}).$$

Corollary I.2.2. *Let X be a fixed space (or manifold). Then*

$$[\mathcal{G}]_J(X) \simeq Bun_{\mathcal{G}}(X).$$

Proof. Since $\tilde{y}(\mathcal{G})$ is a prestack,

$$[\mathcal{G}]_J(X) := a_J(\tilde{y}(\mathcal{G})) \simeq \tilde{y}(\mathcal{G})^+.$$

Therefore, by Proposition I.1.10, this implies

$$[\mathcal{G}]_J(X) \simeq \underset{\mathcal{U}\in cov_J(X)}{\underrightarrow{\mathrm{holim}}}\ \mathrm{Hom}_{\mathbb{TOP}\,Gpd}(X_{\mathcal{U}}, \mathcal{G}).$$

Combining this with Proposition I.2.4, this yields

$$[\mathcal{G}]_J(X) \simeq \underset{\mathcal{U}\in cov_J(X)}{\underrightarrow{\mathrm{holim}}}\ Bun_{\mathcal{G}}(X,\mathcal{U}) \simeq Bun_{\mathcal{G}}(X).$$

□

I.2.6 Morita Equivalences

Definition I.2.20. A continuous functor $\varphi : \mathcal{H} \to \mathcal{G}$ between two topological groupoids is a J-**Morita equivalence** if the following two properties hold:

i) (Essentially Surjective)

The map $t \circ pr_1 : \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{H}_0 \to \mathcal{G}_0$ admits local sections with respect to the topology J, where $\mathcal{G}_1 \times_{G_0} \mathcal{H}_0$ is the fibered product

$$\begin{tikzcd} \mathcal{G}_1 \times_{G_0} \mathcal{H}_0 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{H}_0 \arrow[d, "\varphi"] \\ \mathcal{G}_1 \arrow[r, "s"] & \mathcal{G}_0. \end{tikzcd}$$

ii) (Fully Faithful)

The following is a fibered product

$$\begin{tikzcd} \mathcal{H}_1 \arrow[r, "\varphi"] \arrow[d, "\langle s,t \rangle"'] & \mathcal{G}_1 \arrow[d, "\langle s,t \rangle"] \\ \mathcal{H}_0 \times \mathcal{H}_0 \arrow[r, "\varphi\times\varphi"] & \mathcal{G}_0 \times \mathcal{G}_0. \end{tikzcd}$$

A smooth functor $\varphi : \mathcal{H} \to \mathcal{G}$ between two Lie groupoids is a Morita equivalence if in i), the map $t \circ pr_1$ is a surjective submersion, and ii) holds as stated.

Remark. If $\mathcal{U}$ is a J-cover of the object space $\mathcal{G}_0$ of a topological groupoid $\mathcal{G}$, then the induced map $\mathcal{G}_{\mathcal{U}} \to \mathcal{G}$ is a J-Morita equivalence. The corresponding statement holds in the smooth setting for the open cover topology as well.

Remark. We will again suppress the reference to the Grothendieck topology J; for the rest of the section, a Morita equivalence will implicitly mean a J-Morita equivalence for our fixed topology J, if we are in the topological setting. A Morita equivalence with respect to the open cover topology will be called an ordinary Morita equivalence.

Remark. The property of being a Morita equivalence is weaker than being an equivalence in the 2-category $\mathbb{TOP}Gpd$ (or in $LieGpd$). In fact, a Morita equivalence is an equivalence in if and only if $t \circ pr_1$ admits a global section. However, any Morita equivalence does induce an equivalence in the 2-category Gpd after applying the forgetful 2-functor. Morita equivalences are sometimes referred to as weak equivalences.

Proposition I.2.5. *Let $\varphi : \mathcal{H} \to \mathcal{G}$. Then if φ satisfies condition i) of Definition I.2.20, then the induced map*

$$[\varphi]_J : [\mathcal{H}]_J \to [\mathcal{G}]_J$$

is an epimorphism of stacks. If φ satisfies condition ii), then the induced map is a monomorphism of stacks. (See definitions I.1.33 and I.1.35.)

Proof. Suppose that φ satisfies condition i). Let $f : T \to [\mathcal{G}]_J$ be a map from a representable T. Then, as

$$[\mathcal{G}]_J(T) \simeq \underset{\mathcal{U}\in cov_J(T)}{\underrightarrow{\mathrm{holim}}} \; \mathrm{Hom}_{\mathbb{TOP}\,Gpd}(T_{\mathcal{U}}, \mathcal{G}),$$

by Yoneda, this corresponds to a J-cover $\mathcal{U} = (\lambda_\alpha : U_\alpha \to T)$ of T and a homomorphism

$$T_{\mathcal{U}} \to \mathcal{G}.$$

Now, since φ satisfies condition i), we may find a J-cover $\mathcal{V}$ of $\mathcal{G}_0$

$$(\mu_i : V_i \to \mathcal{G}_0)$$

together with sections

$$\sigma_i : V_i \to \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{H}_0$$

of $t \circ pr_1$. For each pair (α, i), consider the pullback diagram

$$\begin{tikzcd} U_\alpha \times V_i \arrow[r, "pr_i"] \arrow[d, "pr_\alpha"'] & V_i \arrow[d, "f_i"] \\ U_\alpha \arrow[r, "\lambda_\alpha"'] & \mathcal{G}_0. \end{tikzcd}$$

Then we have that

$$\mathcal{W} := (\lambda \circ pr_\alpha : U_\alpha \times V_i \to T)_{\alpha,i}$$

is a J-cover of T refining $\mathcal{U}$. Hence, the composite $T_W \to T_U \to \mathcal{G}$ is equivalent to $T_{\mathcal{U}} \to \mathcal{G}$ in the weak 2-colimit

$$\underset{\mathcal{U}\in cov_J(X)}{\underrightarrow{\mathrm{holim}}} \; \mathrm{Hom}_{\mathbb{TOP}\,Gpd}(X_{\mathcal{U}}, \mathcal{G}).$$

Let us denote this composite as $\bar{f} : T_{\mathcal{W}} \to \mathcal{G}$. Notice for all α and i, the following diagram 2-commutes:

$$\begin{tikzcd} U_\alpha \times V_i \arrow[r] \arrow[dr] & \tilde{y}(T_{\mathcal{W}}) \arrow[r, "\tilde{y}(\bar{f})"] \arrow[d] & \tilde{y}(\mathcal{G}) \arrow[r, "\eta_{\tilde{y}(\mathcal{G})}"] & {[\mathcal{G}]_J}, \\ & T \arrow[urr, "f"'] & & \end{tikzcd}$$

where the map $\eta_{\tilde{y}(\mathcal{G})}$ is the unit of the adjunction $a_J \dashv j$, where j denotes the inclusion of weak presheaves of groupoids into J-stacks. The map

$$U_\alpha \times V_i \to \tilde{y}(\mathcal{G})$$

corresponds under Yoneda to the map

$$\lambda_\alpha \circ pr_\alpha : U_\alpha \times V_i \to \mathcal{G}_0.$$

There is also another canonical map

$$\varphi_0 \circ pr_2 \circ \sigma_i \circ pr_i : U_\alpha \times V_i \to \mathcal{G}_0,$$

where

$$pr_2 : \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{H}_0 \to \mathcal{H}_0$$

is the canonical projection. Viewing these as maps into $\mathcal{G}$,

$$pr_1 \circ \sigma_i \circ pr_i : U_\alpha \times V_i \to \mathcal{G}_1$$

encodes a continuous (smooth) natural isomorphism

$$\omega : \lambda_\alpha \circ pr_\alpha \Rightarrow \varphi_0 \circ pr_2 \circ \sigma_i \circ pr_i.$$

Notice that

$$\varphi_0 \circ pr_2 \circ \sigma_i \circ pr_i$$

factors through $\varphi : \mathcal{H} \to \mathcal{G}$. It follows that the following diagram 2-commutes:

$$\begin{tikzcd}
U_\alpha \times V_i \arrow[r, "{[pr_2 \circ \sigma_i \circ pr_i]_J}"] \arrow[d] & {[\mathcal{H}]_J} \arrow[d, "{[\varphi]_J}"] \\
\tilde{y}\left(T_{\mathcal{W}}\right) \arrow[r] \arrow[d] & {[\mathcal{G}]_J} . \\
T \arrow[ur, "f"] &
\end{tikzcd}$$

Hence f locally factors through $[\varphi]_J$ up to isomorphism, so $[\varphi]_J$ is an epimorphism.

Suppose that φ satisfies condition *ii*) and let $\mathcal{K}$ be another topological (or Lie) groupoid. We will show that the induced functor

$$\mathrm{Hom}\left(\mathcal{K}, \mathcal{H}\right) \to \mathrm{Hom}\left(\mathcal{K}, \mathcal{G}\right)$$

given by composition with φ is full and faithful. Suppose we are given two homomorphisms

$$f, g : \mathcal{K} \to \mathcal{H}$$

and an internal natural isomorphism

$$\alpha : \varphi \circ f \Rightarrow \varphi \circ g.$$

Then α is actually a map

$$\alpha : K_0 \to \mathcal{G}_1$$

making the diagrams commute necessary for it to be a natural transformation from $\varphi \circ f$ and $\varphi \circ g$. In particular, this implies that

$$s \circ \alpha = \varphi \circ f$$

and

$$t \circ \alpha = \varphi \circ g.$$

Since

$$\begin{tikzcd}
\mathcal{H}_1 \arrow[r, "\varphi"] \arrow[d, "\langle s,t \rangle"'] & \mathcal{G}_1 \arrow[d, "\langle s,t \rangle"] \\
\mathcal{H}_0 \times \mathcal{H}_0 \arrow[r, "\varphi \times \varphi"] & \mathcal{G}_0 \times \mathcal{G}_0.
\end{tikzcd}$$

is a pullback diagram, there is a unique $\tilde{\alpha} : \mathcal{K}_0 \to \mathcal{H}_1$ such that

$$s \circ \tilde{\alpha} = f,$$

$$t \circ \tilde{\alpha} = g$$

and

$$\varphi_1 \circ \tilde{\alpha} = \alpha.$$

We claim that this $\tilde{\alpha}$ encodes a natural isomorphism from f to g. Furthermore, any natural transformation

$$\beta : f \Rightarrow g$$

such that $\varphi\beta = \alpha$ must satisfy the above equations, so, by uniqueness must agree with $\tilde{\alpha}$. Hence the induced functor

$$\operatorname{Hom}(\mathcal{K}, \mathcal{H}) \to \operatorname{Hom}(\mathcal{K}, \mathcal{G})$$

is full and faithful. Since $\tilde{y}$ is full and faithful, it follows that the induced map

$$\tilde{y}(\mathcal{H}) \to \tilde{y}(\mathcal{G})$$

is a monomorphism. By the remark immediately following Definition I.1.35, the stackification functor a_J preserves monos, hence

$$[\varphi]_J : [\mathcal{H}]_J \to [\mathcal{G}]_J$$

is a mono as well. $\square$

Corollary I.2.3. *If $\varphi : \mathcal{H} \to \mathcal{G}$ is a Morita equivalence, then the induced map*

$$[\varphi]_J : [\mathcal{H}]_J \to [\mathcal{G}]_J$$

is an equivalence of stacks.

We denote by W_J the class of Morita equivalences. The class W_J admits a right calculus of fractions in the sense of [54]; there is a bicategory $\mathbb{TOP}Gpd\left[W_J^{-1}\right]$ obtained from the 2-category $\mathbb{TOP}Gpd$ by formally inverting the Morita equivalences. A 1-morphism from $\mathcal{H}$ to $\mathcal{G}$ in this bicategory is a diagram of continuous functors

$$\begin{tikzcd} & \mathcal{K} \arrow[dl, "w"'] \arrow[dr] & \\ \mathcal{H} & & \mathcal{G} \end{tikzcd}$$

such that w is a Morita equivalence.

Definition I.2.21. Such a diagram is called a **generalized homomorphism** . There is also a well defined notion of a 2-morphism. For details see [54].

More explicitly, the bicategory $\mathbb{TOP}Gpd\left[W_J^{-1}\right]$ comes equipped with the canonical morphism of bicategories

$$U : \mathbb{TOP}Gpd \to \mathbb{TOP}Gpd\left[W_J^{-1}\right]$$

which is the identity on objects, and sends a morphism $\varphi : \mathcal{H} \to \mathcal{G}$ to the generalized homomorphism

$$\begin{tikzcd} & \mathcal{H} \arrow[dl, "id_{\mathcal{H}}"'] \arrow[dr, "\varphi"] & \\ \mathcal{H} & & \mathcal{G}, \end{tikzcd}$$

and U satisfies the following universal properties:

i) U sends Morita equivalences to equivalences in $\mathbb{TOP}Gpd\left[W_J^{-1}\right]$ and

ii) for any bicategory $\mathscr{D}$, composition with U produces an equivalence of bicategories

$$\mathrm{Hom}\left(\mathbb{TOP}Gpd\left[W_J^{-1}\right], \mathscr{D}\right) \to \mathrm{Hom}_{W_J}\left(\mathbb{TOP}Gpd, \mathscr{D}\right),$$

where the former bicategory is the functor bicategory of homomorphisms

$$\mathbb{TOP}Gpd\left[W_J^{-1}\right] \to \mathscr{D},$$

and where the later is the subbicategory of the functor bicategory from $\mathbb{TOP}Gpd$ to $\mathscr{D}$ on those functors which send Morita equivalences to equivalences.

This construction caries over for Lie groupoids and ordinary Morita equivalences as well. By Corollary I.2.3, from

$$\mathbb{TOP}Gpd \xrightarrow{[\,\cdot\,]} \mathfrak{PresSt}_J\left(\mathbb{TOP}\right),$$

there is an induced functor of bicategories

$$\mathbb{TOP}Gpd\left[W_J^{-1}\right] \to \mathfrak{PresSt}_J\left(\mathbb{TOP}\right).$$

Theorem I.2.1. *The induced map*

$$\mathbb{TOP}Gpd\left[W_J^{-1}\right] \to \mathfrak{PresSt}_J\left(\mathbb{TOP}\right)$$

is an equivalence of bicategories. An analogous statement holds also in the differentiable setting for ordinary Morita equivalences and differentiable stacks.

This theorem is well known. For example, see [54] for the case of étale topological groupoids and topological stacks with an étale atlas, and also for étale Lie groupoids and differentiable stacks with an étale atlas. The preprint [9] contains much of the necessary ingredients for the proof of the case of general Lie groupoids and differentiable stacks in its so-called Dictionary Lemmas. Similar statements in the case of algebraic stacks can be found in [48]. The general theorem follows from an easy application of [54], Section 3.4.

Proposition I.2.6. *Let $p : T \to [\mathcal{G}]_J$ be a map from a space or manifold T. Consider the canonical map $a : \mathcal{G}_0 \to \mathcal{G}$. Let*

$$\begin{tikzcd} \mathcal{G}_1 \arrow[d, shift right] \arrow[d, shift left] & \circlearrowleft & P \arrow[dll, "\mu"'] \arrow[d, "\pi"] \\ \mathcal{G}_0 & & T \end{tikzcd}$$

be the J-principal $\mathcal{G}$-bundle over T to which p corresponds under the 2-Yoneda Lemma. Then the following is a weak 2-pullback diagram of stacks:

$$\begin{tikzcd} P \arrow[r, "\mu"] \arrow[d, "\pi"'] & \mathcal{G}_0 \arrow[d, "{[a]_J}"] \\ T \arrow[r, "p"] & {[\mathcal{G}]_J}. \end{tikzcd}$$

Proof. Choose a J-cover $\mathcal{U}$ of T such that p corresponds to a homomorphism

$$\bar{p} : T_{\mathcal{U}} \to \mathcal{G}.$$

Then

$$P = \mathbb{BUN}_{\mathcal{U}}\left(\bar{p}\right).$$

Let $\pi^*\mathcal{U}$ denote the induced J-cover on P. By direct inspection, there is a canonical isomorphism

$$P_{\pi^*\mathcal{U}} \cong T_{\mathcal{U}} \times_{\mathcal{G}} \mathcal{G}_0.$$

Hence, the following is a weak 2-pullback diagram of topological (or Lie) groupoids:

$$\begin{tikzcd} P_{\pi^*\mathcal{U}} \arrow[r, "\bar{\mu}"] \arrow[d, "\bar{\pi}"'] & \mathcal{G}_0 \arrow[d, "a"] \\ T_{\mathcal{U}} \arrow[r, "\bar{p}"] & \mathcal{G}. \end{tikzcd}$$

One can furthermore check that the composite

$$P_{\pi^*\mathcal{U}} \to P \xrightarrow{\mu} \mathcal{G}_0$$

is equal to $\bar{\mu}$ and that the following diagram commutes on the nose:

$$\begin{tikzcd} P_{\pi^*\mathcal{U}} \arrow[r] \arrow[d, "\bar{\pi}"'] & P \arrow[d, "\pi"] \\ T_{\mathcal{U}} \arrow[r] & T. \end{tikzcd}$$

Since $[\,\cdot\,]_J$ preserves finite weak limits, it follows that the following diagram is a weak 2-pullback diagram of stacks:

$$\begin{tikzcd} {[P_{\pi^*\mathcal{U}}]_J} \arrow[r, "{[\bar{\mu}]_J}"] \arrow[d, "{[\bar{\pi}]_J}"'] & \mathcal{G}_0 \arrow[d, "{[a]_J}"] \\ {[T_{\mathcal{U}}]_J} \arrow[r, "\bar{p}"] & {[\mathcal{G}]_J}\,. \end{tikzcd}$$

Since

$$T_{\mathcal{U}} \to T$$

and

$$P_{\pi^*\mathcal{U}} \to P$$

are J-Morita equivalences, by Corollary I.2.3, we are done. $\square$

Corollary I.2.4. *For any topological or Lie groupoid $\mathcal{G}$, the map*

$$[a]_J : \mathcal{G}_0 \to [\mathcal{G}]_J$$

in Proposition I.2.6 is representable (See Definition I.1.31.)

I.2.7 An intrinsic description of topological and differentiable stacks

We will now present an intrinsic description of topological and differentiable stacks not directly involving groupoids.

Definition I.2.22. An **atlas** for a stack $\mathscr{X}$ over $(\mathscr{C}, J)$ is a *representable* epimorphism $p : C \to \mathscr{X}$ from an object C (see Definition I.1.31).

Remark. For $\mathscr{C}$ the category of manifolds, we will define an atlas $X \to \mathscr{X}$ to be a representable epimorphism, where we consider non-Hausdorff manifolds as being representable objects as well. We will also require that for all maps from a manifold

$$M \to \mathscr{X},$$

the induced map $M \times_{\mathscr{X}} X \to M$ should be a submersion, i.e. we require the atlas to be a representable epimorphic submersion. (See Section I.2.8.)

Proposition I.2.7. *[49] A stack $\mathscr{X}$ over $(\mathbb{TOP}, J)$ is presentable if and only if it has an atlas.*

Proof. Suppose that $p : X \to \mathscr{X}$ is an atlas. Consider the weak 2-pullback

$$\begin{tikzcd} X \times_{\mathscr{X}} X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ X \arrow[r, "p"] & \mathscr{X}. \end{tikzcd}$$

Then, for each space T,

$$\begin{tikzcd} \mathrm{Hom}\,(T, X \times_{\mathscr{X}} X) \arrow[r] \arrow[d] & \mathrm{Hom}\,(T, X) \arrow[d, "p"] \\ \mathrm{Hom}\,(T, X) \arrow[r, "p"] & \mathscr{X}\,(T) \end{tikzcd}$$

is a weak 2-pullback of groupoids. By the ending remarks of Section I.2.2, we know that for each T,

$$\mathrm{Hom}\,(T, X \times_{\mathscr{X}} X) \rightrightarrows \mathrm{Hom}\,(T, X)$$

has the structure of a groupoid. It follows that the assignment

$$T \mapsto \mathrm{Hom}\,(T, X \times_{\mathscr{X}} X) \rightrightarrows \mathrm{Hom}\,(T, X)$$

is a strict presheaf of groupoids, or equivalently, a groupoid object in presheaves. However, as a groupoid object in sheaves, its objects are the sheaf X and its arrows are the sheaf $X \times_{\mathscr{X}} X$, both of which are topological spaces.

Hence, $X \times_{\mathscr{X}} X \rightrightarrows X$ has the structure of a topological groupoid. Moreover, for each T, there is a canonical morphism of groupoids

$$\tilde{p}(T) : X \times_{\mathscr{X}} X \rightrightarrows X \to \mathscr{X}(T),$$

which is fully faithful since

$$X \times_{\mathscr{X}} X(T) \rightrightarrows X(T)$$

is equivalent to $p(T)^*(\mathscr{X}(T))$. This assembles into a monomorphism

$$\tilde{p} : \tilde{y}(X \times_{\mathscr{X}} X \rightrightarrows X) \to \mathscr{X}.$$

By the remark immediately proceeding Definition I.1.35, the induced morphism

$$a_J(\tilde{y}(X \times_{\mathscr{X}} X \rightrightarrows X)) \to \mathscr{X} \tag{I.20}$$

is a monomorphism. Since p is an epimorphism, it is easy to check that $\tilde{p}$ is a J-covering morphism (See Definition I.1.34). It follows that the induced map (I.20) is both an epimorphism and a monomorphism, hence an equivalence. That is,

$$[X \times_{\mathscr{X}} X \rightrightarrows X]_J \simeq \mathscr{X}.$$

Conversely, given a topological groupoid $\mathcal{G}$, Proposition I.2.5 implies that the canonical map of groupoids

$$(\mathcal{G}_0)^{id} \to \mathcal{G},$$

which is the identity on objects and u on arrows, produces an epimorphism $p : \mathcal{G}_0 \to [\mathcal{G}]_J$. By Corollary I.2.4, it is also representable. □

Corollary I.2.5. *A topological stack $\mathscr{X}$ over $\mathbb{TOP}$ with the open cover topology is a topological stack if and only if it has an atlas. A stack $\mathscr{X}$ over Mfd is a differentiable stack if and only if it has an atlas.*

Proof. The topological case follows from Proposition I.2.7 entirely. The proof in the smooth setting is exactly the same except, to show that

$$X \times_{\mathscr{X}} X \rightrightarrows X$$

is a Lie groupoid, one must show that the source and target maps are submersions. However, this is true by definition since they are obtained by pulling back along p, which is a representable submersion. (See Section I.2.8.) □

I.2.8 Representable maps

If $\mathscr{C}$ is any category, we say that a map $f : C \to D$ is representable, if and only if $y(f)$ is representable in $\mathrm{Set}^{\mathscr{C}^{op}}$.

Any map of topological spaces is representable. For manifolds, this is not the case, but all submersions are representable.

Definition I.2.23. Let P be a property of a map of spaces. It is said to be **invariant under change of base** if for all

$$f : Y \to X$$

with property P, if

$$g : Z \to X$$

is any representable map, the induced map

$$Z \times_X Y \to Z$$

also has property P. The property P is said to be **invariant under restriction**, in the topological setting if this holds whenever g is an embedding, and in the differentiable setting if and only if this holds whenever g is an *open* embedding. Note that in either case, being invariant under change of base implies being invariant under restriction. A property P which is invariant under restriction is said to be **local on the target** if any

$$f : Y \to X$$

for which there exists an open cover $(U_\alpha \to X)$ such that the induced map

$$\coprod_\alpha U_\alpha \times_X Y \to \coprod_\alpha U_\alpha$$

has property P, must also have property P.

Examples of such properties are being an open map, étale map, proper map, closed map etc.

Definition I.2.24. Let P be a property of a map of topological spaces or manifolds which is invariant under restriction and local on the target. Then, a representable map

$$f : \mathscr{X} \to \mathscr{Y}$$

of stacks (over $\mathbb{TOP}$ or *Mfd* respectively) is said to have property P if for any map

$$T \to \mathscr{Y}$$

from a space or manifold, the induced map

$$T \times_{\mathscr{Y}} \mathscr{X} \to T$$

has property P.

I.2.9 The bicategory of principal bundles

We will now describe another way of representing the bicategories of topological and differentiable stacks. The term principal bundle refers to a subcanonical Grothendieck topology J, which in the smooth setting we assume to be the standard open cover topology.

Notice that a morphism from a space X to the stack $[\mathcal{G}]_J$, by Yoneda, corresponds to a principal $\mathcal{G}$-bundle over X. This suggests that we should view a principal $\mathcal{G}$-bundle over $\mathcal{H}$ as a morphism from $\mathcal{H}$ to $\mathcal{G}$. Suppose we are given a principal $\mathcal{G}$-bundle over $\mathcal{H}$

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[d, shift right] \arrow[d, shift left] & \circlearrowright & P \arrow[dll, "\mu"'] \arrow[drr, "\nu"] & \circlearrowleft & \mathcal{H}_1 \arrow[d, shift right] \arrow[d, shift left] \\
\mathcal{G}_0 & & & & \mathcal{H}_0,
\end{tikzcd}$$

which we view as a map

$$P : \mathcal{H} \to \mathcal{G}$$

(right now this is just notation). Suppose furthermore that we are given a principal $\mathcal{H}$-bundle over $\mathcal{K}$

$$\begin{tikzcd}
\mathcal{H}_1 \arrow[d, shift right] \arrow[d, shift left] & \circlearrowright & Q \arrow[dll, "\mu'"'] \arrow[drr, "\nu'"] & \circlearrowleft & \mathcal{K}_1 \arrow[d, shift right] \arrow[d, shift left] \\
\mathcal{H}_0 & & & & \mathcal{K}_0,
\end{tikzcd}$$

which we view as a map

$$Q : \mathcal{K} \to \mathcal{H}.$$

If it is actually reasonable to view principal bundles as maps, then we should be able to "compose" these two principal bundles. That is, form their **composition** $P \otimes Q$, which will be a principal $\mathcal{G}$-bundle over $\mathcal{K}$. This can indeed be done; the construction is as follows. First, we endow $P \times_{\mathcal{H}_0} Q$ with the structure of a right $\mathcal{H}$-space along

$$\nu \circ pr_1 = \mu' \circ pr_2,$$

by defining

$$(p, q) \cdot h := \left(p \cdot h, h^{-1} \cdot q\right).$$

We define the total space of $P \otimes Q$ to be the quotient space of this action. For the smooth setting, notice that the action of $\mathcal{H}$ on Q is free, so hence the induced action on $P \times_{\mathcal{H}_0} Q$ is too. Furthermore, $P \times_{\mathcal{H}_0} Q$ is a manifold as ν is a submersion, hence $P \otimes Q$ is a manifold as well.

We denote by $p \otimes q$ the image of the pair (p,q) under the canonical projection onto the quotient space. Define maps

$$\begin{aligned} \mu \otimes \mu' : P \otimes Q &\to \mathcal{G}_0 \\ p \otimes q &\mapsto \mu(p) \end{aligned}$$

and

$$\begin{aligned} \nu \otimes \nu' : P \otimes Q &\to \mathcal{K}_0 \\ p \otimes q &\mapsto \nu'(q) . \end{aligned}$$

There is an induced right $\mathcal{K}$-action given by

$$(p \otimes q) \cdot k := p \otimes (k \cdot q)$$

and an induced left $\mathcal{G}$-action given by

$$g \cdot (p \otimes q) := (g \cdot p) \otimes q.$$

It is easy to check that this defines a principal $\mathcal{G}$-bundle over $\mathcal{K}$.

One may ask if this composition is associative, that is, given R a principal $\mathcal{K}$-bundle over $\mathcal{L}$, do we have

$$(P \otimes Q) \otimes R = P \otimes (Q \otimes R)?$$

This is not the case. However, this is not strictly necessary for this to be a good notion of map; both $(P \otimes Q) \otimes R$ and $P \otimes (Q \otimes R)$ are objects in the groupoid $Bun_{\mathcal{G}}(\mathcal{L})$, so it is not natural to ask if they are equal, but rather one should ask if they are isomorphic. An easy exercise shows that they are indeed canonically isomorphic. Using these isomorphisms as associators, one can construct a bicategory $Bun^J\mathbb{TOP}Gpd$ (or $BunLieGpd$ in the smooth setting). It is in fact a $(2,1)$-category:

The **objects** are topological groupoids (or Lie groupoids). The category of maps between two topological (or Lie) groupoids $\mathcal{H}$ and $\mathcal{G}$ is simply the groupoid $Bun_{\mathcal{G}}(\mathcal{H})$ of principal $\mathcal{G}$-bundles over $\mathcal{H}$. It is easy to check that the composition construction just described is the object part of a functor

$$Bun_{\mathcal{G}}(\mathcal{H}) \times Bun_{\mathcal{H}}(\mathcal{K}) \to Bun_{\mathcal{G}}(\mathcal{K}) .$$

This describes composition in this bicategory. We leave the rest of the details to the reader.

By the remark following definition I.2.18, there is a canonical 2-functor

$$\mathbb{TOP}Gpd \to Bun^J\mathbb{TOP}Gpd$$

which sends Morita equivalences to equivalences. Therefore, there is an induced map

$$\mathbb{TOP}Gpd\left[W_J^{-1}\right] \to Bun^J\mathbb{TOP}Gpd$$

of bicategories.

Theorem I.2.2. *The induced map*

$$\mathbb{TOP}Gpd\left[W_J^{-1}\right] \to Bun^J\mathbb{TOP}Gpd$$

is an equivalence of bicategories.

This theorem is well known. A 1-categorical version of this theorem is proven in [45], Section 2.6. The general result follows easily from [54], Section 3.4. Note, combining this with Theorem I.2.1, this implies that the two bicategories $Bun^J\mathbb{TOP}Gpd$ and $\mathfrak{PresSt}_J(\mathbb{TOP})$ are equivalent. Similarly the bicategory $BunLieGpd$ is equivalent to differentiable stacks.

However, for completeness, we shall give a direct proof. For this, it will help to have a different way of describing principal bundles over groupoids.

Let $\mathcal{H}$ and $\mathcal{G}$ be topological (or Lie) groupoids. Let $\mathcal{Cocycle}(\mathcal{H}, \mathcal{G})$ denote the following groupoid:

The objects are pairs (P, α) with P a principal $\mathcal{G}$-bundle over $\mathcal{H}_0$ and

$$\alpha : d_1^*P \to d_0^*P$$

a morphism of principal $\mathcal{G}$-bundles over $\mathcal{H}_1$ which satisfy a cocycle condition on $\mathcal{H}_2$, i.e. the following diagram commutes:

(I.21)
$$\begin{tikzcd}
 & d_2^*d_0^*P \arrow[r, "\sim"] & d_0^*d_1^*P \arrow[dr, "d_0^*\alpha"] & \\
d_2^*d_1^*P \arrow[ur, "d_2^*\alpha"] \arrow[dr, "\sim"'] & & & d_0^*d_0^*P, \\
 & d_1^*d_1^*P \arrow[r, "d_1^*\alpha"] & d_1^*d_0^*P \arrow[ur, "\sim"'] &
\end{tikzcd}$$

where above we have invoked simplicial notation for the face maps of the simplicial nerve $N(\mathcal{H})$, and the unlabeled isomorphisms are the canonical ones.

An arrow between a pair (P, α) and (Q, β) is a morphism

$$f : P \to Q$$

of principal $\mathcal{G}$-bundles over $\mathcal{H}_0$ such that the following diagram commutes:

(I.22)
$$\begin{tikzcd}
d_1^*P \arrow[r, "\alpha"] \arrow[d, "d_1^*f"'] & d_0^*P \arrow[d, "d_0^*f"] \\
d_1^*Q \arrow[r, "\beta"] & d_0^*Q.
\end{tikzcd}$$

Remark. This groupoid is canonically isomorphic to the groupoid

$$\mathcal{C}ocycle\left(\mathcal{H}, [\mathcal{G}]_J\right)$$

defined in Lemma A.3.2.

Lemma I.2.3. *For topological (or Lie) groupoids $\mathcal{H}$ and $\mathcal{G}$, there is a functorial equivalence of groupoids*

$$F_{\mathcal{H},\mathcal{G}} : Bun_{\mathcal{G}}\left(\mathcal{H}\right) \xrightarrow{\sim} \mathcal{C}ocycle\left(\mathcal{H}, \mathcal{G}\right).$$

Proof. Define $F_{\mathcal{H},\mathcal{G}}$ on objects as follows. Suppose

$$\begin{tikzcd} \mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & P \arrow[dll, "\mu"'] \arrow[drr, "\nu"] & \circlearrowleft & \mathcal{H}_1 \arrow[d, shift left] \arrow[d, shift right] \\ \mathcal{G}_0 & & & & \mathcal{H}_0. \end{tikzcd}$$

is a principal $\mathcal{G}$-bundle over $\mathcal{H}$. Denote by $\underline{P}$ the underlying principal $\mathcal{G}$-bundle over $\mathcal{H}_0$. Define

$$\begin{aligned} \alpha\left(P\right) : d_1^*\underline{P} &\rightarrow d_0^*\underline{P} \\ \left(p, h\right) &\mapsto \left(ph, h\right). \end{aligned}$$

It is $\mathcal{G}$-equivariant, and respects the projection map pr_2, therefore is a morphism of principal $\mathcal{G}$-bundles over $\mathcal{H}_1$. Following the top of diagram I.21 yields

$$\begin{tikzcd} d_2d_1^*\underline{P} \arrow[r, "d_2^*\alpha"] & d_2^*d_0^*\underline{P} \arrow[r, "\sim"] & d_0^*d_1^*\underline{P} \arrow[r, "d_0^*\alpha"] & d_0^*d_0^*\underline{P} \\ \left(\left(p, h'\right), \left(h', h\right)\right) \arrow[r, mapsto] & \left(\left(p \cdot h', h'\right), \left(h', h\right)\right) \arrow[r, mapsto] & \left(\left(ph', h\right), \left(h', h\right)\right) \arrow[r, mapsto] & \left(\left(\left(p \cdot h'\right) \cdot h, h\right)\left(h', h\right)\right) \end{tikzcd}$$

while following the bottom of the same diagram yields

$$\begin{tikzcd} d_2d_1^*\underline{P} \arrow[r, "\sim"] & d_1^*d_1^*\underline{P} \arrow[r, "d_1^*\alpha"] & d_1^*d_0^*\underline{P} \arrow[r, "\sim"] & d_0^*d_0^*\underline{P} \\ \left(\left(p, h'\right), \left(h', h\right)\right) \arrow[r, mapsto] & \left(\left(p, h'h\right), \left(h', h\right)\right) \arrow[r, mapsto] & \left(\left(p \cdot \left(h'h\right), h'h\right), \left(h', h\right)\right) \arrow[r, mapsto] & \left(\left(p \cdot \left(h'h\right), h\right), \left(h', h\right)\right). \end{tikzcd}$$

So the diagram I.21 commutes. Hence $\left(\underline{P}, \alpha\right)$ is an object in $\mathcal{C}ocycle\left(\mathcal{H}, \mathcal{G}\right)$. We denote it by $F_{\mathcal{H},\mathcal{G}}\left(P\right)$.

Suppose now that we are given P and Q, two principal $\mathcal{G}$-bundles over $\mathcal{H}$, and a morphism $f : \underline{P} \rightarrow \underline{Q}$ of their underlying principal $\mathcal{G}$-bundles over $\mathcal{H}_0$. Then the diagram I.22 commutes if and only if for all

$$\left(p, h\right) \in d_0^*P,$$

$$\left(f\left(ph\right), h\right) = \left(f\left(p\right)h, h\right).$$

This equality clearly holds if and only if f is $\mathcal{H}$-equivariant, and since it was already a map of principal $\mathcal{G}$-bundles over $\mathcal{H}_0$, this is if and only if f is a map between the principal $\mathcal{G}$-bundles over $\mathcal{H}$. It follows that $F_{\mathcal{H},G}$ extends to a full and faithful functor.

It suffices to show that $F_{\mathcal{H},G}$ is essentially surjective. Given a pair (P, α), we wish to use the isomorphism α to endow P with the structure of a principal $\mathcal{G}$-bundle over $\mathcal{H}$. In other words, if

$$\begin{tikzcd} \mathcal{G}_1 \arrow[d, shift right] \arrow[d, shift left] & \circlearrowleft & P \arrow[dll, "\mu"'] \arrow[d, "\nu"] \\ \mathcal{G}_0 & & \mathcal{H}_0 \end{tikzcd}$$

is the principal bundle P, we must find a compatible $\mathcal{H}$-action with moment map ν. Consider the following pullback diagram:

$$\begin{tikzcd} d_0^* P \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{H}_1 \arrow[d, "d_0"] \\ P \arrow[r, "\nu"] & \mathcal{H}_0. \end{tikzcd}$$

We define the desired $\mathcal{H}$-action by the map

$$\rho := pr_1 \circ \alpha : d_1^* P = P \times^t_{\mathcal{H}_0} \mathcal{H}_1 \to P.$$

We must check that this indeed encodes a compatible $\mathcal{H}$-action. Let us write $p \cdot h$ for $\rho(p, h)$. Since α is a map over $\mathcal{H}_1$, it follows that, in components, α is given by

$$\alpha(p, h) = (p \cdot h, h).$$

One condition to check in order to show that ρ encodes a right $\mathcal{H}$-action is that

$$\nu(p \cdot h) = s(h).$$

This condition follows since

$$\nu(p \cdot h) = \nu\rho(p, h) = d_0 \circ pr_2 \circ \alpha(p, h) = d_0(h).$$

Consider the following commutative diagram:

$$
\begin{tikzcd}
P \arrow[rr, "\nu"] \arrow[dd, dashed] \arrow[dr, dashed, "r"] \arrow[ddd, bend right=60, "id_P"'] & & \mathcal{H}_0 \arrow[dd, "u"] \arrow[dr, "u"] & \\
& d_1^*P \arrow[rr, dotted, "pr_2"] \arrow[dl, "\alpha"'] \arrow[ddl, dotted, "\rho"] & & \mathcal{H}_1 \arrow[ddl, "d_0"] \\
d_0^*P \arrow[rr, "pr_2"] \arrow[d, "pr_1"'] & & \mathcal{H}_1 \arrow[d, "d_1"'] & \\
P \arrow[rr, "\nu"] & & \mathcal{H}_0, &
\end{tikzcd}
$$

where the two dashed arrows are induced by the pullback diagrams. In components, the map labeled as r is given by

$$r(p) = \left(p, \mathbb{1}_{\nu(p)}\right).$$

It follows that

$$p \cdot \mathbb{1}_{\nu(p)} = p.$$

The same calculation used to show that the diagram I.21 commuted when we defined $F_{\mathcal{H},\mathcal{G}}$ on objects, shows that for ρ satisfies

$$(p \cdot h') \cdot h = p \cdot (h'h).$$

Hence ρ defines a right $\mathcal{H}$-action with moment map ν. Since the map α is a map of principal $\mathcal{G}$-bundles, it is $\mathcal{G}$-equivariant. It is immediate therefore that the right $\mathcal{H}$-action is compatible with this $\mathcal{G}$-action. Hence we get a well defined principal $\mathcal{G}$-bundle over $\mathcal{H}$. It readily follows that $F_{\mathcal{H},\mathcal{G}}$ applied to this principal bundle recovers (P, α), so we are done. $\square$

Proposition I.2.8. *Let P be a principal $\mathcal{G}$-bundle over $\mathcal{H}$. Then the underlying principal $\mathcal{G}$-bundle $\underline{P}$ over $\mathcal{H}_0$ is canonically and naturally isomorphic to $P \otimes 1_{\mathcal{H}}$, where $1_{\mathcal{H}}$ is the unit bundle over $\mathcal{H}$ (See Definition I.2.14).*

Proof. Suppose that

$$
\begin{tikzcd}
\mathcal{G}_1 \arrow[d, shift right] \arrow[d, shift left] & \circlearrowright & P \arrow[dll, "\mu"'] \arrow[drr, "\nu"] & \circlearrowleft & \mathcal{H}_1 \arrow[d, shift right] \arrow[d, shift left] \\
\mathcal{G}_0 & & & & \mathcal{H}_0,
\end{tikzcd}
$$

is a principal $\mathcal{G}$-bundle over $\mathcal{H}$. The total space of $P \otimes 1_{\mathcal{H}}$, is the quotient of the pullback

$$\begin{tikzcd} P \times_{\mathcal{H}_0} \mathcal{H}_1 \arrow[r] \arrow[d] & \mathcal{H}_1 \arrow[d, "t"] \\ P \arrow[r, "\nu"] & \mathcal{H}_0, \end{tikzcd}$$

by the action of $\mathcal{H}$ defined by

$$(p, l) \cdot h = \left(ph, h^{-1}l\right).$$

Define a map $k_P : P \to P \times_{\mathcal{H}_0} \mathcal{H}_1$ by the rule

$$p \mapsto \left(p, \mathbb{1}_{\nu(p)}\right).$$

On one hand, we claim this map hits every orbit. Indeed, notice that

$$(p, l) \cdot l = \left(pl, \mathbb{1}_{\nu(pl)}\right).$$

On the other hand, no non-identity element of $\mathcal{H}$ fixes a point in the image of k_P. It follows that k_P induces an isomorphism

$$k_P : \underline{P} \xrightarrow{\sim} P \otimes 1_{\mathcal{H}},$$

at the level of spaces. One can easily check that this map respects the necessary structure maps for this to be a morphism of principal bundles. $\square$

Theorem I.2.4. *$Bun^J\mathbb{TOP}Gpd$ and $\mathfrak{PresSt}_J(\mathbb{TOP})$ are equivalent bicategories.*

Proof. First, we shall extend the functor

$$\mathbb{TOP}Gpd \xrightarrow{[\cdot]} \mathfrak{PresSt}_J(\mathbb{TOP})$$

to a functor

$$Bun\mathbb{TOP}Gpd \xrightarrow{[\cdot]} \mathfrak{PresSt}_J(\mathbb{TOP}).$$

On objects, it is the same as before, namely

$$\mathcal{G} \mapsto [\mathcal{G}].$$

Notice that given P a principal $\mathcal{G}$-bundle over $\mathcal{H}$, it determines a functor

$$[P](X) : [\mathcal{H}](X) \to [\mathcal{G}](X)$$

which on objects is given by

$$Q \mapsto P \otimes Q.$$

On arrows, if $f : Q \to Q'$ is a bi-equivariant map (i.e. an isomorphism in $[\mathcal{H}](X)$), then it induces a map

$$\begin{aligned} P \otimes f : P \otimes Q &\to P \otimes Q' \\ p \otimes q &\mapsto p \otimes f(q) \end{aligned}$$

which is well-defined as f is equivariant. Now, suppose that $\varphi : X \to Y$ is a map of spaces. We claim that the diagram

$$\begin{tikzcd} {[\mathcal{H}](Y)} \arrow[r, "{[P](Y)}"] \arrow[d, "\varphi^*"'] & {[\mathcal{G}](Y)} \arrow[d, "\varphi^*"] \\ {[\mathcal{H}](X)} \arrow[r, "{[P](X)}"] & {[\mathcal{G}](X)}, \end{tikzcd}$$

2-commutes. To see this, we may choose to view the spaces X and Y as topological groupoids, and regard φ as a principal Y bundle over X with total space X. Let us denote this principal bundle by R_φ. For each $Q \in Bun_{\mathcal{H}}(Y)$,

$$\varphi^*(P \otimes Q) = (P \otimes Q) \otimes R_\varphi,$$

and

$$P \otimes \varphi^*(Q) = P \otimes (Q \otimes R_\varphi).$$

Define

$$\alpha_Q : (P \otimes Q) \otimes R_\varphi \to P \otimes (Q \otimes R_\varphi)$$

to be the canonical associator map. Then the collection $(\alpha_Q)_Q$ assembles into a natural isomorphism

$$[P](\varphi) : \varphi^* \circ [P](Y) \Rightarrow [P](X) \circ \varphi^*.$$

The fact that the composition of principal bundles makes $Bun\mathbb{TOP}Gpd$ into a coherent bicategory guarantees that the necessary pentagon commutes to make $[P]$ a well-define lax natural transformation from $[\mathcal{H}]$ to $[\mathcal{G}]$.

This defines the objects of a functor

$$[\,\cdot\,]_{\mathcal{H},\mathcal{G}} : Bun_{\mathcal{G}}(\mathcal{H}) \to \mathrm{Hom}([\mathcal{H}], [\mathcal{G}]).$$

Suppose that $f : P \to P'$ is a morphism in $Bun_{\mathcal{G}}(\mathcal{H})$. Then for all X, there is a natural transformation

$$[f]_{\mathcal{H},\mathcal{G}}(X) : [P](X) \Rightarrow [P'](X)$$

whose component along the principal $\mathcal{H}$-bundle Q over X is given the map

$$\begin{aligned} P \otimes Q &\to P' \otimes Q \\ p \otimes q &\mapsto f(p) \otimes q. \end{aligned}$$

The necessary square to make this assignment a modifications again commutes by virtue of the coherency of the associators of the bicategory. We leave it to the reader to check that each $[\,\cdot\,]_{\mathcal{H},\mathcal{G}}$ is a functor and that they assemble into a functor of bicategories

$$[\,\cdot\,] : Bun\mathbb{TOP}Gpd \to \mathfrak{PresSt}_J\left(\mathbb{TOP}\right).$$

Notice that by definition, $[\,\cdot\,]$ is essentially surjective. It suffices to show that each functor $[\,\cdot\,]_{\mathcal{H},\mathcal{G}}$ is in fact an equivalence of categories.

By Lemma A.3.2, there is an equivalence of groupoids

$$G : \mathrm{Hom}\left(\left[\mathcal{H}\right],\left[\mathcal{G}\right]\right) \xrightarrow{\sim} \mathit{Cocycle}\left(\mathcal{H},\mathcal{G}\right).$$

We also know, by Lemma I.2.3, that there is an equivalence of groupoids

$$F_{\mathcal{H},\mathcal{G}} : Bun_{\mathcal{G}}\left(\mathcal{H}\right) \xrightarrow{\sim} \mathit{Cocycle}\left(\mathcal{H},\mathcal{G}\right).$$

To show that $[\,\cdot\,]_{\mathcal{H},\mathcal{G}}$ is an equivalence of groupoids, it suffices to show that $F_{\mathcal{H},\mathcal{G}}$ is naturally isomorphic to $G \circ [\,\cdot\,]_{\mathcal{H},\mathcal{G}}$.

For this, we will need some more details of how G is constructed. To fully understand the following argument, one must read Appendix A.3.

By Corollary A.3.1, the atlas $p : \mathcal{H}_0 \to [\mathcal{H}]$ exhibits the following diagram as a weak colimit:

$$\mathcal{H}_2 \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \mathcal{H}_1 \rightrightarrows \mathcal{H}_0 \xrightarrow{p} [\mathcal{H}].$$

Denote this cocone by σ. As it is colimiting, by composition, it induces an equivalence of groupoids

$$\mathrm{Hom}\left(\left[\mathcal{H}\right],\left[\mathcal{G}\right]\right) \xrightarrow{\widehat{\sigma}} \mathit{Cocone}\left(\mathcal{F}_{\mathcal{H}},\left[\mathcal{G}\right]\right).$$

In the proof of Lemma A.3.2, there is an equivalence of groupoids

$$\Theta : \mathit{Cocone}\left(\mathcal{F}_{\mathcal{H}},\left[\mathcal{G}\right]\right) \to \mathit{Cocycle}\left(\mathcal{H},\mathcal{G}\right).$$

G is then the composite of $\widehat{\sigma}$ and Θ.

Now, let P be a principal $\mathcal{G}$-bundle over $\mathcal{H}$. The functor $[\,\cdot\,]_{\mathcal{H},\mathcal{G}}$ sends this to the morphism

$$P \otimes (\,\cdot\,) : [\mathcal{H}] \to [\mathcal{G}].$$

The functor $\widehat{\sigma}$ produces from this a cocone for $\mathcal{F}_{\mathcal{H}}$ with vertex $[\mathcal{G}]$ by composing $\Delta_{P\otimes(\,\cdot\,)}$ with σ. Denote this cocone by ρ. The component of ρ along $[0]$ (i.e. the map $\mathcal{H}_0 \to [\mathcal{G}]$) corresponds to the composite

$$\mathcal{H}_0 \xrightarrow{p} [\mathcal{H}] \xrightarrow{P\otimes(\,\cdot\,)} [\mathcal{G}].$$

By Yoneda, this corresponds to the principal $\mathcal{G}$-bundle over $\mathcal{H}_0$,

$$P \otimes 1_{\mathcal{H}},$$

where $1_{\mathcal{H}}$ is the unit bundle (See Definition I.2.14). This corresponds to the bundle part of the pair in $\mathit{Cocycle}(\mathcal{H},\mathcal{G})$ to which G sends $P \otimes (\,\cdot\,)$. The morphism

$$\beta : d_1^*(P \otimes 1_{\mathcal{H}}) \to d_0^*(P \otimes 1_{\mathcal{H}}),$$

given by G, is described as follows:
Consider the two pullback diagrams

$$\begin{tikzcd}
\mathcal{H}_1 {}^s\times^t_{\mathcal{H}_0} \mathcal{H}_1 \arrow[r] \arrow[d] & \mathcal{H}_1 \arrow[d, "t"] \\
\mathcal{H}_1 \arrow[r, "s"] & \mathcal{H}_0,
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{H}_1 {}^s\times^s_{\mathcal{H}_0} \mathcal{H}_1 \arrow[r] \arrow[d] & \mathcal{H}_1 \arrow[d, "s"] \\
\mathcal{H}_1 \arrow[r, "s"] & \mathcal{H}_0.
\end{tikzcd}$$

These are the total spaces of $d_1^*1_{\mathcal{H}}$ and $d_0^*1_{\mathcal{H}}$ respectively. There is a canonical isomorphism of principal $\mathcal{H}$-bundles over $\mathcal{H}_1$ given by:

$$\begin{aligned}
\gamma : \mathcal{H}_1 {}^s\times^t_{\mathcal{H}_0} \mathcal{H}_1 &\to \mathcal{H}_1 {}^s\times^s_{\mathcal{H}_0} \mathcal{H}_1 \\
(h', h) &\mapsto (h'h, h).
\end{aligned}$$

This corresponds, under Yoneda, to the composite

$$\sigma_0 d_1 \xRightarrow{\sigma(d_1)} \sigma_1 \xRightarrow{\sigma(d_0)^{-1}} \sigma_0 d_0.$$

It follows that

$$\rho_0 d_1 \xRightarrow{\rho(d_1)} \rho_1 \xRightarrow{\rho(d_0)^{-1}} \rho_0 d_0,$$

corresponds to the dashed map

$$\begin{tikzcd}
d_1^*(P \otimes 1_{\mathcal{H}}) \arrow[r, equal] \arrow[d, dashed] & (P \otimes 1_{\mathcal{H}}) \otimes R_{d_1} \arrow[r, "\sim"] & P \otimes (1_{\mathcal{H}} \otimes R_{d_1}) \arrow[r, equal] & P \otimes d_1^* 1_{\mathcal{H}}, \arrow[d, "P\otimes\gamma"] \\
d_0^*(P \otimes 1_{\mathcal{H}}) \arrow[r, equal] & (P \otimes 1_{\mathcal{H}}) \otimes R_{d_0} & P \otimes (1_{\mathcal{H}} \otimes R_{d_0}) \arrow[l, "\sim"'] \arrow[r, equal] & P \otimes d_0^* 1_{\mathcal{H}}
\end{tikzcd}$$

under Yoneda. Explicitly, this is map is given by

$$(h, p \otimes l) \mapsto (h, p \otimes lh).$$

Denote this map by β.

To summarize, so far we have shown that

$$G \circ [P]_{\mathcal{H},\mathcal{G}} = (P \otimes 1_{\mathcal{H}}, \beta).$$

By Proposition I.2.8, there is a canonical isomorphism $k_P : \underline{P} \to P \otimes 1_{\mathcal{H}}$. One easily checks that the following diagram commutes:

$$\begin{tikzcd} d_1^*\underline{P} \arrow[r, "\alpha(P)"] \arrow[d, "d_1(k_P)"'] & d_0^*\underline{P} \arrow[d, "d_0(k_P)"] \\ d_1^*\left(P \otimes 1_{\mathcal{H}}\right) \arrow[r, "\beta"] & d_0^*\left(P \otimes 1_{\mathcal{H}}\right), \end{tikzcd}$$

where

$$\alpha\left(P\right) : \left(h,p\right) \mapsto \left(h,ph\right)$$

as in Lemma I.2.3. Hence, k_P is an isomorphism in $\mathcal{C}ocycle\left(\mathcal{H},\mathcal{G}\right)$ between $F_{\mathcal{H},\mathcal{G}}\left(P\right)$ and $G \circ \left[P\right]_{\mathcal{H},\mathcal{G}}$. Since the isomorphism k_P depends naturally on P, this defines a natural isomorphism

$$k : F_{\mathcal{H},\mathcal{G}} \Rightarrow G \circ \left[\,\cdot\,\right]_{\mathcal{H},\mathcal{G}}.$$

$\square$

Chapter II

Compactly Generated Stacks

This chapter is the main body of my paper [16], which has been accepted for publication in *Advances in Mathematics*. The research was conducted during the first two years of my PhD studies. It is included in its entirety, other than the appendices which have been incorporated into Chapter I. The version to appear in *Advances in Mathematics* shall be nearly identical.

Brief Summary It is well-known that the category of topological spaces is not well behaved. In particular, it is not Cartesian closed. Recall that if a category $\mathscr{C}$ is Cartesian closed then for any two objects X and Y of $\mathscr{C}$, there exists a mapping object $\mathbb{M}\mathrm{ap}\,(X, Y)$, such that for every object Z of $\mathscr{C}$, there is a natural isomorphism

$$\mathrm{Hom}\,(Z, \mathbb{M}\mathrm{ap}\,(X, Y)) \cong \mathrm{Hom}\,(Z \times X, Y)\,.$$

The category of topological spaces is not Cartesian closed; one can topologize the set of maps from X to Y with the compact-open topology, but this space will not always satisfy the above universal property. In a 1967 paper [57], Norman Steenrod set forth compactly generated Hausdorff spaces as a convenient category of topological spaces in which to work. In particular, compactly generated spaces are Cartesian closed. Though technical in nature, history showed this paper to be of great importance; it is now standard practice to work within the framework of compactly generated spaces.

Unfortunately, topological stacks are not as nicely behaved as topological spaces, even when considering only those associated to compactly generated Hausdorff topological groupoids. The bicategory of topological stacks is deficient in two ways as it appears to be neither complete, nor Cartesian closed; that is, mapping stacks need not exist. Analogously to the definition of mapping spaces, if $\mathscr{X}$ and $\mathscr{Y}$ are two topological stacks, a mapping stack $\mathbb{M}\mathrm{ap}\,(\mathscr{X}, \mathscr{Y})$ (if it exists), would be a topological stack such that there is a natural equivalence of groupoids

$$\mathrm{Hom}\,(\mathscr{Z}, \mathbb{M}\mathrm{ap}\,(\mathscr{X}, \mathscr{Y})) \simeq \mathrm{Hom}\,(\mathscr{Z} \times \mathscr{X}, \mathscr{Y})\,,$$

for every topological stack $\mathscr{Z}$. The mapping stack $\mathsf{Map}(\mathscr{X}, \mathscr{Y})$ always exists as an abstract stack, but it may not be a *topological* stack, so we may not be able to apply all the tools of topology to it. This problem can be fixed however, as there exists a nicer bicategory of topological stacks, which I call “compactly generated stacks”, which is Cartesian closed and complete as a bicategory. This bicategory provides the topologist with a convenient bicategory of topological stacks in which to work. The aim of this chapter is to introduce this theory.

II.1 Introduction

The aim of this chapter is to introduce the 2-category of compactly generated stacks. Compactly generated stacks are “essentially the same” as topological stacks, however, their associated 2-category is Cartesian closed and complete, whereas the 2-category of topological stacks appears to enjoy neither of these properties. In this chapter, we show that these categorical shortcomings can be overcame by refining the open cover Grothendieck topology to take into account compact generation.

The study of the mapping stack between topological stacks has been done in many different settings. The special case of differentiable maps between orbifolds has been studied in [17], and is restricted to the case where the domain orbifold is compact. André Haefliger has studied the case of smooth maps between étale Lie groupoids (which correspond to differentiable stacks with an étale atlas) in [26]. In [34], Ernesto Lupercio and Bernardo Uribe showed that the free loop stack (the stack of maps from S^1 to the stack) of an arbitrary topological stack is again a topological stack. In [52], Behrang Noohi addressed the general case of maps between topological stacks. He showed that under a certain compactness condition on the domain stack, the stack of maps between two topological stacks is a topological stack, and if this compactness condition is replaced with a local compactness condition, the mapping stack is “not very far” from being topological.

In order to obtain a Cartesian closed 2-category of topological stacks, we first restrict to stacks over a Cartesian closed subcategory of the category $\mathbb{TOP}$ of all topological spaces. For instance, all of the results of [52] about mapping stacks are about stacks over the category of compactly generated spaces with respect to the open cover Grothendieck topology. We choose to work over the category of compactly generated Hausdorff spaces since, in addition to being Cartesian closed, every compact Hausdorff space is locally compact Hausdorff, which is crucial in defining the compactly generated Grothendieck topology.

There are several equivalent ways of describing compactly generated stacks. The description that substantiates most clearly the name “compactly generated” is the description in terms of topological groupoids and principal bundles. Recall that the 2-category of topological stacks is equivalent to the bicategory of topological groupoids and principal bundles. Classically, if X is a topological space and $\mathcal{G}$ is a topological groupoid, the map π of a (left) principal $\mathcal{G}$-bundle P over X

$$\begin{tikzcd} \mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] \arrow[r, phantom, "\circlearrowleft"] & P \arrow[dl, "\mu"'] \arrow[d, "\pi"] \\ \mathcal{G}_0 & X \end{tikzcd}$$

must admit local sections. If instead π only admits local sections *over each compact subset of* X, then one arrives at the definition of a compactly generated principal bundle. With this notion of compactly generated principal bundles, one can define a bicategory of topological groupoids in an obvious way. This bicategory is equivalent to compactly generated stacks.

There is another simple way of defining compactly generated stacks. Given any stack $\mathscr{X}$ over the category of compactly generated Hausdorff spaces, it can be restricted to the category of compact Hausdorff spaces $\mathbb{CH}$. This produces a 2-functor

$$j^* : \mathfrak{TSt} \to St\left(\mathbb{CH}\right)$$

from the 2-category of topological stacks to the 2-category of stacks over compact Hausdorff spaces. Compactly generated stacks are (equivalent to) the essential image of this 2-functor.

Finally, the simplest description of compactly generated stacks is that compactly generated stacks are classical topological stacks (over compactly generated Hausdorff spaces) which admit a locally compact atlas. In this description, the mapping stack of two spaces is usually not a space, but a stack!

For technical reasons, neither of the three previous concepts of compactly generated stacks are put forth as the definition. Instead, a Grothendieck topology $\mathscr{CG}$ is introduced on the category $\mathbb{CGH}$ of compactly generated Hausdorff spaces which takes into account the compact generation of this category. It is in fact the Grothendieck topology induced by geometrically embedding the topos $Sh\left(\mathbb{CH}\right)$ of sheaves over $\mathbb{CH}$ into the topos of presheaves $\mathrm{Set}^{\mathbb{CGH}^{op}}$. Compactly generated stacks are defined to be presentable stacks (see Definition I.2.9) with respect to this Grothendieck topology. The equivalence of all four notions of compactly generated stacks is shown in Section II.4.1.

II.1.1 Why are Compactly Generated Hausdorff Spaces Cartesian Closed?

In order to obtain a Cartesian closed 2-category of topological stacks, we start with a Cartesian closed category of topological spaces. We choose to work with the aforementioned category $\mathbb{CGH}$ of compactly generated Hausdorff spaces (also known as Kelley spaces).

Definition II.1.1. A topological space X is **compactly generated** if it has the final topology with respect to all maps into X with compact Hausdorff domain. When X is Hausdorff, this is equivalent to saying that a subset A of X is open if and only if its intersection which every compact subset of X is open.

The inclusion $\mathbb{CGH} \hookrightarrow \mathbb{HAUS}$ of the category of compactly generated Hausdorff spaces into the category of Hausdorff spaces admits a right adjoint, called the Kelley functor, which replaces the topology of a space X with the final topology with respect to all maps into X with compact Hausdorff domain. Limits in $\mathbb{CGH}$ are computed by first computing the limit in $\mathbb{HAUS}$, and then applying the Kelley functor. (In this way the compactly generated product topology differs from the ordinary product topology).) In particular, $\mathbb{CGH}$ is a complete category.

Although the fact that this category is Cartesian closed is a classical result (see: [57]), we will recall briefly the key reasons why this is true in order to gain insight into how one could construct a Cartesian closed theory of topological stacks.

1. In $\mathbb{TOP}$, if K is compact Hausdorff, then for any space X, the space of maps endowed with the compact-open topology serves as an exponential object $\mathsf{Map}\,(K, X)$.

2. A Hausdorff space Y is compactly generated if and only if it is the colimit of all its compact subsets:

$$Y = \varinjlim_{K_\alpha \hookrightarrow Y} K_\alpha.$$

i) For a fixed Y, for all X, $\mathsf{Map}\,(K_\alpha, X)$ exists for each compact subset K_α of Y.

ii) $\mathbb{CGH}$ has all limits

So by general properties of limits and colimits, the space

$$\mathsf{Map}\,(Y, X) := \varprojlim_{K_\alpha \hookrightarrow Y} \mathsf{Map}\,(K_\alpha, X)$$

is a well defined exponential object (with the correct universal property).

The story starts the same for topological stacks:

Let Y be as above and let $\mathscr{X}$ be a topological stack. Then $\mathsf{Map}\,(K_\alpha, \mathscr{X})$ is a topological stack for each compact subset $K_\alpha \subset Y$ (see [52]).

One might therefore be tempted to claim:

$$\mathbb{M}\mathsf{ap}\,(Y, \mathscr{X}) := \underset{\overleftarrow{K_\alpha \hookrightarrow Y}}{\mathrm{holim}}\, \mathbb{M}\mathsf{ap}\,(K_\alpha, \mathscr{X})\,,$$

but there are some problems with this. First of all, this weak 2-limit may not exist as a topological stack, since topological stacks are only known to have finite weak limits. There is also a more technical problem related to the fact that the Yoneda embedding does not preserve colimits (see Section II.3 for details). The main task of this chapter is to show that both of these difficulties can be surmounted by using a more suitable choice of Grothendieck topology on the category of compactly generated Hausdorff spaces. The resulting 2-category of presentable stacks with respect to this topology will be the 2-category of compactly generated stacks and turn out to be both Cartesian closed and complete.

II.1.2 Organization and Main Results

Section II.2 is a review of some recent developments in topological groupoids and topological stacks, including some results of David Gepner and André Henriques in [24] which are crucial for the proof of the completeness and Cartesian closedness of compactly generated stacks. In this section, we also extend Behrang Noohi's results to show that the mapping stack of two topological stacks is topological if the domain stack admits a compact atlas and "nearly topological" if the domain stack admits a locally compact atlas.

Section II.3 details the construction of the compactly generated Grothendieck topology $\mathscr{CG}$ on the category of compactly generated Hausdorff spaces $\mathbb{CGH}$. This is the Grothendieck topology whose associated presentable stacks are precisely compactly generated stacks. Many properties of the associated categories of sheaves and stacks are derived.

Section II.4 is dedicated to compactly generated stacks. In Section II.4.1, it is shown that compactly generated stacks are equivalent to two bicategories of topological groupoids. Also, it is shown that these are in turn equivalent to the restriction of topological stacks to compact Hausdorff spaces. Finally, it is shown that compactly generated stacks are equivalent to ordinary topological stacks (over compactly generated Hausdorff spaces) which admit a locally compact atlas.

Section II.4.1 also contains one of the main results of the chapter:

Theorem II.1.1. *The 2-category of compactly generated stacks is closed under arbitrary small weak limits.*

(See Corollary II.4.3).

Section II.4.2 is dedicated to the proof of *the* main result of the chapter:

Theorem II.1.2. *If $\mathscr{X}$ and $\mathscr{Y}$ are arbitrary compactly generated stacks, then $\mathsf{Map}\left(\mathscr{Y},\mathscr{X}\right)$ is a compactly generated stack.*

(See Theorem II.4.8).

This of course proves that classical topological stacks (over compactly generated Hausdorff spaces) which admit a locally compact atlas form a Cartesian closed and complete 2-category. We also give a concrete description of a topological groupoid presentation for the mapping stack of two compactly generated stacks.

Section II.4.3 uses techniques developed in [51] to assign to each compactly generated stack a weak homotopy type.

Finally, in Section II.4.4, there is a series of results showing how compactly generated stacks are "essentially the same" as topological stacks. In particular we extend the construction of a weak homotopy type to a wider class of stacks which include all topological stacks and all compactly generated stacks, so that their corresponding homotopy types can be compared.

For instance, the following theorems are proven:

Theorem II.1.3. *For every topological stack $\mathscr{X}$, there is a canonical compactly generated stack $\bar{\mathscr{X}}$ and a map*

$$\mathscr{X} \to \bar{\mathscr{X}}$$

which induces an equivalence of groupoids

$$\mathscr{X}\left(Y\right) \to \bar{\mathscr{X}}\left(Y\right)$$

for all locally compact Hausdorff spaces Y. Moreover, this map is a weak homotopy equivalence.

(See Corollary II.4.10).

Theorem II.1.4. *Let $\mathscr{X}$ and $\mathscr{Y}$ be topological stacks such that $\mathscr{Y}$ admits a locally compact atlas. Then $\mathsf{Map}\left(\mathscr{Y},\mathscr{X}\right)$ is "nearly topological", $\mathsf{Map}\left(\bar{\mathscr{Y}},\bar{\mathscr{X}}\right)$ is a compactly generated stack, and there is a canonical weak homotopy equivalence*

$$\mathsf{Map}\left(\mathscr{Y},\mathscr{X}\right) \to \mathsf{Map}\left(\bar{\mathscr{Y}},\bar{\mathscr{X}}\right).$$

Moreover, $\mathsf{Map}\left(\mathscr{Y},\mathscr{X}\right)$ and $\mathsf{Map}\left(\bar{\mathscr{Y}},\bar{\mathscr{X}}\right)$ restrict to the same stack over locally compact Hausdorff spaces.

(See Theorem II.4.19).

We end this chapter by showing in what way compactly generated stacks are to topological stacks what compactly generated spaces are to topological spaces:

Recall that there is an adjunction

$$\mathbb{CG} \underset{v}{\overset{k}{\leftrightarrows}} \mathbb{TOP},$$

exhibiting compactly generated spaces as a co-reflective subcategory of the category of topological spaces, and for any space X, the co-reflector

$$vk\left(X\right) \to X$$

is a weak homotopy equivalence.

We prove the 2-categorical analogue of this statement:

Theorem II.1.5. *There is a 2-adjunction*

$$\mathscr{C}\mathscr{G}\mathfrak{TSt} \underset{v}{\overset{k}{\leftrightarrows}} \mathfrak{TSt},$$

$$v \dashv k,$$

exhibiting compactly generated stacks as a co-reflective sub-2-category of topological stacks, and for any topological stack $\mathscr{X}$, the co-reflector

$$vk\left(\mathscr{X}\right) \to \mathscr{X}$$

is a weak homotopy equivalence. A topological stack is in the essential image of the 2-functor

$$v : \mathscr{C}\mathscr{G}\mathfrak{TSt} \to \mathfrak{TSt}$$

if and only if it admits a locally compact atlas.

(See Theorem II.4.20.)

II.2 Topological Stacks

A review of the basics of topological groupoids and topological stacks including many notational conventions used in this section can be found in Section I.2.

Remark. In this chapter, we will denote the 2-category of topological stacks by $\mathfrak{TSt}$.

II.2.1 Fibrant Topological Groupoids

The notion of fibrant topological groupoids was introduced in [24]. Roughly speaking, fibrant topological groupoids are topological groupoids which "in the eyes of paracompact spaces are stacks." The fact that every topological groupoid is Morita equivalent to a fibrant one is essential to the existence of arbitrary weak limits of compactly generated stacks. Since this concept is relatively new, in this subsection, we summarize the basic facts about fibrant topological groupoids. All details may be found in [24].

Definition II.2.1. The **classifying space** of a topological groupoid $\mathcal{G}$ is the fat geometric realization of its simplicial nerve (regarded as a simplicial space) and is denoted by $\|\mathcal{G}\|$.

For any topological groupoid $\mathcal{G}$, the classifying space of its translation groupoid $\|\mathbb{E}\mathcal{G}\|$ (see definition I.2.12) admits the structure of a principal $\mathcal{G}$-bundle over the classifying space $\|\mathcal{G}\|$.

Definition II.2.2. [24] Let $\mathcal{G}$ be a topological groupoid. A principal $\mathcal{G}$-bundle E over a space B is **universal** if every principal $\mathcal{G}$-bundle P over a paracompact base X admits a $\mathcal{G}$-bundle map

$$\begin{tikzcd} P \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] & B, \end{tikzcd}$$

unique up to homotopy.

Lemma II.2.1. *[24] The principal $\mathcal{G}$-bundle*

$$\begin{tikzcd} \mathcal{G}_1 \arrow[d, shift left] \arrow[d, shift right] & \circlearrowright & \|\mathbb{E}\mathcal{G}\| \arrow[dll, "\mu"'] \arrow[d, "\pi"] \\ \mathcal{G}_0 & & \|\mathcal{G}\| \end{tikzcd}$$

is universal.

Definition II.2.3. [24] A topological groupoid $\mathcal{G}$ is **fibrant** if the unit principal $\mathcal{G}$-bundle is universal. (See Definition I.2.14.)

Definition II.2.4. [24] The **fibrant replacement** of a topological groupoid $\mathcal{G}$ is the gauge groupoid of the universal principal $\mathcal{G}$-bundle $\|\mathbb{E}\mathcal{G}\|$, denoted by $Fib\left(\mathcal{G}\right)$. (See Definition I.2.17.)

Remark. If $\mathcal{G}$ is compactly generated Hausdorff, then so is $Fib\left(\mathcal{G}\right)$.

Lemma II.2.2. *[24] $Fib\left(\mathcal{G}\right)$ is fibrant.*

Lemma II.2.3. *[24] There is a canonical groupoid homomorphism*

$$\xi_{\mathcal{G}} : \mathcal{G} \to Fib\left(\mathcal{G}\right)$$

which is a Morita equivalence for all topological groupoids $\mathcal{G}$.

The following theorem will be of importance later:

Theorem II.2.4. *[24] Let $\mathcal{G}$ be a fibrant topological groupoid. Then for any topological groupoid $\mathcal{H}$ with paracompact Hausdorff object space $\mathcal{H}_0$, there is an equivalence of groupoids*

$$\mathrm{Hom}_{\mathfrak{TSt}}\left(\left[\mathcal{H}\right],\left[\mathcal{G}\right]\right) \simeq \mathrm{Hom}_{\mathbb{TOP}Gpd}\left(\mathcal{H},\mathcal{G}\right)$$

natural in $\mathcal{H}$. In particular, the restriction of $\tilde{y}\left(\mathcal{G}\right)$ to the full-subcategory of paracompact Hausdorff spaces agrees with the restriction of $\left[\mathcal{G}\right]$. (See Section I.2.4 for the notation.)

II.2.2 Paratopological Stacks and Mapping Stacks

Stacks on $\mathbb{TOP}$ (with respect to the open cover topology) come in many different flavors. Of particular importance of course are topological stacks, which are those stacks coming from topological groupoids. However, this class of stacks seems to be too restrictive since many natural stacks, for instance the stack of maps between two topological stacks, appear to not be topological.

A topological stack is a stack $\mathscr{X}$ which admits a representable epimorphism $X \to \mathscr{X}$ from a topological space X. This implies:

i) Any map $T \to \mathscr{X}$ from a topological space is representable (equivalently, the diagonal $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is representable) [49].

ii) If $T \to \mathscr{X}$ is a continuous map, then the induced map $T \times_{\mathscr{X}} X \to T$ admits local sections (i.e. is an epimorphism in $\mathbb{TOP}$).

If the second condition is slightly weakened, the result is a stack which is "nearly topological".

Definition II.2.5. [51] A **paratopological** stack is a stack $\mathscr{X}$ on $\mathbb{TOP}$ (with respect to the open cover topology), satisfying condition i) above, and satisfying condition ii) for all maps $T \to \mathscr{X}$ from a paracompact space.

Paratopological stacks are very nearly topological stacks:

Proposition II.2.1. *[51] A stack $\mathscr{X}$ with representable diagonal is paratopological if and only if there exists a topological stack $\bar{\mathscr{X}}$ and a morphism $q : \bar{\mathscr{X}} \to \mathscr{X}$ such that for any paracompact space T, q induces an equivalence of groupoids*

$$q(T) : \bar{\mathscr{X}}(T) \to \mathscr{X}(T). \tag{II.1}$$

The idea of the proof can be found in [51], but is enlightening, so we include it for completeness:

If q is as in (II.1), and $p : X \to \bar{\mathscr{X}}$ is an atlas for $\bar{\mathscr{X}}$, then

$$q \circ p : X \to \mathscr{X}$$

satisfies condition ii) of Definition II.2.5, hence $\mathscr{X}$ is paratopological. Conversely, if $\mathscr{X}$ is paratopological, take

$$p : X \to \mathscr{X}$$

as in condition ii) of Definition II.2.5. Form the weak fibered product

$$\begin{tikzcd} X \times_{\mathscr{X}} X \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ X \arrow[r, "p"] & \mathscr{X}. \end{tikzcd}$$

Let $\bar{\mathscr{X}}$ be the topological stack associated with the topological groupoid

$$X \times_{\mathscr{X}} X \rightrightarrows X,$$

and $q : \bar{\mathscr{X}} \to \mathscr{X}$ the canonical map.

Definition II.2.6. Any two stacks $\mathscr{X}$ and $\mathscr{Y}$ on $\mathbb{TOP}$ have an exponential stack $\mathscr{X}^{\mathscr{Y}}$ such that

$$\mathscr{X}^{\mathscr{Y}}(T) = \mathrm{Hom}\,(\mathscr{Y} \times T, \mathscr{X}).$$

We will from here on in denote $\mathscr{X}^{\mathscr{Y}}$ by $\mathsf{Map}\,(\mathscr{Y}, \mathscr{X})$ and refer to it as the **mapping stack** from $\mathscr{Y}$ to $\mathscr{X}$.

For the rest of this section, we work in the category $\mathbb{CGH}$ of compactly generated Hausdorff spaces, which is Cartesian closed.

In [52] Noohi proved:

Theorem II.2.5. *If $\mathscr{X}$ and $\mathscr{Y}$ are topological stacks with $\mathscr{Y} \simeq [\mathcal{H}]$ with $\mathcal{H}_0$ and $\mathcal{H}_1$ compact, then $\mathsf{Map}\,(\mathscr{Y}, \mathscr{X})$ is a topological stack.*

It appears that when $\mathscr{Y}$ does not satisfy this rather rigid compactness condition, that this may fail (however, we will shortly release the condition for the arrow space). Noohi also proved that when $\mathscr{Y}$ is instead locally compact, then $\mathsf{Map}(\mathscr{Y}, \mathscr{X})$ is at least paratopological:

Theorem II.2.6. *[52] If $\mathscr{X}$ and $\mathscr{Y}$ are paratopological stacks with $\mathscr{Y} \simeq [\mathcal{H}]$ such that $\mathcal{H}_0$ and $\mathcal{H}_1$ are locally compact, then $\mathsf{Map}(\mathscr{Y}, \mathscr{X})$ is a paratopological stack.*

We end this section by extending Noohi's results to work without imposing conditions on the arrow space. Firstly, Noohi's proof of Theorem II.2.5 easily extends to the case when $\mathcal{H}_1$ need not be compact:

First, we will need a lemma from [52]:

Lemma II.2.7. *Let $\mathscr{X}$ and $\mathscr{Y}$ be topological stacks. Then the diagonal of the stack $\mathsf{Map}(\mathscr{Y}, \mathscr{X})$ is representable, i.e., every morphism $T \to \mathsf{Map}(\mathscr{Y}, \mathscr{X})$, with T a topological space, is representable.*

Theorem II.2.8. *If $\mathscr{X}$ and $\mathscr{Y}$ are topological stacks and $\mathscr{Y}$ admits a compact atlas, then $\mathsf{Map}(\mathscr{Y}, \mathscr{X})$ is a topological stack.*

Proof. Fix a compact Hausdorff atlas $K \to \mathscr{Y}$ for $\mathscr{Y}$. Let $\mathcal{K}$ denote the corresponding topological groupoid

$$K \times_{\mathscr{Y}} K \rightrightarrows K.$$

Let $\mathcal{U}$ denote the set of finite open covers of K. For each $U \in \mathcal{U}$, consider the Čech groupoid $\mathcal{K}_U$ and $\mathcal{G}^{\mathcal{K}_U}$, the internal exponent of groupoid objects in compactly generated Hausdorff spaces.

Set $R_U := \left(G^{\mathcal{K}_U}\right)_0$. By adjunction, the canonical map

$$R_U \to \mathcal{G}^{\mathcal{K}_U}$$

induced by the unit, produces a homomorphism

$$R_U \times \mathcal{K}_U \to \mathcal{G}.$$

Suppose U is given by $U = (N_i)$, and let V be the open cover of $R_U \times K$, given by

$$(R_U \times N_i).$$

Then this homomorphism is a map

$$(R_U \times \mathcal{K})_V \to \mathcal{G},$$

and in particular, a generalized homomorphism from $R_U \times \mathcal{K}$ to $\mathcal{G}$ (See Definition I.2.21). This corresponds to a map of stacks

$$R_U \times \mathscr{Y} \to \mathscr{X},$$

which by adjunction is a morphism

$$p_U : R_U \to \mathbb{M}\mathrm{ap}\left(\mathscr{Y}, \mathscr{X}\right).$$

Let

$$R := \coprod_{U \in \mathcal{U}} R_U.$$

Then we can conglomerate these morphisms to a morphism

$$p : R \to \mathbb{M}\mathrm{ap}\left(\mathscr{Y}, \mathscr{X}\right).$$

We will show that p is an epimorphism. By Lemma II.2.7, it is representable.

Suppose that $f : T \to \mathbb{M}\mathrm{ap}\left(\mathscr{Y}, \mathscr{X}\right)$ is any morphism from a space T. We will show that f locally factors through p up to isomorphism. By adjunction, f corresponds to a map

$$\bar{f} : T \times \mathscr{Y} \to \mathscr{X},$$

which corresponds to a generalized homomorphism

$$\tilde{f} : (T \times \mathcal{K})_W \to \mathcal{G}$$

for some open cover W of $T \times K$. Let $t \in T$ be an arbitrary point. Since K is compact Hausdorff, $T \times K$, with the classical product topology, is already compactly generated, so we can assume, without loss of generality, that each element W_j of the cover W is of the form

$$V_j \times U_j$$

for open subsets V_j and U_j of T and K respectively. Let $t \in T$ be an arbitrary point. Then as W covers the slice $\{t\} \times K$, which is compact, there exists a finite collection $\left(U_j \times V_j\right)_{j \in A_t}$ which covers it. Let

$$\mathcal{O}_t := \bigcap_{j \in A_t} U_j.$$

Then $\mathcal{O}_t$ is a neighborhood of t in T such that for all $j \in A_t$,

$$\mathcal{O}_t \times V_j \subset W_j.$$

Let $U_t = \left(V_j\right)_{j \in A_t}$ be the corresponding finite cover of K, and

$$N_t = \left(\mathcal{O}_t \times V_j\right)_{j \in A_t}$$

the cover of $\mathcal{O}_t \times K$. Denote the composite

$$\mathcal{O}_t \times \mathcal{K}_{U_t} \cong \left(\mathcal{O}_t \times \mathcal{K}\right)_{N_t} \to \left(T \times \mathcal{K}\right)_W \xrightarrow{\tilde{f}} \mathcal{G}$$

by $\tilde{f}_t$. By adjunction, this corresponds to a homomorphism

$$\mathcal{O}_t \to G^{\mathcal{K}_{U_t}},$$

so the induced map of stacks

$$\mathcal{O}_t \to \mathsf{Map}\left(\mathscr{Y}, \mathscr{X}\right)$$

factors through p. This induced map of stacks is the same as the map adjoint to the one induced by $\tilde{f}_t$,

$$\mathcal{O}_t \times \mathscr{Y} \to T \times \mathscr{Y} \xrightarrow{\bar{f}} \mathscr{X},$$

so we are done. □

Now, we will recall some basic notions from topology:

Definition II.2.7. A **shrinking** of an open cover $(U_\alpha)_{\alpha \in A}$ is another open cover $(V_\alpha)_{\alpha \in A}$ *indexed by the same set* such that for each α, the closure of V_α is contained in U_α. A topological space X is a **shrinking space** if and only if every open cover of X admits a shrinking.

The following proposition is standard:

Proposition II.2.2. *Every paracompact Hausdorff space is a shrinking space.*

Theorem II.2.9. *If $\mathscr{X}$ and $\mathscr{Y}$ are topological stacks such that $\mathscr{Y}$ admits a locally compact atlas, then $\mathsf{Map}\left(\mathscr{Y}, \mathscr{X}\right)$ is paratopological.*

Proof. It is proven in [52] that if $\mathscr{X}$ and $\mathscr{Y}$ are topological stacks, then $\mathsf{Map}\left(\mathscr{Y}, \mathscr{X}\right)$ has a representable diagonal. Therefore, by Proposition II.2.1, it suffices to prove that there exists a topological stack $\mathscr{Z}$ and a map

$$q : \mathscr{Z} \to \mathsf{Map}\left(\mathscr{Y}, \mathscr{X}\right)$$

which induces an equivalence of groupoids

$$q\left(T\right) : \mathscr{Z}\left(T\right) \to \mathsf{Map}\left(\mathscr{Y}, \mathscr{X}\right)\left(T\right)$$

along every paracompact space T.

Let $\mathcal{G}$ be a topological groupoid presenting $\mathscr{X}$. Let $Y \to \mathscr{Y}$ be a locally compact atlas for $\mathscr{Y}$. Then we may find a covering of Y by compact neighborhoods, (Y_α). It follows that

$$\coprod_\alpha Y_\alpha \to Y \to \mathscr{Y}$$

is an atlas. Hence we can choose a topological groupoid $\mathcal{H}$ presenting $\mathscr{Y}$ such that $\mathcal{H}_0$ is a disjoint union of compact Hausdorff spaces. Let $Fib\left(\mathcal{G}\right)^{\mathcal{H}}$ denote the internal exponent of groupoid objects in compactly generated Hausdorff spaces. Let

$$\mathcal{K} := \left[Fib\left(\mathcal{G}\right)^{\mathcal{H}}\right].$$

Note that there is a canonical map

$$\varphi : \mathcal{K} \to \mathbb{M}\mathrm{ap}\left(\mathscr{Y}, \mathscr{X}\right)$$

which sends any generalized homomorphism $T_{\mathcal{U}} \to Fib\left(\mathcal{G}\right)^{\mathcal{H}}$ to the induced generalized homomorphism from $T \times \mathcal{H}$, $T_{\mathcal{U}} \times \mathcal{H} \to Fib\left(\mathcal{G}\right)$ (which may be viewed as object in $\mathrm{Hom}_{\mathfrak{TSt}}\left(T \times \mathscr{Y}, \mathscr{X}\right)$ since $\mathcal{G}$ and $Fib\left(\mathcal{G}\right)$ are Morita equivalent).

Suppose that T is a paracompact Hausdorff space. Then:

$$\begin{aligned} \mathcal{K}\left(T\right) &\simeq \underset{\underset{\mathcal{U}}{\longrightarrow}}{\mathrm{holim}}\ \mathrm{Hom}_{\mathbb{CGH}Gpd}\left(T_{\mathcal{U}}, Fib\left(\mathcal{G}\right)^{\mathcal{H}}\right) \\ &\simeq \underset{\underset{\mathcal{U}}{\longrightarrow}}{\mathrm{holim}}\ \mathrm{Hom}_{\mathbb{CGH}Gpd}\left(T_{\mathcal{U}} \times \mathcal{H}, Fib\left(\mathcal{G}\right)\right). \end{aligned}$$

Note that any paracompact Hausdorff space is a shrinking space so without loss of generality we may assume that each cover $\mathcal{U}$ of T is a topological covering by closed neighborhoods. Since any closed subset of a paracompact space is paracompact, this means that the groupoid $T_{\mathcal{U}}$ has paracompact object space. Moreover, the object space of $T_{\mathcal{U}} \times \mathcal{H}$ is the product of the compactly generated space $T_{\mathcal{U}}$ and the locally compact Hausdorff space $\mathcal{H}_0$, hence compactly generated. Since $T_{\mathcal{U}}$ has paracompact object space, and $\mathcal{H}_0$ is a disjoint union of compact Hausdorff spaces, the product is in fact paracompact by [46]. Finally, by Theorem II.2.4, we have that

$$\mathrm{Hom}_{\mathbb{CG}Gpd}\left(T_{\mathcal{U}} \times \mathcal{H}, Fib\left(\mathcal{G}\right)\right) \simeq \mathrm{Hom}_{\mathfrak{TSt}}\left(T \times \mathscr{Y}, \mathscr{X}\right).$$

Hence $\mathcal{K}\left(T\right) \simeq \mathbb{M}\mathrm{ap}\left(\mathscr{Y}, \mathscr{X}\right)\left(T\right)$. □

II.3 The Compactly Generated Grothendieck Topology

Recall from Section II.1.1 that if Y is a compactly generated Hausdorff space and $\mathscr{X}$ a topological stack, then one might be tempted to claim that

$$\mathbb{M}\mathrm{ap}\left(Y, \mathscr{X}\right) := \underset{\underset{K_\alpha \hookrightarrow Y}{\longleftarrow}}{\mathrm{holim}}\ \mathbb{M}\mathrm{ap}\left(K_\alpha, \mathscr{X}\right),$$

where the weak limit is taken over all compact subsets K_α of Y. However, there are some immediate problems with this temptation:

- This weak 2-limit may not exist as a topological stack.

- The fact that Y is the colimit of its compact subsets in $\mathbb{CGH}$ does not imply that Y is the weak colimit of its compact subsets as a topological stack since the Yoneda embedding does not preserve arbitrary colimits.

Recall however that for an arbitrary subcanonical Grothendieck site $(\mathscr{C}, J)$, the Yoneda embedding $y : \mathscr{C} \hookrightarrow Sh_J(\mathscr{C})$ preserves colimits of the form

$$C = \varinjlim_{C_\alpha \to C} C_\alpha$$

where $\left(C_\alpha \xrightarrow{f_\alpha} C\right)$ is a J-cover. We therefore shall construct a Grothendieck topology $\mathscr{CG}$ on $\mathbb{CGH}$, called the compactly generated Grothendieck topology, such that for all Y, the inclusion of all compact subsets $(K_\alpha \hookrightarrow Y)$ is a $\mathscr{CG}$-cover. As it shall turn out, in addition to being Cartesian closed, the 2-category of presentable stacks for this Grothendieck topology will also be complete.

In this subsection, we give a geometric construction of the compactly generated Grothendieck topology on $\mathbb{CGH}$. Those readers not familiar with topos theory may wish to skip to Definition II.3.1 for the concrete definition of a $\mathscr{CG}$-cover. Some important properties of $\mathscr{CG}$-covers are summarized as follows:

i) Every open cover is a $\mathscr{CG}$-cover (Proposition II.3.2)

ii) For any space, the inclusion of all its compact subsets is a $\mathscr{CG}$-cover (Corollary II.3.1)

iii) Every $\mathscr{CG}$-cover of a locally compact space can be refined by an open one (Proposition II.3.4)

iv) The category of $\mathscr{CG}$-sheaves over compactly generated Hausdorff spaces is equivalent to the category of ordinary sheaves over compact Hausdorff spaces (Theorem II.3.2).

II.3.1 The Compactly Generated Grothendieck Topology

Let

$$j : \mathbb{CH} \hookrightarrow \mathbb{CGH}$$

be the full and faithful inclusion of compact Hausdorff spaces into compactly generated Hausdorff spaces. This induces a geometric morphism (j^*, j_*)

$$\mathrm{Set}^{\mathbb{CH}^{op}} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathrm{Set}^{\mathbb{CGH}^{op}}$$

which is an embedding (i.e. j_* is full and faithful [36])[1].

Denote by

$$y_{\mathbb{CH}} : \mathbb{CGH} \to \mathrm{Set}^{\mathbb{CH}^{op}}$$

the functor which assigns a compactly generated Hausdorff space X the presheaf $T \mapsto \mathrm{Hom}_{\mathbb{CGH}}(T, X)$ and by

$$y_{\mathbb{CGH}} : \mathbb{CH} \to \mathrm{Set}^{\mathbb{CGH}^{op}}$$

the functor which assigns a $T \in \mathbb{CH}$ the presheaf $X \mapsto \mathrm{Hom}_{\mathbb{CGH}}(X, T)$. Note that $y_{\mathbb{CH}}$ is a fully faithful embedding. The pair (j^*, j_*) can be constructed as the adjoint pair induced by left Kan-extending $y_{\mathbb{CH}}$ along the Yoneda embedding. (See Definition I.1.14.) Explicitly:

$$j^*(F)(T) = F(T)$$

and

$$j_*(G)(X) = \mathrm{Hom}_{\mathrm{Set}^{\mathbb{CH}^{op}}}(y_{\mathbb{CH}}(X), G).$$

From the general theory of adjoint functors, j_* restricts to an equivalence between, on one hand the full subcategory of $\mathrm{Set}^{\mathbb{CH}^{op}}$ whose objects are those for which the co-unit $\varepsilon(j)$ is an isomorphism, and on the other hand, the full subcategory of $\mathrm{Set}^{\mathbb{CGH}^{op}}$ whose objects are those for which the unit $\eta(j)$ is an isomorphism. However, since j is fully faithful, the co-unit is always an isomorphism, which verifies that j_* is fully faithful and gives us a way of describing its essential image.

In fact, j^* also has a left adjoint $j_!$. The adjoint pair $j_! \dashv j^*$ is the one induced by $y_{\mathbb{CGH}}$. Hence, $j_!$ is the left Kan extension of $y_{\mathbb{CGH}}$. We conclude that $j_!$ is also fully faithful (see: [36]).

Denote by $Sh(\mathbb{CGH})$ the topos of sheaves on compactly generated Hausdorff spaces with respect to the open cover topology. We define $Sh(\mathbb{CH})$ as the unique topos fitting in the following pullback diagram:

$$\begin{tikzcd} Sh(\mathbb{CH}) \arrow[r, tail] \arrow[d, tail] & \mathrm{Set}^{\mathbb{CH}^{op}} \arrow[d, tail] \\ Sh(\mathbb{CGH}) \arrow[r, tail] & \mathrm{Set}^{\mathbb{CGH}^{op}} \end{tikzcd}$$

Due to the factorization theorem of geometric morphisms in topos theory [36], the geometric embedding $Sh(\mathbb{CH}) \rightarrowtail \mathrm{Set}^{\mathbb{CH}^{op}}$ corresponds to a Grothendieck topology $\mathscr{K}$ on $\mathbb{CH}$. (See Section I.1.6.) It is easy to verify that since the functor j is fully faithful, the covering sieves in $\mathscr{K}$ for a compact Hausdorff

[1] Technically, these categories are not well defined due to set-theoretic issues, however, this can be overcame by careful use of Grothendieck universes. We will not dwell on this and all such similar size issues in this thesis.

space K are precisely those subobjects $S \rightarrowtail y(K)$ which are obtained by restricting a covering sieve of K with respect to the open cover topology on $\mathbb{CGH}$ to $\mathbb{CH}$ via the functor j^*. In this sense, the covering sieves are "the same as in the open cover topology".

Proposition II.3.1. *The Grothendieck topology $\mathscr{K}$ on $\mathbb{CH}$ has a basis of finite covers of the form $(T_i \hookrightarrow T)_{i=1}^n$ by compact neighborhoods (i.e. their interiors form an open cover).*

Proof. Compact Hausdorff spaces are locally compact in the strong sense that every point has a local base of compact neighborhoods. Hence covers by compact neighborhoods generate the same sieves as open covers. □

Consider the geometric embedding

$$Sh(\mathbb{CH}) \underset{i}{\overset{a}{\leftrightarrows}} \mathrm{Set}^{\mathbb{CH}^{op}},$$

where a denotes the sheafification with respect to $\mathscr{K}$.

Remark. It is clear that for any presheaf F in $\mathrm{Set}^{\mathbb{CGH}^{op}}$, the $\mathscr{K}$-sheafification of the restriction of F to $\mathbb{CH}$ is the same as the restriction of the sheafification of F.

By composition, we get an embedding of topoi

$$Sh(\mathbb{CH}) \underset{j_* \circ i}{\overset{a \circ j^*}{\leftrightarrows}} \mathrm{Set}^{\mathbb{CGH}^{op}}.$$

Again by the factorization theorem [36], there exists a unique Grothendieck topology $\mathscr{CG}$ on $\mathbb{CGH}$ such that the category of sheaves $Sh_{\mathscr{CG}}(\mathbb{CGH})$ is $j_*(Sh(\mathbb{CH}))$. We will construct it and give some of its properties.

There is a very general construction [36] that shows how to extract the unique Grothendieck topology corresponding to this embedding.

First, we define a universal closure operation on $\mathrm{Set}^{\mathbb{CGH}^{op}}$. (For details, see [36] V.1 on Lawvere-Tierney topologies.) Let F be a presheaf over $\mathbb{CGH}$ and let $m : A \rightarrowtail F$ be a representative for a subobject A of F. Then a representative for the subobject $\bar{A}$ is given by the left-hand side of the following pullback diagram

$$\begin{tikzcd}
\bar{A} \arrow[r] \arrow[d] & j_* a j^*(A) \arrow[d, "j_* a j^*(m)"] \\
F \arrow[r, "\eta_F"] & j_* a j^*(F),
\end{tikzcd}$$

where η is the unit of the adjunction $a \circ j^* \dashv j_* \circ i$.

To describe the covering sieves of $\mathscr{CG}$, it suffices to describe the universal closure operation on representables.

Let X be a compactly generated Hausdorff space.

Claim. The unit η_X is an isomorphism.

Proof. The restriction j^*X is a $\mathscr{K}$-sheaf. Hence

$$j_*aj^*X \cong j_*y_{\mathbb{CH}}(X).$$

Furthermore, for any compactly generated Hausdorff space Y,

$$j_*y_{\mathbb{CH}}(X)(Y) = \mathrm{Hom}_{\mathrm{Set}^{\mathbb{CH}^{op}}}\left(y_{\mathbb{CH}}(Y), y_{\mathbb{CH}}(X)\right) \cong \mathrm{Hom}_{\mathbb{CGH}}(Y, X)$$

since $y_{\mathbb{CH}}$ is fully faithful. $\square$

Now, let $m : A \rightarrowtail X$ a sieve. Then, since the unit η_X is an isomorphism, $\bar{A}$ is represented by the monomorphism

$$j_*aj^*(A) \rightarrowtail X.$$

The covering sieves in $\mathscr{CG}$ of X are exactly those sieves on X whose closure is equal to the maximal sieve, i.e. X. So

$$m : A \rightarrowtail X$$

is a covering sieve if and only if

$$\bar{m} : j_*aj^*(A) \to j_*j^*(X) \cong X$$

is an isomorphism. Since j_* is fully faithful, this is if and only if

$$\tilde{m} : aj^*(A) \to j^*X$$

is an isomorphism. In other words,

$$m : A \rightarrowtail X$$

is a covering sieve if and only if the $\mathscr{K}$-sheafification of $j^*(A)$ is isomorphic to $y_{\mathbb{CH}}(X)$

Definition II.3.1. Let X be a compactly generated Hausdorff space and let

$$(\alpha_i : V_i \hookrightarrow X)_{i \in I}$$

be family of inclusions of subsets V_i of X. Such a family is called a $\mathscr{CG}$-**cover** if for any compact subset K of X, there exists a (finite) subset $J(K) \subseteq I$ such that the collection $(V_j \cap K)_{j \in J(K)}$ can by refined by an open cover of K. Denote the set of $\mathscr{CG}$-covers of X by $\mathcal{B}(X)$.

Lemma II.3.1. *$\mathcal{B}$ is a basis for the Grothendieck topology $\mathscr{CG}$.*

Proof. Let $(f_i : Q_i \to X)$ be a class of maps into X. We denote the sieve it generates by S_f. For any compactly generated Hausdorff space Y, we have

$$S_f(Y) = \{h : Y \to X \text{ such that } h \text{ factors through } f_i \text{ for some } i\}.$$

So, S_f is a covering sieve if and only if when restricted to $\mathbb{CH}$, its $\mathscr{K}$-sheafification is isomorphic to $y_{\mathbb{CH}}(X)$. We first note that $j^*(S_f)$ is clearly a $\mathscr{K}$-separated presheaf. Hence, its sheafification is the same as $j^*(S_f)^+$. Since $y_{\mathbb{CH}}(X)$ is a sheaf, the canonical map

$$j^*(S_f) \rightarrowtail y_{\mathbb{CH}}(X)$$

factors uniquely as

$$\begin{tikzcd} j^*(S_f) \arrow[r, tail] \arrow[d] & y_{\mathbb{CH}}(X) . \\ j^*(S_f)^+ \arrow[ur, tail] & \end{tikzcd}$$

It suffices to see when the map $j^*(S_f)^+ \rightarrowtail y_{\mathbb{CH}}(X)$ is an epimorphism.

Let $\tilde{S}_f$ be the presheaf on $\mathbb{CH}$

$$\tilde{S}_f(T) = \left\{\mathcal{U} = (U_i)_{j=1}^n, (a_j \in S_f(U_j))_{j=1}^n \;\mid\; a_i|_{U_{ij}} = a_j|_{U_{ij}} \text{ for all i,j}\right\}.$$

Then the map $j^*(S_f)^+ \rightarrowtail y_{\mathbb{CH}}(X)$ fits in a diagram

$$\begin{tikzcd} \tilde{S}_f \arrow[r] \arrow[d, two heads] & y_{\mathbb{CH}}(X) . \\ j^*(S_f)^+ \arrow[ur, tail] & \end{tikzcd}$$

It suffices to see when the canonical map $\tilde{S}_f \to y_{\mathbb{CH}}(X)$ is point-wise surjective. This is precisely when for any map $h : K \to X$ from a compact space $K \in \mathbb{CH}$, there exists an open cover $(U_j)_j$ of K such that for all j, $h|_{U_j}$ factors through f_i for some i. Classes of maps with codomain X with this property constitute a large basis for the Grothendieck topology $\mathscr{CG}$. It is in fact maximal in the sense that S is a covering sieve if and only if it is one generated by a large cover of this form. We will now show that any such large covering family has a refinement by one of the form of the lemma.

Let $(f_i : Q_i \to X)$ denote such a (possibly large) family and let

$$(i_\alpha : K_\alpha \hookrightarrow X)$$

denote the inclusion of all compact subsets of X. Then for each α, there exists a finite open cover of K_α, $\left(\mathcal{O}_j^\alpha\right)$, such that the inclusion of each $\mathcal{O}_j^\alpha$ into X factors through some

$$f_{j,\alpha} : Q_{j,\alpha} \to X$$

via a map

$$\lambda_j^\alpha : \mathcal{O}_j^\alpha \to Q_{j,\alpha}.$$

Let $\mathcal{U} := \left(\mathcal{O}_j^\alpha \hookrightarrow X\right)_{j,\alpha}$. Let $g : L \to X$ be a map with $L \in \mathbb{CH}$. Then $g(L) = K_\alpha$ for some α. Let

$$\mathcal{V}_L^g := \left(g^{-1}\left(\mathcal{O}_j^\alpha\right)\right).$$

Then $\mathcal{V}_L^g$ is an open cover of L such that the restriction of g to any element of the cover factors through the inclusion of some $\mathcal{O}_j^\alpha$ into X. Hence the sieve generated by $\mathcal{U}$ is a covering sieve for $\mathscr{CG}$ which refines the sieve generated by $(f_i : Q_i \to X)$.

□

We have the following obvious proposition whose converse is not true:

Proposition II.3.2. *Any open cover of a space is also a $\mathscr{CG}$ cover.*

In particular, one cover that is quite useful is the following.

Corollary II.3.1. *For any compactly generated Hausdorff space, the inclusion of all compact subsets is a $\mathscr{CG}$-cover.*

However, for the category $\mathbb{LCH}$ of locally compact Hausdorff spaces, it suffices to work with open covers:

Proposition II.3.3. *Every $\mathscr{CG}$-cover of a locally compact Hausdorff space $X \in \mathbb{LCH}$ can be refined by an open covering.*

Proof. Let $X \in \mathbb{LCH}$ and let $\mathcal{V} = (\alpha_i : V_i \hookrightarrow X)_{i \in I}$ be a $\mathscr{CG}$-cover of X. Let (K_l) be a topological covering of X by compact subsets such that the interiors $int(K_l)$ constitute an open cover for X. Then for each K_l, there exists a finite subset $J(K_l) \subseteq I$ such that

$$(V_j \cap K_l)_{j \in J(K_l)}$$

can be refined by an open cover $(W_j)_{j \in J(K_l)}$ for K_l such that the inclusion of each W_j into X factors through the inclusion of V_j. Let $\mathcal{U} := (W_j)_{l, j \in J(K_l)}$. Then $\mathcal{U}$ is an open cover of X refining $\mathcal{V}$. □

We can now define the $\mathscr{CG}$-sheafification functor $a_{\mathscr{CG}}$ either by the covering sieves, or by using the basis $\mathcal{B}$ (i.e. both will give naturally isomorphic functors). Let $Sh_{\mathscr{CG}}(\mathbb{CGH})$ denote the category of $\mathscr{CG}$-sheaves. Then we have an embedding of topoi given by

$$Sh_{\mathscr{CG}}(\mathbb{CGH}) \underset{\ell}{\overset{a_{\mathscr{CG}}}{\leftrightarrows}} \mathrm{Set}^{\mathbb{CGH}^{op}},$$

where $\ell : Sh_{\mathscr{CG}}(\mathbb{CGH}) \hookrightarrow \mathrm{Set}^{\mathbb{CGH}^{op}}$ is the inclusion of the category of sheaves.

By the previous observation that open covers are $\mathscr{CG}$-covers, we also have

$$Sh_{\mathscr{CG}}(\mathbb{CGH}) \subset Sh(\mathbb{CGH}),$$

where $Sh(\mathbb{CGH})$ is the category of sheaves on $\mathbb{CGH}$ with respect to the open cover topology.

By construction, we have the following theorem:

Theorem II.3.2. *There is an equivalence of topoi*

$$Sh(\mathbb{CH}) \underset{a_{\mathscr{CG}} \circ j_* \circ i}{\overset{a \circ j^* \circ \ell}{\leftrightarrows}} Sh_{\mathscr{CG}}(\mathbb{CGH})$$

such that

$$Sh(\mathbb{CH}) \underset{a_{\mathscr{CG}} \circ j_* \circ i}{\overset{a \circ j^* \circ \ell}{\leftrightarrows}} Sh_{\mathscr{CG}}(\mathbb{CGH}) \underset{\ell}{\overset{Sh_{\mathscr{CG}}}{\leftrightarrows}} \mathrm{Set}^{\mathbb{CGH}^{op}}$$

is a factorization of

$$Sh(\mathbb{CH}) \underset{j_* \circ i}{\overset{a \circ j^*}{\leftrightarrows}} \mathrm{Set}^{\mathbb{CGH}^{op}}$$

(up to natural isomorphism).

Note that the essential image of ℓ is the same as the essential image of $j_* \circ i$. Hence, a presheaf F in $\mathrm{Set}^{\mathbb{CGH}^{op}}$ is a $\mathscr{CG}$-sheaf if and only if the unit η of $a \circ j^* \dashv j_* \circ i$ is an isomorphism at F. We have the following immediate corollary.

Corollary II.3.2. *The Grothendieck topology $\mathscr{CG}$ is subcanonical.*

Lemma II.3.3. *If F is a $\mathscr{CG}$-sheaf, then $j^*(F)$ is a $\mathscr{K}$-sheaf.*

Proof. Since F is a $\mathscr{CG}$-sheaf, it is in the essential image of $j_* \circ i$, hence, via η_F, we have

$$F \cong j_* a j^*(F).$$

By applying j^* we have

$$j^*(F) \cong j^* j_* a j^*(F).$$

Since the co-unit $\varepsilon(j)$ of $j^* \dashv j_*$ is an isomorphism, this yields

$$j^*(F) \cong aj^*(F).$$

$\square$

Corollary II.3.3. *If F is a presheaf in* $\mathrm{Set}^{\mathbb{CGH}^{op}}$,

$$a_{\mathscr{CG}}(F) \cong j_*a(j^*F).$$

If $F \in Sh(\mathbb{CGH})$ is a sheaf in the open cover topology then its $\mathscr{CG}$-sheafification is given by j_*j^*F.

Proof.
$$j_*a(j^*F) \cong (j_* \circ i) \circ a(j^*F),$$
so $j_*a(j^*F)$ is in the image of $j_* \circ i$, hence a $\mathscr{CG}$-sheaf. Now, let G be any $\mathscr{CG}$-sheaf. Then:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Set}^{\mathbb{CGH}^{op}}}(j_*a(j^*F), G) &\cong \mathrm{Hom}_{\mathrm{Set}^{\mathbb{CH}^{op}}}(aj^*(F), j^*(G)) \\
&\cong \mathrm{Hom}_{\mathrm{Set}^{\mathbb{CH}^{op}}}(j^*(F), j^*(G)) \\
&\cong \mathrm{Hom}_{\mathrm{Set}^{\mathbb{CGH}^{op}}}(F, j_*j^*(G)) \\
&\cong \mathrm{Hom}_{\mathrm{Set}^{\mathbb{CGH}^{op}}}(F, G).
\end{aligned}$$

$\square$

We end this subsection by noting ordinary sheaves and $\mathscr{CG}$-sheaves agree on locally compact Hausdorff spaces:

Let ς denote the unit of the adjunction $j^* \dashv j_*$.

Proposition II.3.4. *Let $F \in Sh(\mathbb{CGH})$ be a sheaf in the open cover topology and X in $\mathbb{LCH}$ a locally compact Hausdorff space. Then the map*

$$\varsigma_F(X) : F(X) \to j_*j^*F(X) \cong a_{\mathscr{CG}}(F)(X)$$

is a bijection. In particular, F and $a_{\mathscr{CG}}(F)$ agree on locally compact Hausdorff spaces.

Proof. This follows immediately from Proposition II.3.3 $\square$

II.3.2 Stacks for the Compactly Generated Grothendieck Topology

Denote again by $y_{\mathbb{CH}}$ the 2-functor

$$y_{\mathbb{CH}} : \mathbb{CGH} \to Gpd^{\mathbb{CH}^{op}}$$

induced by the inclusion

$$(\,\cdot\,)^{(id)} : \mathrm{Set}^{\mathbb{CH}^{op}} \hookrightarrow Gpd^{\mathbb{CH}^{op}}.$$

Then, it produces a 2-adjoint pair, which we will also denote by $j^* \dashv j_*$, by constructing j^* as the weak left Kan extension of $y_{\mathbb{CH}}$, and by letting

$$j_*\mathscr{Y}(X) := \mathrm{Hom}_{Gpd^{\mathbb{CH}^{op}}}\left(y_{\mathbb{CH}}(X), \mathscr{Y}\right).$$

By setting

$$j^*(\mathscr{X})(T) := \mathscr{X}(T),$$

we get a 2-functor which is weak-colimit preserving and whose restriction to representables is the same as $y_{\mathbb{CH}}$, hence, by uniqueness, the above equation for j^* must be correct. Note that the co-unit is an equivalence, hence j_* is fully faithful. Similarly, denote again by $y_{\mathbb{CGH}}$ the 2-functor

$$y_{\mathbb{CGH}} : \mathbb{CH} \to Gpd^{\mathbb{CGH}^{op}}$$

induced by the inclusion

$$(\,\cdot\,)^{(id)} : \mathrm{Set}^{\mathbb{CGH}^{op}} \hookrightarrow Gpd^{\mathbb{CGH}^{op}}.$$

Let $j_!$ be the 2-functor obtained by its weak left Kan extension. Just as before, $j_!$ is left 2-adjoint to j^*. Similarly, $j_!$ is fully faithful.

To justify the use of the same symbols, we have

$$\begin{aligned}
(\,\cdot\,)^{(id)} \circ j^* &= j^* \circ (\,\cdot\,)^{(id)} \\
(\,\cdot\,)^{(id)} \circ j_* &= j_* \circ (\,\cdot\,)^{(id)} \\
(\,\cdot\,)^{(id)} \circ j_! &= j_! \circ (\,\cdot\,)^{(id)}
\end{aligned}$$

where, the j^*, j_*, $j_!$ appearing on the left-hand side are 1-functors.

Let $St(\mathbb{CH})$ denote the 2-category of stacks on $\mathbb{CH}$ with respect to the Grothendieck topology $\mathscr{K}$. Then we have a 2-adjoint pair $a \dashv i$

$$St(\mathbb{CH}) \underset{i}{\overset{a}{\leftrightarrows}} Gpd^{\mathbb{CH}^{op}},$$

where a is the stackification 2-functor (Definition I.1.32) and i is the inclusion. Then, by composition, we get a 2-adjoint pair $a \circ j^* \dashv j_* \circ i$

$$St(\mathbb{CH}) \underset{j_* \circ i}{\overset{a \circ j^*}{\leftrightarrows}} Gpd^{\mathbb{CGH}^{op}}.$$

Definition II.3.2. A stack with respect to the Grothendieck topology $\mathscr{CG}$ on $\mathbb{CGH}$ will be called a $\mathscr{CG}$-stack.

Let $St_{\mathscr{CG}}(\mathbb{CGH})$ denote the full sub-2-category of $Gpd^{\mathbb{CGH}^{op}}$ consisting of $\mathscr{CG}$-stacks, and let $a_{\mathscr{CG}}$ denote the associated stackification 2-functor, and ℓ the inclusion, so $a_{\mathscr{CG}} \dashv \ell$.

Just as before, since every open covering is a $\mathscr{CG}$-cover,

$$St_{\mathscr{CG}}(\mathbb{CGH}) \subset St(\mathbb{CGH}).$$

The following results and their proofs follow naturally from those of the previous section when combined with the Comparison Lemma for stacks, a straight-forward stacky analogue of the theorem in [2] III:

Corollary II.3.4. *There is an equivalence of 2-categories*

$$St(\mathbb{CH}) \underset{a_{\mathscr{CG}} \circ j_* \circ i}{\overset{a \circ j^* \circ \ell}{\rightleftarrows}} St_{\mathscr{CG}}(\mathbb{CGH}),$$

such that

$$St(\mathbb{CH}) \underset{a_{\mathscr{CG}} \circ j_* \circ i}{\overset{a \circ j^* \circ \ell}{\rightleftarrows}} St_{\mathscr{CG}}(\mathbb{CGH}) \underset{\ell}{\overset{a_{\mathscr{CG}}}{\rightleftarrows}} Gpd^{\mathbb{CGH}^{op}}$$

is a factorization of

$$St(\mathbb{CH}) \underset{j_* \circ i}{\overset{a \circ j^*}{\rightleftarrows}} Gpd^{\mathbb{CGH}^{op}}$$

(up to natural equivalence).

Lemma II.3.4. *If $\mathscr{X}$ is a $\mathscr{CG}$-stack, then $j^*(\mathscr{X})$ is a $\mathscr{K}$-stack.*

Corollary II.3.5. *If $\mathscr{X}$ is a weak presheaf in $Gpd^{\mathbb{CGH}^{op}}$,*

$$a_{\mathscr{CG}}(\mathscr{X}) \simeq j_* a(j^* \mathscr{X}).$$

If $\mathscr{X} \in St(\mathbb{CGH})$ is a stack in the open cover topology then its $\mathscr{CG}$-stackification is given by $j_* j^* \mathscr{X}$.

Again, let ς denote the unit of the 2-adjunction $j^* \dashv j_*$.

Proposition II.3.5. *Let $\mathscr{X} \in St(\mathbb{CGH})$ be a stack in the open cover topology and X in $\mathbb{LCH}$ a locally compact Hausdorff space. Then the map*

$$\varsigma_{\mathscr{X}}(X) : \mathscr{X}(X) \to j_* j^* \mathscr{X}(X) \simeq a_{\mathscr{CG}}(\mathscr{X})(X)$$

is an equivalence of groupoids. In particular, $\mathscr{X}$ and $a_{\mathscr{CG}}(\mathscr{X})$ agree on locally compact Hausdorff spaces.

II.4 Compactly Generated Stacks

II.4.1 Compactly Generated Stacks

Definition II.4.1. A **compactly generated stack** is a presentable $\mathscr{CG}$-stack (see Definition I.2.9).

We denote the full sub-2-category of $St_{\mathscr{CG}}(\mathbb{CGH})$ of compactly generated stacks by $\mathscr{CG}\mathfrak{TSt}$.

Intrinsically, a compactly generated stack is a $\mathscr{CG}$-stack $\mathscr{X}$ such that there exists a compactly generated Hausdorff space X and a representable $\mathscr{CG}$-epimorphism

$$p : X \to \mathscr{X}.$$

The map above is a $\mathscr{CG}$-**atlas** for $\mathscr{X}$.

Let

$$\tilde{y}_{\mathbb{CH}} : \mathbb{CGH}Gpd \to Gpd^{\mathbb{CH}^{op}}$$

denote the 2-functor

$$\mathcal{G} \mapsto \operatorname{Hom}_{\mathbb{CGH}Gpd}\left(\left(\cdot\right)^{id}, \mathcal{G}\right).$$

Given a topological groupoid $\mathcal{G}$ in $\mathbb{CGH}Gpd$, denote by $[\mathcal{G}]_{\mathscr{K}}$ the associated $\mathscr{K}$-stack $a \circ \tilde{y}_{\mathbb{CH}}(\mathcal{G})$.

Let $\mathscr{CG}\mathfrak{TSt}'$ denote the essential image in $Gpd^{\mathbb{CH}^{op}}$ of $a \circ \tilde{y}_{\mathbb{CH}}$, i.e., it is the full sub-2-category of consisting of $\mathscr{K}$-stacks equivalent to $[\mathcal{G}]_{\mathscr{K}}$ for some compactly generated topological groupoid $\mathcal{G}$. It is immediate from Theorem II.3.4 that this 2-category is equivalent to $\mathscr{CG}\mathfrak{TSt}$. In fact, the functor j_* restricts to an equivalence $j_* : \mathscr{CG}\mathfrak{TSt}' \to \mathscr{CG}\mathfrak{TSt}$ of 2-categories. Hence we have proven:

Theorem II.4.1. *The 2-category of compactly generated stacks, $\mathscr{CG}\mathfrak{TSt}$, is equivalent to the essential image of*

$$j^* : \mathfrak{TSt} \to St(\mathbb{CH}).$$

Note that from Theorems I.2.1 and I.2.4, $\mathscr{CG}\mathfrak{TSt}$ is also equivalent to the bicategory of fractions $\mathbb{CGH}Gpd\left[W^{-1}_{\mathscr{CG}}\right]$ of compactly generated Hausdorff topological groupoids with inverted $\mathscr{CG}$-Morita equivalences, and also to the bicategory $Bun^{\mathscr{CG}}\mathbb{CGH}Gpd$ of compactly generated Hausdorff topological groupoids with left $\mathscr{CG}$-principal bundles as morphisms:

Theorem II.4.2. *The 2-functor*

$$a_{\mathscr{CG}} \circ \tilde{y} : \mathbb{CGH}Gpd \to \mathscr{CG}\mathfrak{TSt}$$

induces an equivalence of bicategories

$$P_{\mathscr{CG}} : \mathbb{CGH}Gpd\left[W^{-1}_{\mathscr{CG}}\right] \xrightarrow{\sim} \mathscr{CG}\mathfrak{TSt}.$$

Theorem II.4.3. *The 2-functor*

$$a_{\mathscr{C}\mathscr{G}} \circ \tilde{y} : \mathbb{CGH}Gpd \to \mathscr{C}\mathscr{G}\mathfrak{T}\mathfrak{S}\mathfrak{t}$$

induces an equivalence of bicategories

$$P'_{\mathscr{C}\mathscr{G}} : Bun^{\mathscr{C}\mathscr{G}}\mathbb{CGH}Gpd \xrightarrow{\sim} \mathscr{C}\mathscr{G}\mathfrak{T}\mathfrak{S}\mathfrak{t}.$$

We note that the principal bundles in $Bun^{\mathscr{C}\mathscr{G}}\mathbb{CGH}Gpd$ have a very simple description:

Recall that our notion of principal bundle depends on a Grothendieck topology. When the projection map of a principal bundle admits local sections (with respect to the open cover topology), it is called **ordinary**.

Proposition II.4.1. *If $\mathcal{G}$ is a topological groupoid in $\mathbb{CGH}Gpd$, X is a compactly generated Hausdorff space, and*

$$\begin{tikzcd} \mathcal{G}_1 \arrow[d, shift right] \arrow[d, shift left] \arrow[r, phantom, "\circlearrowright"] & P \arrow[dl, "\mu"'] \arrow[d, "\pi"] \\ \mathcal{G}_0 & X \end{tikzcd}$$

is a left $\mathcal{G}$-space over π, then it is a $\mathscr{C}\mathscr{G}$-principal bundle if and only if the restriction of P to K is an ordinary principal $\mathcal{G}$-bundle over K, for every compact subset $K \subseteq X$.

Proof. Suppose that we are given a left $\mathcal{G}$-space $\pi : P \to X$, whose restriction to K is an ordinary principal $\mathcal{G}$-bundle over K, for every compact subset $K \subseteq X$. Then, for each compact subset $K_\alpha \subseteq X$, we can choose an open cover $(U^i_\alpha \hookrightarrow K_\alpha)_{i=1}^{N_\alpha}$ over which P admits local sections. Then P admits local sections with respect to the $\mathscr{C}\mathscr{G}$-cover

$$\mathcal{U} := \left(U^i_\alpha \hookrightarrow X\right).$$

The converse is trivial. □

Corollary II.4.1. *If $\mathcal{G}$ and $\mathcal{H}$ are topological groupoids in $\mathbb{CGH}Gpd$ and $\mathcal{H}_0$ is locally compact, then*

$$Bun^{\mathscr{C}\mathscr{G}}_{\mathcal{G}}(\mathcal{H}) \simeq Bun_{\mathcal{G}}(\mathcal{H}),$$

where $Bun^{\mathscr{C}\mathscr{G}}_{\mathcal{G}}(\mathcal{H})$ denotes the groupoid of $\mathscr{C}\mathscr{G}$-principal $\mathcal{G}$-bundles over $\mathcal{H}$, and $Bun_{\mathcal{G}}(\mathcal{H})$ denotes the groupoid of ordinary principal $\mathcal{G}$-bundles over $\mathcal{H}$.

Equivalently:

If $\mathscr{X}$ and $\mathscr{Y}$ are topological stacks, and $\mathscr{Y}$ admits a locally compact Hausdorff atlas $Y \to \mathscr{Y}$, then the map

$$\mathrm{Hom}_{\mathrm{St}(\mathbb{CGH})}(\mathscr{Y}, \mathscr{X}) \to \mathrm{Hom}_{\mathrm{St}(\mathbb{CGH})}\left(a_{\mathscr{C}\mathscr{G}}(\mathscr{Y}), a_{\mathscr{C}\mathscr{G}}(\mathscr{X})\right)$$

induced by the unit $\varsigma_{\mathscr{X}} : \mathscr{X} \to a_{\mathscr{C}\mathscr{G}}(\mathscr{X})$, and the 2-adjunction $a_{\mathscr{C}\mathscr{G}} \dashv \ell$ is an equivalence of groupoids.

Corollary II.4.2. *The $\mathscr{CG}$-stackification functor restricted to the sub-2-category of topological stacks consisting of those topological stacks which admit a locally compact atlas, is 2-categorically full and faithful.*

Theorem II.4.4. *The 2-category of compactly generated stacks is equivalent to the sub-2-category of topological stacks consisting of those topological stacks which admit a locally compact atlas.*[2]

Proof. Denote the sub-2-category of topological stacks consisting of those topological stacks which admit a locally compact atlas by $\mathfrak{TSt}_{\mathbb{LCH}}$. Note that the image of

$$a_{\mathscr{CG}}|_{\mathfrak{TSt}_{\mathbb{LCH}}} : \mathfrak{TSt}_{\mathbb{LCH}} \to \mathrm{St}_{\mathscr{CG}}(\mathbb{CGH})$$

lies entirely in $\mathscr{CG}\mathfrak{TSt}$. By Corollary II.4.2 this 2-functor is full and faithful. It suffices to show it is essentially surjective. Notice that the essential image is those compactly generated stacks which admit a locally compact atlas. To complete the proof, note that if $X \to \mathscr{X}$ is any atlas of a compactly generated stack, then the inclusion of all compact subsets of X is a $\mathscr{CG}$-cover, hence

$$\coprod_{\alpha} K_{\alpha} \to X \to \mathscr{X}$$

is a $\mathscr{CG}$-atlas for $\mathscr{X}$ which is locally compact. $\square$

Theorem II.4.5. *The 2-category of compactly generated stacks has arbitrary products.*

Proof. Let $\mathscr{X}_i$ be an arbitrary family of compactly generated stacks. Then we can choose topological groupoids $\mathcal{G}_i$ in $\mathbb{CGH}Gpd$ such that

$$\mathscr{X}_i \simeq [\mathcal{G}_i]_{\mathscr{CG}}.$$

Note that

$$[\mathcal{G}_i]_{\mathscr{CG}} \simeq j_* [\mathcal{G}_i]_{\mathscr{K}}.$$

In light of Lemma II.2.3, we may assume without loss of generality that each $\mathcal{G}_i$ is fibrant. Under this assumption, by Theorem II.2.4, it follows that

$$[\mathcal{G}_i]_{\mathscr{CG}} \simeq j_* \tilde{y}_{\mathbb{CH}}(\mathcal{G}_i).$$

Note that the product $\prod_i \mathscr{X}_i$ is a $\mathscr{CG}$-stack, as any 2-category of stacks is complete. It suffices to show that this product is still presentable.

Recall that $\tilde{y}_{\mathbb{CH}}$ preserves small weak limits. Moreover, j_* does as well as it is a right 2-adjoint. Hence

[2] When we work over compactly generated Hausdorff spaces

$$\prod_i \mathscr{X}_i \simeq \prod_i j_* \tilde{y}_{\mathbb{CH}}(\mathcal{G}_i) \simeq j_* \tilde{y}_{\mathbb{CH}}\left(\prod_i \mathcal{G}_i\right).$$

It follows that

$$\begin{aligned}
\prod_i \mathscr{X}_i \simeq a_{\mathscr{CG}}\left(\prod_i \mathscr{X}_i\right) &\simeq a_{\mathscr{CG}}\left(j_* \tilde{y}_{\mathbb{CH}}\left(\prod_i \mathcal{G}_i\right)\right) \\
&\simeq (j_* \circ a \circ j^*) \circ j_* \circ (j^* \tilde{y})\left(\prod_i \mathcal{G}_i\right) \\
&\simeq (j_* \circ a \circ j^*) \circ (\tilde{y})\left(\prod_i \mathcal{G}_i\right) \\
&\simeq \left[\prod_i \mathcal{G}_i\right]_{\mathscr{CG}}.
\end{aligned}$$

□

Corollary II.4.3. *The 2-category of compactly generated stacks is closed under arbitrary small weak limits.*

Proof. Since $\mathbb{CGH}Gpd$ is closed under binary weak fibered products and the stackification 2-functor $a_{\mathscr{CG}}$ preserves finite weak limits, the 2-category $\mathscr{CG}\mathfrak{TSt}$ is closed under binary weak fibered products. By Theorem II.4.5, this 2-category has arbitrary small products. Since $\mathscr{CG}\mathfrak{TSt}$ is a $(2,1)$-category, by [35] it follows that $\mathscr{CG}\mathfrak{TSt}$ has all limits and hence is complete. □

II.4.2 Mapping Stacks of Compactly Generated Stacks

Recall that if $\mathscr{X}$ and $\mathscr{Y}$ are any stacks over $\mathbb{CGH}$, they have a mapping stack

$$\mathbb{M}\mathrm{ap}(\mathscr{Y}, \mathscr{X})(T) = \mathrm{Hom}_{Gpd^{\mathbb{CGH}^{op}}}(\mathscr{Y} \times T, \mathscr{X}).$$

It is the goal of this section to prove that if $\mathscr{X}$ and $\mathscr{Y}$ are compactly generated stacks, then so is $\mathbb{M}\mathrm{ap}(\mathscr{Y}, \mathscr{X})$.

Lemma II.4.6. *If $\mathscr{Y} \simeq [\mathcal{H}]_{\mathscr{CG}}$ is a compactly generated stack with $\mathcal{H}_0$ compact, and $\mathscr{X} \simeq [\mathcal{G}]_{\mathscr{CG}}$ an arbitrary compactly generated stack, then $\mathbb{M}\mathrm{ap}(\mathscr{Y}, \mathscr{X})$ is a compactly generated stack. More specifically, if $\mathcal{K}$ is a presentation for the topological stack $\mathbb{M}\mathrm{ap}([\mathcal{H}], [\mathcal{G}])$ ensured by Theorem II.2.8, then*

$$\mathbb{M}\mathrm{ap}(\mathscr{Y}, \mathscr{X}) \simeq [\mathcal{K}]_{\mathscr{CG}}.$$

In particular, $\mathbb{M}\mathrm{ap}(\mathscr{Y}, \mathscr{X})$ and $\mathbb{M}\mathrm{ap}([\mathcal{H}], [\mathcal{G}])$ restrict to the same stack over locally compact Hausdorff spaces.

Proof. Since any $\mathscr{CG}$-stack is completely determined by its restriction to $\mathbb{CH}$, it suffices to show that for any compact Hausdorff space $T \in \mathbb{CH}$,

$$[\mathcal{K}]_{\mathscr{CG}}(T) \simeq \mathrm{Hom}_{Gpd^{\mathbb{CGH}^{op}}}(\mathscr{Y} \times T, \mathscr{X}).$$

But, since T is compact Hausdorff

$$[\mathcal{K}]_{\mathscr{CG}}(T) \simeq [\mathcal{K}](T)$$

and because of the definition of $\mathcal{K}$

$$[\mathcal{K}](T) \simeq \mathrm{Hom}_{Gpd^{\mathbb{CGH}^{op}}}([\mathcal{H}] \times T, [\mathcal{G}]).$$

Furthermore, since $\mathscr{Y} \times T \simeq [\mathcal{H} \times T]_{\mathscr{CG}}$ and $\mathcal{H} \times T$ has compact Hausdorff object space, by Corollary II.4.1,

$$\begin{aligned} \mathrm{Hom}_{Gpd^{\mathbb{CGH}^{op}}}([\mathcal{H}] \times T, [\mathcal{G}]) &\simeq \mathrm{Hom}_{Gpd^{\mathbb{CGH}^{op}}}([\mathcal{H}]_{\mathscr{CG}} \times T, [\mathcal{G}]_{\mathscr{CG}}) \\ &\simeq \mathrm{Hom}_{Gpd^{\mathbb{CGH}^{op}}}(\mathscr{Y} \times T, \mathscr{X}). \end{aligned}$$

Hence

$$[\mathcal{K}]_{\mathscr{CG}}(T) \simeq \mathrm{Hom}(\mathscr{Y} \times T, \mathscr{X}).$$

□

Corollary II.4.4. *If K is a compact Hausdorff space and $\mathscr{X}$ an arbitrary compactly generated stack, then $\mathbb{M}\mathrm{ap}(K, \mathscr{X})$ is a compactly generated stack.*

Lemma II.4.7. *If X is a compactly generated Hausdorff space and $\mathscr{X}$ an arbitrary compactly generated stack, then $\mathbb{M}\mathrm{ap}(X, \mathscr{X})$ is a compactly generated stack.*

Proof. Let $\left(K_\alpha \xhookrightarrow{i_\alpha} X\right)$ denote the inclusion of all compact subsets of X. This is a $\mathscr{CG}$-cover for X. Let Y be an arbitrary compactly generated Hausdorff space. Note that $\left(K_\alpha \times Y \xhookrightarrow{i_\alpha \times id_Y} X \times Y\right)$ is also a $\mathscr{CG}$-cover.

By Proposition I.1.6, we have that in $\mathscr{CG}\mathfrak{T}\mathfrak{S}\mathfrak{t}$

$$X \simeq \underset{K_\alpha \hookrightarrow X}{\underrightarrow{\mathrm{holim}}}\, K_\alpha$$

and

$$X \times Y \simeq \underset{K_\alpha \times Y \hookrightarrow X \times Y}{\underrightarrow{\mathrm{holim}}} (K_\alpha \times Y).$$

Hence

$$\begin{aligned}
\mathsf{Map}(X,\mathscr{X})(Y) &\simeq \mathrm{Hom}_{Gpd^{\mathbb{CGH}^{op}}}(X\times Y,\mathscr{X})\\
&\simeq \mathrm{Hom}_{Gpd^{\mathbb{CGH}^{op}}}\left(\underset{K_\alpha\times Y\hookrightarrow X\times Y}{\underrightarrow{\mathrm{holim}}}(K_\alpha\times Y),\mathscr{X}\right)\\
&\simeq \underset{K_\alpha\times Y\hookrightarrow X\times Y}{\underleftarrow{\mathrm{holim}}}\ \mathrm{Hom}_{Gpd^{\mathbb{CGH}^{op}}}(K_\alpha\times Y,\mathscr{X})\\
&\simeq \underset{K_\alpha\times Y\hookrightarrow X\times Y}{\underleftarrow{\mathrm{holim}}}\ \mathrm{Hom}_{Gpd^{\mathbb{CGH}^{op}}}(Y,\mathsf{Map}(K_\alpha,\mathscr{X}))\\
&\simeq \mathrm{Hom}_{Gpd^{\mathbb{CGH}^{op}}}\left(Y,\underset{K_\alpha\hookrightarrow X}{\underleftarrow{\mathrm{holim}}}\,\mathsf{Map}(K_\alpha,\mathscr{X})\right)\\
&\simeq \left(\underset{K_\alpha\hookrightarrow X}{\underleftarrow{\mathrm{holim}}}\,\mathsf{Map}(K_\alpha,\mathscr{X})\right)(Y).
\end{aligned}$$

Therefore

$$\mathsf{Map}(X,\mathscr{X})\simeq \underset{K_\alpha\hookrightarrow X}{\underleftarrow{\mathrm{holim}}}\,\mathsf{Map}(K_\alpha,\mathscr{X}).$$

So by Corollary II.4.3, $\mathsf{Map}(X,\mathscr{X})$ is a compactly generated stack. □

Theorem II.4.8. *If $\mathscr{X}$ and $\mathscr{Y}$ are arbitrary compactly generated stacks, then $\mathsf{Map}(\mathscr{Y},\mathscr{X})$ is a compactly generated stack.*

Proof. Let $\mathscr{Y}$ be presented by a topological groupoid $\mathcal{H}$. By Lemma A.3.1, we can write $\mathscr{Y}$ as the weak colimit of the following diagram:

$$\mathcal{H}_1\times_{\mathcal{H}_0}\mathcal{H}_1 \rightrightarrows\!\!\!\to \mathcal{H}_1 \rightrightarrows \mathcal{H}_0\ ,$$

where the three parallel arrows are the first and second projections and the composition map.

Furthermore, let X be any compactly generated Hausdorff space. Then $\mathscr{Y}\times X$ is the weak colimit of

$$(\mathcal{H}_1\times_{\mathcal{H}_0}\mathcal{H}_1)\times X \rightrightarrows\!\!\!\to \mathcal{H}_1\times X \rightrightarrows \mathcal{H}_0\times X\ .$$

With this in mind, in much the same way as Lemma II.4.7, some simple calculations allow one to express $\mathsf{Map}(\mathscr{Y},\mathscr{X})$ as a weak limit of a diagram involving $\mathsf{Map}(\mathcal{H}_1\times_{\mathcal{H}_0}\mathcal{H}_1,\mathscr{X})$, $\mathsf{Map}(\mathcal{H}_1,\mathscr{X})$, and $\mathsf{Map}(\mathcal{H}_0,\mathscr{X})$, all of which are compactly generated stacks by Lemma II.4.7. □

We presented the proof of Cartesian-closure in this way to emphasize the role of completeness and compact generation. We will now give a concrete description of a topological groupoid presenting the mapping stack of two compactly generated stacks. Note that since the inclusion of all compact subsets of a space is a $\mathscr{CG}$-cover, every compactly generated stack has a locally compact, paracompact Hausdorff atlas.

Theorem II.4.9. *Let $\mathscr{X} \simeq [\mathcal{G}]_{\mathscr{CG}}$ and $\mathscr{Y} \simeq [\mathcal{H}]_{\mathscr{CG}}$ be two compactly generated stacks. Assume (without loss of generality) that $\mathcal{H}_0$ is locally compact and paracompact Hausdorff. Then*

$$\left[Fib\,(\mathcal{G})^{\mathcal{H}}\right]_{\mathscr{CG}} \simeq \mathtt{Map}\,(\mathscr{Y}, \mathscr{X})\,.$$

Proof. It suffices to check that $\left[Fib\,(\mathcal{G})^{\mathcal{H}}\right]_{\mathscr{CG}}$ and $\mathtt{Map}\,(\mathscr{Y}, \mathscr{X})$ agree on every compact Hausdorff space T. Following the same proof as Theorem II.2.9, one only has to realize that the product of a compact Hausdorff space with a paracompact space is paracompact [46]. The rest of the proof is identical. □

Remark. This implies that the 2-category of topological stacks (in compactly generated Hausdorff spaces) with locally compact atlases is Cartesian closed. This might seem surprising since, after all, locally compact Hausdorff spaces are quite far from being Cartesian closed. What is happening is that the exponential of two locally compact spaces is not a space in this description, but a stack! (This stack is actually a sheaf.) In fact, the category of compactly generated Hausdorff spaces embeds into this 2-category by sending a space X to the stack associated to the topological groupoid $(X)^{(id)}_{\mathcal{K}}$, where $\mathcal{K}$ denotes the $\mathscr{CG}$-cover which is the inclusion of all compact subsets of X.

II.4.3 Homotopy Types of Compactly Generated Stacks

In [51], Noohi constructs a functorial assignment to each topological stack $\mathscr{X}$, a weak homotopy type. For $\mathscr{X}$ a topological stack, its weak homotopy type turns out to be the weak homotopy type of $\|\mathcal{G}\|$ for any topological groupoid $\mathcal{G}$ for which $\mathscr{X} \simeq [\mathcal{G}]$. Moreover, each topological stack $\mathscr{X}$ admits at atlas which is also a weak homotopy equivalence; the canonical atlas

$$\varphi : \|\mathcal{G}\| \to \mathscr{X}$$

coming from Lemma II.2.3 is a weak homotopy equivalence.

A particular corollary is:

Corollary II.4.5. *If $\mathcal{G} \to \mathcal{H}$ is a Morita equivalence, the induced map*

$$\|\mathcal{G}\| \to \|\mathcal{H}\|$$

is a weak homotopy equivalence.

This is a classical result. For instance, it is proven for the case of étale topological groupoids in [40] and [42]. [3]

In this section, we extend these results to the setting of compactly generated stacks. We begin with the technical notion of a shrinkable map, which will prove quite useful.

Definition II.4.2. [21] A continuous map $f : X \to B$ is **shrinkable** if admits a section $s : B \to X$ together with a homotopy

$$H : I \times X \to X$$

from sf to id_X over B, i.e. for all t, the map

$$H_t : X \to X$$

is a map in Top/B from f to f.

Remark. Every shrinkable map is in particular a homotopy equivalence.

Definition II.4.3. A continuous map $f : X \to B$ is **locally shrinkable** [51] if there exists an open cover (U_α) of B such that for each α, the induced map

$$f|_{U_\alpha} : f^{-1}(U_\alpha) \to U_\alpha$$

is shrinkable. A map $f : X \to B$ is called $\mathscr{CG}$-locally shrinkable if there exists a $\mathscr{CG}$-cover (V_i) of B such that the same condition holds.

Clearly, shrinkable $\Rightarrow$ locally shrinkable $\Rightarrow$ $\mathscr{CG}$-locally shrinkable.

Definition II.4.4. A continuous map $f : X \to B$ is **quasi-shrinkable** if for every map $T \to B$ from a locally compact, paracompact (Hausdorff) space T, the induced map

$$X \times_Y T \to T$$

is shrinkable.

Lemma II.4.10. *Every $\mathscr{CG}$-locally shrinkable map is quasi-shrinkable.*

Proof. Since every $\mathscr{CG}$-cover of a locally compact space can be refined by an open one, and every open cover of a paracompact Hausdorff space can be refined by a numerable one, the result follows from [21], Corollary 3.2. □

Definition II.4.5. A map $f : X \to Y$ of spaces is a **universal weak equivalence** if for any map $T \to Y$, the induced map $T \times_Y X \to T$ is a weak homotopy equivalence.

[3] I would like to thank Ieke Moerdijk for explaining to me how to extend his method of proof to any topological groupoid whose object and arrow spaces have a basis of contractible open sets.

Remark. Shrinkable, locally shrinkable, and $\mathscr{CG}$-locally shrinkable maps are invariant under change of base. (See Definition I.2.23.) Furthermore, this is true for universal weak equivalences by definition.

The following lemma is a useful characterization of universal weak equivalences:

Lemma II.4.11. *A map $f : X \to B$ is a universal weak equivalence if and only if, for all $n \geq 0$, for any map $D^n \to B$ from the n-disk, the induced map $X \times_B D^n \to D^n$ is a weak homotopy equivalence (i.e. $X \times_B D^n$ is weakly contractible).*

Proof. One direction is clear by definition.

Conversely, let $f : X \to B$ be a map satisfying the stated hypothesis for each n-disk. We will first show that f is a weak equivalence.

Denote by I the unit interval $[0,1]$, and let $E(f)$ denote the homotopy fiber of f,

$$\begin{tikzcd} E(f) \arrow[r] \arrow[d] & B^I \arrow[d, "ev(0)"] \\ X \arrow[r, "f"] & B, \end{tikzcd}$$

where $ev(0)$ is evaluation at 0.

We may factor $X \to B$ as

$$X \to E(f) \xrightarrow{ev(1)} B$$

where $X \to E(f)$ is a homotopy equivalence and the evaluation map (at 1) is a fibration. From the long-exact sequence of homotopy groups resulting from the fiber sequence

$$E(f)_b \to E_f \to B,$$

it suffices to show that for each $b \in B$ in the image of f, the homotopy fiber $E(f)_b = ev(1)^{-1}(b)$ is weakly contractible.

Suppose we are given a based map $l : S^{n-1} \to E(f)_b$. Identifying D^n with the cone on S^{n-1}, this is the same as giving a commutative diagram

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "l_1"'] & D^n \arrow[d, "l_2"] \\ X \arrow[r, "f"] & B, \end{tikzcd}$$

where $S^{n-1} \to D^n$ is the canonical map.

Let $\tilde{f}$ denote the induced map

$$\begin{tikzcd} X \times_B D^n \arrow[r, "\tilde{f}"] \arrow[d] & D^n \arrow[d, "l_2"] \\ X \arrow[r, "f"] & B. \end{tikzcd}$$

By our hypothesis, $\tilde{f}$ is a weak homotopy equivalence. Moreover, it is easy to check that the map

$$l : S^{n-1} \to E(f)_b$$

factors through the canonical map

$$q : E(\tilde{f})_b \to E(f)_b,$$

say $l = ql'$ for some

$$l' : S^{n-1} \to E(\tilde{f})_b.$$

As $\tilde{f}$ is a weak equivalence, $E(\tilde{f})_b$ is weakly contractible. So, l' is null-homotopic, and hence so is l. It follows that $E(f)_b$ is also weakly contractible, and thus f is a weak equivalence.

Moreover, f is in fact a universal weak equivalence since if $T \to X$ is any map, the induced map $X \times_B T$ satisfies the same hypothesis that f does. $\square$

Corollary II.4.6. *Every quasi-shrinkable map is a universal weak equivalence.*

Definition II.4.6. A representable map $\mathscr{X} \to \mathscr{Y}$ of stacks on $\mathbb{CGH}$ (with respect to either the $\mathscr{CG}$-topology or the open cover topology) is said to be shrinkable, locally-shrinkable, $\mathscr{CG}$-locally shrinkable, quasi-shrinkable, or a universal weak equivalence, if and only if for every map $T \to \mathscr{X}$ from a topological space, the induced map $\mathscr{X} \times_{\mathscr{Y}} T \to T$ is.

Lemma II.4.12. *Let $f : \mathscr{X} \to \mathscr{Y}$ and $g : \mathscr{Y}' \to \mathscr{Y}$ be morphisms in* $\mathrm{St}_{\mathscr{CG}}(\mathbb{CGH})$ *such that g is a $\mathscr{CG}$-epimorphism and the induced map*

$$\mathscr{X} \times_{\mathscr{Y}} \mathscr{Y}' \to \mathscr{Y}'$$

is a representable $\mathscr{CG}$-locally shrinkable map. Then f is also representable and $\mathscr{CG}$-locally shrinkable.

Proof. Let $h : T \to \mathscr{Y}$ be arbitrary. Choose a $\mathscr{CG}$-cover (V_α) of T such that for all α, there is a 2-commutative diagram

$$\begin{tikzcd} V_\alpha \arrow[r] \arrow[d, "h_\alpha"'] & T \arrow[d, "h"] \\ \mathscr{Y}' \arrow[r, "g"] & \mathscr{Y}. \end{tikzcd}$$

Note, by assumption, the induced maps

$$\mathscr{X} \times_{\mathscr{Y}} V_\alpha \to V_\alpha$$

are $\mathscr{CG}$-locally shrinkable maps of topological spaces. By refining this $\mathscr{CG}$-cover if necessary, we can arrange that each of these maps is in fact shrinkable. It follows that $\mathscr{X} \times_{\mathscr{Y}} T$ is a topological space, and that the collection $(\mathscr{X} \times_{\mathscr{Y}} V_\alpha)$ is a $\mathscr{CG}$-cover of it. Since the restriction of

$$\mathscr{X} \times_{\mathscr{Y}} T \to T$$

to each element of this cover is

$$\mathscr{X} \times_{\mathscr{Y}} V_\alpha \to V_\alpha,$$

it follows that

$$\mathscr{X} \times_{\mathscr{Y}} T \to T$$

is $\mathscr{CG}$-locally shrinkable. $\square$

Theorem II.4.13. *Let $\mathscr{X} \simeq [\mathcal{G}]_{\mathscr{CG}}$ be a compactly generated stack. Then the atlas $\|\mathcal{G}\| \to \mathscr{X}$ is $\mathscr{CG}$-locally shrinkable.*

Proof. In [51], it is shown that we have a 2-Cartesian diagram

$$\begin{tikzcd} \|\mathbb{E}\mathcal{G}\| \arrow[r, "f"] \arrow[d] & \mathcal{G}_0 \arrow[d] \\ \|\mathcal{G}\| \arrow[r, "\varphi"] & {[\mathcal{G}]} \end{tikzcd}$$

with f shrinkable. Now, the stackification 2-functor $a_{\mathscr{CG}}$ commutes with finite weak limits, hence, the following is also a 2-Cartesian diagram:

$$\begin{tikzcd} \|\mathbb{E}\mathcal{G}\| \arrow[r, "f"] \arrow[d] & \mathcal{G}_0 \arrow[d] \\ \|\mathcal{G}\| \arrow[r, "\bar{\varphi}"] & \mathscr{X}. \end{tikzcd}$$

Since the map $\mathcal{G}_0 \to \mathscr{X}$ is a $\mathscr{CG}$-epimorphism and f is shrinkable, it follows from Lemma II.4.12 that $\bar{\varphi}$ is $\mathscr{CG}$-locally shrinkable. $\square$

Corollary II.4.7. *Let $\mathscr{X} \simeq [\mathcal{G}]_{\mathscr{CG}}$ be a compactly generated stack. Then the atlas $\|\mathcal{G}\| \to \mathscr{X}$ is a universal weak equivalence.*

Proof. This follows immediately from Lemma II.4.10 and Corollary II.4.6. $\square$

Corollary II.4.8. *Let $\phi : \mathcal{G} \to \mathcal{H}$ be a $\mathscr{CG}$-Morita equivalence between two topological groupoids $\mathcal{G}$ and $\mathcal{H}$. Then ϕ induces a weak homotopy equivalence*

$$\|\phi\| : \|\mathcal{G}\| \to \|\mathcal{H}\|.$$

Proof. Let $\mathscr{X} := [\mathcal{G}]_{\mathscr{CG}} \simeq [\mathcal{H}]_{\mathscr{CG}}$. Then each atlas $\|\mathcal{G}\| \to \mathscr{X}$ and $\|\mathcal{H}\| \to \mathscr{X}$ is a universal weak equivalence. The following diagram 2-commutes (with the outer square Cartesian):

$$\begin{tikzcd} \|\mathcal{G}\| \times_{\mathscr{X}} \|\mathcal{H}\| \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \|\mathcal{H}\| \arrow[d] \\ \|\mathcal{G}\| \arrow[r] \arrow[ur, dotted, "\|\varphi\|"'] & \mathscr{X}. \end{tikzcd}$$

Since each atlas is a universal weak equivalence, α and β are weak equivalences, and hence so is $\|\varphi\|$. $\square$

Example 14. Let X be a topological space. Consider the inclusion of all its compact subsets $(K_\alpha \hookrightarrow X)$. This is a $\mathscr{CG}$-cover, so the associated groupoid

$$\mathcal{K} = \left(\coprod K_\alpha \cap K_\beta \rightrightarrows \coprod K_\alpha\right)$$

is $\mathscr{CG}$-Morita equivalent to X. It follows from Corollary II.4.8 that $\|\mathcal{K}\|$ is weakly homotopy equivalent to X.

We now copy Noohi in [51] to give a functorial assignment to each compactly generated stack a weak homotopy type.

Given a 2-category $\mathscr{C}$, denote the 1-category obtained by identifying equivalent 1-morphisms by $\tau_1(\mathscr{C})$. Suppose we are given a full sub-2-category B of $\mathscr{C}$ which is in fact (equivalent to) a 1-category, and is closed under pullbacks. For example, consider $\mathscr{C}$ to be the 2-category of compactly generated stacks, $\mathscr{CG}\mathfrak{TSt}$, and for B to the category of compactly generated Hausdorff spaces $\mathbb{CGH}$. Let R be a class of morphisms in B which contains all isomorphisms, and is stable under pullback. Let $\tilde{R}$ denote the class of morphisms $f : \mathscr{X} \to \mathscr{Y}$ in $\mathscr{C}$ such that for every $h : T \to \mathscr{Y}$, with T in B, the weak pullback $\mathscr{X} \times_{\mathscr{Y}} T$ is in B and the induced map

$$\mathscr{X} \times_{\mathscr{Y}} T \to T$$

is in R. In the case there $\mathscr{C} = \mathscr{CG}\mathfrak{TSt}$ and $B = \mathbb{CGH}$, then f is a representable map with property R.

Lemma II.4.14. *[51] In the set up just described, if for every object $\mathscr{X}$ of $\mathscr{C}$ there exists a morphism*

$$\varphi(\mathscr{X}) : \Theta(\mathscr{X}) \to \mathscr{X}$$

in $\tilde{R}$ from an object $\Theta(\mathscr{X})$ of B, then there is an induced adjunction

$$\tilde{R}^{-1}\tau_1(\mathscr{C}) \underset{\Theta}{\overset{y}{\leftrightarrows}} R^{-1}B$$

with $y \dashv \Theta$, and with y fully-faithful. Moreover, the components of the co-unit of this adjunction are in $\tilde{R}$.

Theorem II.4.15. *There exists a functor $\Omega : \mathscr{CG}\mathfrak{TSt} \to \mathrm{Ho}(\mathbb{TOP})$ assigning to each compactly generated stack $\mathscr{X}$, a weak homotopy type. Moreover, for each $\mathscr{X}$, there is a $\mathscr{CG}$-atlas $X \to \mathscr{X}$, which is a universal weak equivalence from a space X whose homotopy type is $\Omega(\mathscr{X})$.*

Proof. In the previous lemma, let $\mathscr{C} = \mathscr{C}\mathscr{G}\mathfrak{T}\mathfrak{S}\mathfrak{t}$, $B = \mathbb{CGH}$, and let R be the class of universal weak equivalences. Use Corollary II.4.7 to pick for each compactly generated stack $\mathscr{X}$, a $\mathscr{C}\mathscr{G}$-atlas which is a universal weak equivalence. Lemma II.4.14 implies that there is an induced adjunction

$$\tilde{R}^{-1}\tau_1\left(\mathscr{C}\mathscr{G}\mathfrak{T}\mathfrak{S}\mathfrak{t}\right) \underset{\Theta}{\overset{y}{\rightleftarrows}} R^{-1}\mathbb{CGH}\,,$$

where the unit of this adjunction is an equivalence, and the co-unit for $\mathscr{X}$,

$$y\Theta\left(\mathscr{X}\right) \to \mathscr{X},$$

is the chosen atlas.

Let S denote the class of weak homotopy equivalences in $\mathbb{CGH}$. Let T denote $y\left(S\right)$. Then, since

$$y\left(S\right) = T,$$

and

$$\Theta\left(T\right) = S,$$

it follows that there is an induced adjunction

$$S^{-1}\left(\tilde{R}^{-1}\tau_1\left(\mathscr{C}\mathscr{G}\mathfrak{T}\mathfrak{S}\mathfrak{t}\right)\right) \underset{\bar{\Theta}}{\overset{\bar{y}}{\rightleftarrows}} S^{-1}\left(R^{-1}\mathbb{CGH}\right).$$

Note that there are canonical equivalences

$$S^{-1}\left(\tilde{R}^{-1}\tau_1\left(\mathscr{C}\mathscr{G}\mathfrak{T}\mathfrak{S}\mathfrak{t}\right)\right) \simeq S^{-1}\tau_1\left(\mathscr{C}\mathscr{G}\mathfrak{T}\mathfrak{S}\mathfrak{t}\right),$$

and

$$S^{-1}\left(R^{-1}\mathbb{CGH}\right) \simeq S^{-1}\mathbb{CGH}.$$

Moreover, since every topological space has the weak homotopy type of a compactly generated Hausdorff space, $S^{-1}\mathbb{CGH}$ is equivalent to the homotopy category of spaces, $\mathrm{Ho}\left(\mathbb{TOP}\right)$. Note that the following diagram 2-commutes

$$\begin{tikzcd}
 & \tilde{R}^{-1}\tau_1\left(\mathscr{C}\mathscr{G}\mathfrak{T}\mathfrak{S}\mathfrak{t}\right) \arrow[r, "\Theta"] \arrow[d] & R^{-1}\mathbb{CGH} \arrow[d] \\
\mathscr{C}\mathscr{G}\mathfrak{T}\mathfrak{S}\mathfrak{t} \arrow[r] \arrow[ur] & S^{-1}\tau_1\left(\mathscr{C}\mathscr{G}\mathfrak{T}\mathfrak{S}\mathfrak{t}\right) \arrow[r, "\bar{\Theta}"] & \mathrm{Ho}\left(\mathbb{TOP}\right).
\end{tikzcd}$$

Denote either (naturally isomorphic) composite by $\Omega : \mathscr{C}\mathscr{G}\mathfrak{T}\mathfrak{S}\mathfrak{t} \to \mathrm{Ho}\left(\mathbb{TOP}\right)$. $\square$

II.4.4 Comparison with Topological Stacks

In this subsection, we will extend the results of the previous section to give a functorial assignment of a weak homotopy type to a wider class of stacks, which includes all compactly generated stacks and all topological stacks. We will then show that for a given topological stack $\mathscr{X}$, the induced map to its associated compactly generated stack $a_{\mathscr{CG}}(\mathscr{X})$ is a weak homotopy equivalence. Finally, we will show in what sense compactly generated stacks are to topological stacks what compactly generated spaces are to topological spaces (Theorem II.4.20).

Proposition II.4.2. *For a stack $\mathscr{X}$ over compactly generated Hausdorff spaces (with respect to the open cover topology) whose diagonal*

$$\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$$

is representable, the following conditions are equivalent:

i) the $\mathscr{CG}$-stackification of $\mathscr{X}$ is a compactly generated stack,

ii) there exists a topological space X and a morphism $X \to \mathscr{X}$ such that, for all spaces T, the induced map

$$T \times_{\mathscr{X}} X \to T$$

admits local sections with respect to the topology $\mathscr{CG}$ (i.e. is a $\mathscr{CG}$-covering morphism),

iii) there exists a topological space X and a morphism $X \to \mathscr{X}$ such that, for all compact Hausdorff spaces T, the induced map

$$T \times_{\mathscr{X}} X \to T$$

admits local sections (i.e. is an epimorphism),

iv) there exists a topological space X and a morphism $X \to \mathscr{X}$ such that, for all locally compact Hausdorff spaces T, the induced map

$$T \times_{\mathscr{X}} X \to T$$

admits local sections (i.e. is an epimorphism),

v) there exists a topological stack $\bar{\mathscr{X}}$ and a map $q : \bar{\mathscr{X}} \to \mathscr{X}$ such that, for all locally compact Hausdorff spaces T,

$$q(T) : \bar{\mathscr{X}}(T) \to \mathscr{X}(T)$$

is an equivalence of groupoids.

Proof. Suppose that condition $i)$ is satisfied. Note that condition $ii)$ is equivalent to saying that there exists a $\mathscr{C}\mathscr{G}$-covering morphism $X \to \mathscr{X}$ from a topological space (See Definition I.1.34.) Let

$$p : X \to a_{\mathscr{C}\mathscr{G}}(\mathscr{X})$$

be a locally compact atlas for the compactly generated stack $a_{\mathscr{C}\mathscr{G}}(\mathscr{X})$. Then it factors (up to equivalence) as

$$X \xrightarrow{x} \mathscr{X} \to a_{\mathscr{C}\mathscr{G}}(\mathscr{X}),$$

for some map

$$x : X \to \mathscr{X}.$$

Note that the $\mathscr{C}\mathscr{G}$-stackification of x is (equivalent to) p, hence is an epimorphism in $\mathrm{St}_{\mathscr{C}\mathscr{G}}(\mathbb{CGH})$. This implies x is a $\mathscr{C}\mathscr{G}$-covering morphism. So $i) \Rightarrow ii)$.

Since any $\mathscr{C}\mathscr{G}$-cover of a locally compact space can be refined by an open one, $ii)\, (\Rightarrow iv)) \Rightarrow iii)$.

From Corollary II.3.4, it follows that there is an equivalence of 2-categories

$$\Lambda : \mathrm{St}(\mathbb{CH}) \to \mathrm{St}_{\mathscr{C}\mathscr{G}}(\mathbb{CGH}),$$

such that for every stack $\mathscr{X}$ on $\mathbb{CGH}$ with respect to the open cover topology,

$$\Lambda(j^*\mathscr{X}) \simeq a_{\mathscr{C}\mathscr{G}}(\mathscr{X}).$$

Hence $iii)\, (\Rightarrow ii)) \Rightarrow iv)$.

Suppose $iv)$ holds. Then $iv) \Rightarrow iii)$ trivially, and $iii) \Rightarrow ii)$ by the above argument. So there exists a $\mathscr{C}\mathscr{G}$-covering map $X \to \mathscr{X}$. Hence, the induced map $X \to a_{\mathscr{C}\mathscr{G}}(\mathscr{X})$ is an epimorphism (in particular, this implies $i)$). Consider the induced map

$$\alpha : [X \times_{\mathscr{X}} X \rightrightarrows X] \to \mathscr{X}.$$

It is a monomorphism, and since $X \to \mathscr{X}$ is a $\mathscr{C}\mathscr{G}$-covering morphism, α is too. Since stackification preserves monomorphisms, $a_{\mathscr{C}\mathscr{G}}(\alpha)$ is an equivalence between the compactly generated stack

$$[X \times_{\mathscr{X}} X \rightrightarrows X]_{\mathscr{C}\mathscr{G}}$$

and $a_{\mathscr{C}\mathscr{G}}(\mathscr{X})$. Proposition II.3.5 implies that α satisfies $v)$. Hence, $iv) \Rightarrow v)$.

Suppose that $v)$ holds for a morphism $q : \bar{\mathscr{X}} \to \mathscr{X}$ from a topological stack. Then

$$j^*\left(\bar{\mathscr{X}}\right) \to j^*\mathscr{X}$$

is an equivalence. Hence, $a_{\mathscr{C}\mathscr{G}}(q)$ is an equivalence between $a_{\mathscr{C}\mathscr{G}}(\mathscr{X})$ and the compactly generated stack $a_{\mathscr{C}\mathscr{G}}\left(\bar{\mathscr{X}}\right)$. Hence $v) \Rightarrow i)$. $\square$

Definition II.4.7. A stack $\mathscr{X}$ over compactly generated Hausdorff spaces (with respect to the open cover topology) whose diagonal

$$\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$$

is representable, is **quasi-topological** if any of the equivalent conditions of Proposition II.4.2 hold. Denote the full sub-2-category of $\mathrm{St}\left(\mathbb{CGH}\right)$ consisting of the quasi-topological stacks by $\mathfrak{QuasiTSt}$.

The following proposition is immediate.

Proposition II.4.3. *If $\mathscr{X}$ is a stack over compactly generated Hausdorff spaces which is topological, paratopological, or compactly generated, then it is quasi-topological.*

Lemma II.4.16. *Let $\mathscr{X}$ be a quasi-topological stack, and let*

$$h : T \to a_{\mathscr{CG}}\left(\mathscr{X}\right)$$

be a map from a locally compact Hausdorff space. Then $T \times_{a_{\mathscr{CG}}(\mathscr{X})} \mathscr{X}$ is a topological space and the induced map

$$T \times_{a_{\mathscr{CG}}(\mathscr{X})} \mathscr{X} \to T$$

is a homeomorphism.

Proof. Let $X \to a_{\mathscr{CG}}\left(\mathscr{X}\right)$ be a locally compact $\mathscr{CG}$-atlas. Then it factors (up to equivalence) as

$$X \xrightarrow{x} \mathscr{X} \xrightarrow{\varsigma_{\mathscr{X}}} a_{\mathscr{CG}}\left(\mathscr{X}\right).$$

Moreover, as T is locally compact, there is a 2-commutative lift h'

$$\begin{tikzcd} & \mathscr{X} \arrow[d, "\varsigma_{\mathscr{X}}"] \\ T \arrow[ur, dotted, "h'"] \arrow[r, "h"'] & a_{\mathscr{CG}}\left(\mathscr{X}\right). \end{tikzcd}$$

Consider the weak pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d, "x"] \\ T \arrow[r, "h'"] & \mathscr{X}. \end{tikzcd}$$

As a sheaf, P assigns each space Z, the set of triples (f, g, α) with

$$f : Z \to T,$$

$$g : Z \to X,$$

and

$$\alpha : h'f \Rightarrow xg.$$

Since the diagonal of $\mathscr{X}$ is representable, P is represented by a compactly generated Hausdorff space.

Consider now the weak pullback diagram

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & X \arrow[d, "\varsigma_{\mathscr{X}} \circ x"] \\ T \arrow[r, "\varsigma_{\mathscr{X}} \circ h'"] & a_{\mathscr{CG}}(\mathscr{X}). \end{tikzcd}$$

The sheaf P' is again representable, and it assigns each space Z the set of triples (f, g, β) with

$$f : Z \to T,$$

$$g : Z \to X,$$

and

$$\beta : \varsigma_{\mathscr{X}} \circ h'f \Rightarrow \varsigma_{\mathscr{X}} \circ xg.$$

Consider the induced map $P \to P'$ given by composition with $\varsigma_{\mathscr{X}}$. Since for every locally compact Hausdorff space S, $\varsigma_{\mathscr{X}}(S)$ is an equivalence of groupoids, it follows that the induced map $y_{\mathbb{CH}}(P) \to y_{\mathbb{CH}}(P')$ is an isomorphism, where

$$y_{\mathbb{CH}} : \mathbb{CGH} \to \mathrm{Set}^{\mathbb{CH}^{op}}$$

is the functor which assigns a compactly generated Hausdorff space X the presheaf $S \mapsto \mathrm{Hom}_{\mathbb{CGH}}(S, X)$. Since $y_{\mathbb{CH}}$ is fully-faithful, it follows that the induced map $P \to P'$ is an isomorphism.

Finally, consider the following 2-commutative diagram:

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r, "h'"] \arrow[d, "id_T"'] & \mathscr{X} \arrow[d, "\varsigma_{\mathscr{X}}"] \\ T \arrow[r, "h"] & a_{\mathscr{CG}}(\mathscr{X}). \end{tikzcd}$$

The outer square is Cartesian, and so is the upper-square. It follows that

$$\begin{tikzcd} T \arrow[r, "h'"] \arrow[d, "id_T"'] & \mathscr{X} \arrow[d, "\varsigma_{\mathscr{X}}"] \\ T \arrow[r, "h"] & a_{\mathscr{CG}}(\mathscr{X}) \end{tikzcd}$$

is Cartesian as well. $\square$

Corollary II.4.9. *For every quasi-topological stack $\mathscr{X}$, there exists a representable universal weak equivalence*

$$\Theta(\mathscr{X}) \to \mathscr{X},$$

from a topological space $\Theta(\mathscr{X})$.

Proof. Let $\mathscr{X}$ be a quasi-topological stack, and let $X \to a_{\mathscr{C}\mathscr{G}}(\mathscr{X})$ be a locally compact $\mathscr{C}\mathscr{G}$-atlas. Then it factors (up to equivalence) as

$$X \xrightarrow{x} \mathscr{X} \xrightarrow{\varsigma_{\mathscr{X}}} a_{\mathscr{C}\mathscr{G}}(\mathscr{X}).$$

Denote by $\mathcal{G}$ the topological groupoid

$$X \times_{\mathscr{X}} X \rightrightarrows X.$$

There is a canonical map $[\mathcal{G}] \to \mathscr{X}$ and the unit map $\varsigma_{[\mathcal{G}]}$ factors as

$$[\mathcal{G}] \to \mathscr{X} \xrightarrow{\varsigma_{\mathscr{X}}} a_{\mathscr{C}\mathscr{G}}(\mathscr{X}).$$

The composite

$$\|\mathcal{G}\| \to [\mathcal{G}] \to \mathscr{X} \xrightarrow{\varsigma_{\mathscr{X}}} a_{\mathscr{C}\mathscr{G}}(\mathscr{X})$$

is a representable quasi-shrinkable morphism, by Theorem II.4.13. From Lemma II.4.16, it follows that

$$\|\mathcal{G}\| \to [\mathcal{G}] \to \mathscr{X}$$

is a representable quasi-shrinkable morphism as well, and in particular, a representable universal weak equivalence, by Corollary II.4.6. □

Theorem II.4.17. *There exists a functor $\Omega : \mathfrak{QuasiTSt} \to \mathrm{Ho}(\mathbb{TOP})$ assigning to each quasi-topological stack $\mathscr{X}$, a weak homotopy type. Moreover, for each $\mathscr{X}$, there is a representable universal weak equivalence*

$$X \to \mathscr{X},$$

from a space X whose homotopy type is $\Omega(\mathscr{X})$. Furthermore, we can arrange for the functor Ω to restrict to the one of Theorem II.4.15 on compactly generated stacks.

Proof. Using the notation of Lemma II.4.14, let $\mathscr{C} = \mathfrak{QuasiTSt}$, $B = \mathbb{CGH}$, and let R be the class of universal weak equivalences. Using the notation of the proof of Corollary II.4.9, for each quasi-topological stack $\mathscr{X}$, let

$$\varphi(\mathscr{X}) : \Theta(\mathscr{X}) = \|\mathcal{G}\| \to [\mathcal{G}] \to \mathscr{X}.$$

The rest is identical to the proof of Theorem II.4.15. □

Remark. This agrees with the functorial construction of the weak homotopy type of topological and paratopological stacks given in [51], by construction.

Definition II.4.8. A morphism $f : \mathscr{X} \to \mathscr{Y}$ in $\mathfrak{QuasiTSt}$ is a **weak homotopy equivalence** if $\Omega(f)$ is an isomorphism.

Theorem II.4.18. *Let $\mathscr{X}$ be a quasi-topological stack. Then the unit map*

$$\varsigma_{\mathscr{X}} : \mathscr{X} \to a_{\mathscr{CG}}(\mathscr{X})$$

induces an equivalence of groupoids

$$\mathscr{X}(Y) \to a_{\mathscr{CG}}(\mathscr{X})(Y)$$

for all locally compact Hausdorff spaces Y, and $\varsigma_{\mathscr{X}}$ is a weak homotopy equivalence.

Proof. The first statement follows immediately from Proposition II.3.5. For the second, letting R denote the class of universal weak equivalences, we can factor Ω as

$$\mathfrak{QuasiTSt} \to R^{-1}\mathfrak{QuasiTSt} \xrightarrow{\Theta} R^{-1}\mathbb{CGH} \to \mathrm{Ho}(\mathbb{TOP}).$$

To show $\Omega(\varsigma_{\mathscr{X}})$ is an isomorphism, it suffices to show $\Theta(\varsigma_{\mathscr{X}})$ is. From [51], an arrow between two spaces X and Y in $R^{-1}\mathbb{CGH}$ is a span (r, g) of the form

$$\begin{tikzcd} & T \arrow[dl, "r"'] \arrow[dr, "g"] & \\ X & & Y, \end{tikzcd}$$

with r a universal weak equivalence. Moreover, if $f : \mathscr{X} \to \mathscr{Y}$ is a morphism of quasi-topological stacks, $\Theta(f)$ is given by the span (w, f') provided by the diagram

$$\begin{tikzcd} \Theta(\mathscr{X}) \times_{\mathscr{Y}} \Theta(\mathscr{Y}) \arrow[rr, "f'"] \arrow[d, "w"'] & & \Theta(\mathscr{Y}) \arrow[d, "\varphi(\mathscr{Y})"] \\ \Theta(\mathscr{X}) \arrow[r, "\varphi(\mathscr{X})"] & \mathscr{X} \arrow[r, "f"] & \mathscr{Y}. \end{tikzcd}$$

Such a span is an isomorphism if and only if f' is a universal weak equivalence. It follows that $\Theta(\varsigma_{\mathscr{X}})$ is given by the span defined by the diagram

$$\begin{tikzcd} \Theta(\mathscr{X}) \times_{\mathscr{Y}} \Theta(a_{\mathscr{CG}}(\mathscr{X})) \arrow[rr, "f'"] \arrow[d, "w"'] & & \Theta(a_{\mathscr{CG}}(\mathscr{X})) \arrow[d, "\varphi(a_{\mathscr{CG}}(\mathscr{X}))"] \\ \Theta(\mathscr{X}) \arrow[r, "\varphi(\mathscr{X})"] & \mathscr{X} \arrow[r, "\varsigma_{\mathscr{X}}"] & a_{\mathscr{CG}}(\mathscr{X}). \end{tikzcd}$$

Notice that

$$\Theta(\mathscr{X}) \xrightarrow{\varphi(\mathscr{X})} \mathscr{X} \xrightarrow{\varsigma_{\mathscr{X}}} a_{\mathscr{CG}}(\mathscr{X})$$

is a representable universal weak equivalence. It follows that f' is as well, so we are done. $\square$

Corollary II.4.10. *Let $\mathscr{X}$ be a topological or paratopological stack. Then the unit map*

$$\varsigma_{\mathscr{X}} : \mathscr{X} \to a_{\mathscr{C}\mathscr{G}}\left(\mathscr{X}\right)$$

induces an equivalence of groupoids

$$\mathscr{X}\left(Y\right) \to a_{\mathscr{C}\mathscr{G}}\left(\mathscr{X}\right)\left(Y\right)$$

for all locally compact Hausdorff spaces Y. Moreover, $\varsigma_{\mathscr{X}}$ is a weak homotopy equivalence.

In particular, to any topological stack, there is a canonically associated compactly generated stack of the same weak homotopy type which restricts to the same stack over locally compact Hausdorff spaces. Conversely, if

$$\mathscr{Y} \simeq [\mathcal{H}]_{\mathscr{C}\mathscr{G}}$$

is a compactly generated stack, $[\mathcal{H}]$ is an associated topological stack for which the same is true.

Theorem II.4.19. *Let $\mathscr{X}$ and $\mathscr{Y}$ be topological stacks such that $\mathscr{Y}$ admits a locally compact atlas. Then $\mathbb{M}\mathrm{ap}\left(\mathscr{Y}, \mathscr{X}\right)$ is a paratopological stack, $\mathbb{M}\mathrm{ap}\left(a_{\mathscr{C}\mathscr{G}}\left(\mathscr{Y}\right), a_{\mathscr{C}\mathscr{G}}\left(\mathscr{X}\right)\right)$ is a compactly generated stack, and there is a canonical weak homotopy equivalence*

$$\mathbb{M}\mathrm{ap}\left(\mathscr{Y}, \mathscr{X}\right) \to \mathbb{M}\mathrm{ap}\left(a_{\mathscr{C}\mathscr{G}}\left(\mathscr{Y}\right), a_{\mathscr{C}\mathscr{G}}\left(\mathscr{X}\right)\right).$$

Moreover, $\mathbb{M}\mathrm{ap}\left(\mathscr{Y}, \mathscr{X}\right)$ and $\mathbb{M}\mathrm{ap}\left(a_{\mathscr{C}\mathscr{G}}\left(\mathscr{Y}\right), a_{\mathscr{C}\mathscr{G}}\left(\mathscr{X}\right)\right)$ restrict to the same stack over locally compact Hausdorff spaces.

Proof. The fact that $\mathbb{M}\mathrm{ap}\left(\mathscr{Y}, \mathscr{X}\right)$ is a paratopological stack follows from Theorem II.2.9, and that $\mathbb{M}\mathrm{ap}\left(a_{\mathscr{C}\mathscr{G}}\left(\mathscr{Y}\right), a_{\mathscr{C}\mathscr{G}}\left(\mathscr{X}\right)\right)$ is a compactly generated stack follows from Theorem II.4.8. To prove the rest, it suffices to prove that

$$a_{\mathscr{C}\mathscr{G}}\left(\mathbb{M}\mathrm{ap}\left(\mathscr{X}, \mathscr{Y}\right)\right) \simeq \mathbb{M}\mathrm{ap}\left(a_{\mathscr{C}\mathscr{G}}\left(\mathscr{X}\right), a_{\mathscr{C}\mathscr{G}}\left(\mathscr{Y}\right)\right).$$

For this, it is enough to show that they restrict to the same stack over compact Hausdorff spaces. Let T be a compact Hausdorff space. Then

$$a_{\mathscr{C}\mathscr{G}}\left(\mathbb{M}\mathrm{ap}\left(\mathscr{X}, \mathscr{Y}\right)\right)$$

assigns T the groupoid

$$\mathrm{Hom}\left(T \times \mathscr{X}, \mathscr{Y}\right),$$

since it agrees with $\mathbb{M}\mathrm{ap}\left(\mathscr{X}, \mathscr{Y}\right)$ along locally compact Hausdorff spaces. From Corollary II.4.1, since $T \times \mathscr{X}$ admits a locally compact atlas, this is in turn equivalent to the groupoid

$$\mathrm{Hom}\left(a_{\mathscr{C}\mathscr{G}}\left(T \times \mathscr{X}\right), a_{\mathscr{C}\mathscr{G}}\left(\mathscr{Y}\right)\right) \simeq \mathbb{M}\mathrm{ap}\left(a_{\mathscr{C}\mathscr{G}}\left(\mathscr{X}\right), a_{\mathscr{C}\mathscr{G}}\left(\mathscr{Y}\right)\right)\left(T\right).$$

$\square$

We end this chapter by showing compactly generated stacks are to topological stacks what compactly generated spaces are to topological spaces:

Recall that there is an adjunction

$$\mathbb{CG} \underset{v}{\overset{k}{\leftrightarrows}} \mathbb{TOP},$$

exhibiting compactly generated spaces as a co-reflective subcategory of the category of topological spaces, and for any space X, the co-reflector

$$vk\,(X) \to X$$

is a weak homotopy equivalence.

We now present the 2-categorical analogue of this statement:

Theorem II.4.20. *There is a 2-adjunction*

$$\mathscr{CG}\mathfrak{TSt} \underset{v}{\overset{k}{\leftrightarrows}} \mathfrak{TSt},$$

$$v \dashv k,$$

exhibiting compactly generated stacks as a co-reflective sub-2-category of topological stacks, and for any topological stack $\mathscr{X}$, the co-reflector

$$vk\,(\mathscr{X}) \to \mathscr{X}$$

is a weak homotopy equivalence. A topological stack is in the essential image of the 2-functor

$$v : \mathscr{CG}\mathfrak{TSt} \to \mathfrak{TSt}$$

if and only if it admits a locally compact atlas.

Proof. Let us first start with the 2-functor

$$v : \mathfrak{TSt} \to \mathscr{CG}\mathfrak{TSt}.$$

We define it to be the restriction of the stackification 2-functor

$$a_{\mathscr{CG}} : Gpd^{\mathbb{CGH}^{op}} \to St_{\mathscr{CG}}\,(\mathbb{CGH})$$

to $\mathfrak{TSt}$. Note that since every open cover is a $\mathscr{CG}$-cover, for all topological groupoids $\mathcal{G}$ and $\mathcal{H}$, there is a canonical full and faithful functor

$$Bun_{\mathcal{H}}\,(\mathcal{G}) \to Bun^{\mathscr{CG}}_{\mathcal{H}}\,(\mathcal{G})\,.$$

In fact, it is literally an inclusion on the level of objects. These assemble into a homomorphism of bicategories

$$k : Bun\mathbb{CGH}Gpd \to Bun^{\mathscr{CG}}\mathbb{CGH}Gpd$$

which is the identify on objects, 1-morphisms, and 2-morphisms (but it is not 2-categorically full and faithful). Composing with the canonical equivalences,

$$\mathfrak{TSt} \xrightarrow{\sim} Bun\mathbb{CGH}Gpd \xrightarrow{k} Bun^{\mathscr{CG}}\mathbb{CGH}Gpd \xrightarrow{\sim} \mathscr{CG}\mathfrak{TSt}$$

is a factorization (up to equivalence) of $a_{\mathscr{CG}}|_{\mathfrak{TSt}}$. We will construct a left 2-adjoint to k. For all topological groupoids $\mathcal{G}$, let $v(\mathcal{G})$ be the Čech groupoid $\mathcal{G}_{\mathcal{K}}$ of $\mathcal{G}$ with respect to the $\mathscr{CG}$-cover of $\mathcal{G}_0$ by all its compact subsets. In particular, $v(\mathcal{G})$ has a locally compact object space. The canonical map $\mathcal{G}_{\mathcal{K}} \to \mathcal{G}$ is a $\mathscr{CG}$-Morita equivalence. Denote the associated $\mathscr{CG}$-principal $\mathcal{G}$-bundle over $\mathcal{G}_{\mathcal{K}}$ by $1^k{}_{\mathcal{G}}$. Since $(\mathcal{G}_{\mathcal{K}})_0$ is locally compact, $1^k{}_{\mathcal{G}}$ is an ordinary principal bundle

$$1^k_{\mathcal{G}} \in Bun_{\mathcal{G}}(v(\mathcal{G}))_0 .$$

Since $\mathcal{G}_{\mathcal{K}} \to \mathcal{G}$ is a $\mathscr{CG}$-Morita equivalence, there exists a $\mathscr{CG}$-principal $\mathcal{G}_{\mathcal{K}}$-bundle over $\mathcal{G}$, $r_{\mathcal{G}}$ and isomorphisms

$$\begin{aligned} \alpha_{\mathcal{G}} : r_{\mathcal{G}} \otimes 1^k{}_{\mathcal{G}} &\xrightarrow{\sim} id_{\mathcal{G}} \\ \beta_{\mathcal{G}} : id_{\mathcal{G}_{\mathcal{K}}} &\xrightarrow{\sim} 1^k{}_{\mathcal{G}} \otimes r_{\mathcal{G}}. \end{aligned}$$

The assignment $\mathcal{G} \mapsto v(\mathcal{G}) = \mathcal{G}_{\mathcal{K}}$ extends to a homomorphism of bicategories

$$v : Bun^{\mathscr{CG}}\mathbb{CGH}Gpd \longrightarrow Bun\mathbb{CGH}Gpd,$$

by

$$\begin{aligned} v_{\mathcal{H},\mathcal{G}} : Bun^{\mathscr{CG}}_{\mathcal{G}}(\mathcal{H})_0 &\to Bun_{v(\mathcal{G})}(v(\mathcal{H}))_0 \\ P &\mapsto r_{\mathcal{H}} \otimes P \otimes 1^k{}_{\mathcal{G}}, \end{aligned}$$

and similarly on 2-cells. Note that for $\mathcal{G}$ and $\mathcal{H}$ topological groupoids, there is a natural equivalence of groupoids

$$\begin{aligned} \mathrm{Hom}(v(\mathcal{G}), \mathcal{H}) &= Bun_{\mathcal{H}}(\mathcal{G}_{\mathcal{K}}) \\ &= Bun^{\mathscr{CG}}_{\mathcal{H}}(\mathcal{G}_{\mathcal{K}}) \text{ (since } \mathcal{G}_{\mathcal{K}} \text{ has locally compact object space)} \\ &\simeq Bun^{\mathscr{CG}}_{\mathcal{H}}(\mathcal{G}) \text{ (since } \mathcal{G}_{\mathcal{K}} \text{ is } \mathscr{CG}\text{-Morita equivalent to } \mathcal{G}) \\ &= \mathrm{Hom}(\mathcal{G}, k(\mathcal{H})), \end{aligned}$$

which sends a principal $\mathcal{G}$-bundle P over $\mathcal{H}_{\mathcal{K}}$ to $P \otimes r_{\mathcal{G}}$. An inverse for this equivalence is given by sending

$$P \in Bun^{\mathscr{CG}}_{\mathcal{G}}(\mathcal{H})$$

to $P \otimes 1^k_{\mathcal{H}}$. These equivalences define an adjunction of bicategories, $v \dashv k$. The unit of this adjunction is given by

$$r_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}_{\mathcal{K}} = kv(\mathcal{G}) \text{ in } Bun^{\mathscr{CG}}\mathbb{CGH}Gpd,$$

which is an equivalence. It follows that v is bicategorically full and faithful. The co-unit is given by

$$1^{k}{}_{\mathcal{G}} : vk\left(\mathcal{G}\right) = \mathcal{G}_{\mathcal{K}} \to \mathcal{G} \text{ in } Bun\mathbb{CGH}Gpd.$$

By abuse of notation, denote by v and k the induced adjunction

$$\mathscr{C}\mathscr{G}\mathfrak{T}\mathfrak{S}\mathfrak{t} \underset{v}{\overset{k}{\leftrightarrows}} \mathfrak{T}\mathfrak{S}\mathfrak{t}.$$

The 2-functor v sends a compactly generated stack equivalent to $[\mathcal{G}]_{\mathscr{C}\mathscr{G}}$ to $[\mathcal{G}_{\mathcal{K}}]$, which has a locally compact atlas. From the general theory of adjunctions, the essential image is precisely the sub-2-category of topological stacks over which $k = a_{\mathscr{C}\mathscr{G}}$ restricts to a full and faithful 2-functor. We have already shown that the essential image is contained in those topological stacks which admit a locally compact atlas. By Corollary II.4.2, $a_{\mathscr{C}\mathscr{G}}$ restricted to this sub-2-category is full and faithful, hence the essential image of v is topological stacks which admit a locally compact atlas.

It remains to show that the co-unit is a weak homotopy equivalence. Let $[\mathcal{G}]$ be a topological stack. Then the co-unit is given by the canonical map

$$\varepsilon_{[\mathcal{G}]} : [\mathcal{G}_K] \to [\mathcal{G}].$$

Notice that the following diagram is 2-commutative:

$$\begin{tikzcd} {[\mathcal{G}_{\mathcal{K}}]} \arrow[r, "\varepsilon_{[\mathcal{G}]}"] \arrow[dr, "\varsigma_{[\mathcal{G}_{\mathcal{K}}]}"'] & {[\mathcal{G}]} \arrow[d, "\varsigma_{[\mathcal{G}]}"] \\ & a_{\mathscr{C}\mathscr{G}}\left([\mathcal{G}]\right). \end{tikzcd}$$

By Corollary II.4.10, the maps $\varsigma_{[\mathcal{G}]}$ and $\varsigma_{[\mathcal{G}_{\mathcal{K}}]}$ are weak homotopy equivalences. It follows that so is $\varepsilon_{[\mathcal{G}]}$. $\square$

Chapter III

Small Sheaves, Stacks, and Gerbes over Étale Topological and Differentiable Stacks

This chapter is the main body of my preprint [15], which is currently posted on arXiv. The research was conducted during the final two years of my PhD studies. It is included in its entirety, other than some of the preliminaries and one of the appendices, both of which have been incorporated into Chapter I.

Motivation Differentiable stacks show up naturally in many contexts. I was first led to their study through foliation theory [44]. Roughly speaking, a foliation of codimension q of a n-dimensional smooth manifold M is a smooth partitioning of M into (immersed) q-codimensional connected submanifolds called leaves. Foliation theory is intimately linked with the theory of étale differentiable stacks. Étale differentiable stacks are those differentiable stacks all of whose points have discrete automorphism groups. This includes all orbifolds. Given any submersion $f : M \to N$ between manifolds, the connected components of the fibers of f form a foliation of M. It is not true that every foliation arises this way since any foliation of this form has no holonomy. However, this is *almost* true since every foliation on M arises from a submersion $f : M \to \mathscr{X}$, where $\mathscr{X}$ is allowed to be an étale differentiable stack. One can even make a stronger statement, namely, that for every positive integer q, there exists a q-dimensional étale stack Γ^q, called the Haefliger stack, which is universal for foliations of codimension q in the sense that for any M, there is a bijective correspondence between isomorphism classes of submersions $f : M \to \Gamma^q$ and foliations of codimension q on M [41].

Any étale stack $\mathscr{X}$ of dimension q has a canonical map $\mathscr{X} \to \Gamma^q$, and if

$$f : M \to \mathscr{X}$$

is a submersion, the foliation induced on M from f is the one classified by the composite

$$M \to \mathscr{X} \to \Gamma^q.$$

If $\mathscr{X}$ is effective, then all the information in $\mathscr{X}$ is encoded in its image in Γ^q, so all the information in $f : M \to \mathscr{X}$ is encoded in the composite $M \to \mathscr{X} \to \Gamma^q$, i.e. in the induced foliation. However, if $\mathscr{X}$ is not effective, then some information is lost, which means that a submersion $f : M \to \mathscr{X}$ induces more structure on M than just a foliation. The question as to the nature of this extra structure is what led to this chapter of my thesis.

In this chapter I show that étale stacks are the same as effective étale stacks equipped with a small gerbe. This implies that the extra structure, besides a foliation, induced on M from a submersion $M \to \mathscr{X}$, is a gerbe which is compatible with the foliation. The theory of gerbed foliations and their holonomy will be explored in another paper; we will not elaborate these connections to foliation theory any further in this thesis.

Although the motivation for this chapter stemmed from foliation theory and trying to give meaning to the ineffective data of étale stacks, the theoretical framework developed is much more far-reaching; this chapter develops the theory of small sheaves and stacks over étale topological and differentiable stacks. Recall that for a topological space X, a *small sheaf* over X is a sheaf over its category of open subsets, $\mathcal{O}(X)$, where the arrows are inclusions. The corresponding topos is denoted as $Sh(X)$. This is in contrast to the topos of *large sheaves* which is the slice topos $Sh(\mathbb{TOP})/X$, where $\mathbb{TOP}$ is the category of topological spaces. For small sheaves over X, there is an étalé space construction. That is, there is a pair of adjoint functors

$$\mathrm{Set}^{\mathcal{O}(X)^{op}} \underset{L}{\overset{\Gamma}{\leftrightarrows}} \mathbb{TOP}/X\ .$$

Here, L takes a presheaf to its étalé space and Γ takes a space $T \to X$ over X to its sheaf of sections. This adjunction restricts to an equivalence

$$Sh(X) \underset{L}{\overset{\Gamma}{\leftrightarrows}} Et(X)\ ,$$

between the category of small sheaves over X, and the category of local homeomorphisms over X. In this chapter, we extend this result to the setting of small stacks of groupoids over étale topological and differentiable stacks

For an étale topological or differentiable stack $\mathscr{X}$, we can define a small sheaf in a similar way as for spaces, by finding an appropriate substitute for a Grothendieck site of open subsets. Sheaves over this site are what we call small sheaves over $\mathscr{X}$. Similarly, stacks and gerbes over this site are what we call small stacks and gerbes over $\mathscr{X}$. This is in contrast with the 2-topos of *large stacks* over $\mathscr{X}$, which is the slice 2-topos

$$\mathrm{St}(\mathbb{TOP})/\mathscr{X},$$

in the case of topological stacks and

$$\mathrm{St}\,(Mfd)\,/\,\mathscr{X}\,,$$

in the case of differentiable stacks, where *Mfd* is the category of smooth manifolds.

Small sheaves, stacks, and gerbes need to be distinguished from their large counterparts. This distinction is highlighted in [38]. A small sheaf, stack, or gerbe over a space or stack should be thought of as algebraic data attached to that space or stack, whereas a large sheaf, stack, or gerbe should be thought of as a geometric object "sitting over it". It should be noted that nearly all applications in the literature of gerbes in differential geometry are applications of *large* gerbes, moreover large gerbes with band $U(1)$, so-called bundle-gerbes (see e.g. [47, 9]). Not every large gerbe is a small gerbe, nor is every large gerbe a bundle gerbe. To the author's knowledge, there has been, as of yet, little application of small gerbes in differentiable geometry or topology. However, the classification of extensions of regular Lie groupoids given in [43] may be interpreted in terms of small gerbes over étale stacks. Nonetheless, there are plenty of examples of small gerbes right under everyone's noses, in the disguise of ineffective data: e.g. every ineffective orbifold gives rise to a small gerbe as does any almost-free action of a Lie group on a manifold. One aim of this chapter is to establish the technical tools necessary to begin the study of these objects.

III.1 Introduction

The purpose of this chapter is to extend the theory of small sheaves of sets over spaces to a theory of small stacks of groupoids over étale topological and differentiable stacks. We provide a construction analogous to the étalé space construction in this context and establish an equivalence of 2-categories between small stacks over an étale stack and local homeomorphisms over it. This theory provides an interpretation of the ineffective data of any orbifold, or more generally of any étale stack, as a small gerbe over its effective part: we show that any étale stack $\mathscr{X}$ encodes a small gerbe over its effective part $\mathrm{Eff}\,(\mathscr{X})$, and moreover, every small gerbe over an effective étale stack $\mathscr{Y}$ arises uniquely from some étale stack $\mathscr{Z}$ whose effective part is equivalent to $\mathscr{Y}$.

Étale stacks are an important class of stacks as they include all orbifolds, and more generally, all "stacky leaf-spaces" of foliated manifolds. The passage from spaces to étale stacks is a natural one as such a passage circumvents many obstructions to geometric problems. For example, it is not true that every foliation of a manifold M arises from a submersion $f : M \to N$ of manifolds, however, it is true that every foliation on M arises from a submersion $M \to \mathscr{X}$, where $\mathscr{X}$ is allowed to be an étale differentiable stack [41].

Similarly, it is not true that every Lie algebroid over a manifold M integrates to a Lie groupoid $\mathcal{G} \rightrightarrows M$, [18], however it is true when the arrow space $\mathcal{G}$ is allowed to be an étale differentiable stack [59]. Étale stacks are also a natural setting to consider small sheaves (and more generally small stacks), as the results of [54] imply that étale stacks are faithfully represented by their topos of small sheaves.

III.1.1 Small gerbes and ineffective isotropy data

Besides establishing a theory of small sheaves and stacks over étale stacks, this chapter unravels the mystery behind "ineffective data" of étale stacks. Suppose that G is a finite group acting on a manifold M. The "stacky-quotient" $M//G$ is an étale differentiable stack, and in particular, an orbifold. Points of this stacky-quotient are the same as points of the naive quotient, that is, orbits of the action. These are precisely images of points of M under the quotient map $M \to M//G$. For a particular point $x \in M$, if $[x]$ denotes the point in $M//G$ which is its image, then

$$Aut\left([x]\right) \cong G_x.$$

If this action is not faithful, then there exists a non-trivial kernel K of the homomorphism

$$\rho : G \to Diff\left(M\right). \tag{III.1}$$

In this case, any element k of K acts trivially and is "tagged-along" as extra data in the automorphism group

$$Aut\left([x]\right) \cong G_x$$

of each point $[x]$ of the stack $M//G$. In fact,

$$\bigcap_{x \in M} \left(G\right)_x = Ker\left(\rho\right).$$

In particular, ρ restricted to $Aut\left([x]\right)$ becomes a homomorphism

$$\left(\rho\right)_x : Aut\left([x]\right) \to Diff\left(M\right)_x \tag{III.2}$$

to the group of diffeomorphisms of M which fix x. This homomorphism is injective for all x if and only if the kernel of ρ is trivial. The kernel of each of these homomorphisms is the "inflated" part of each automorphism group, and is called the *ineffective isotropy group* of $[x]$. Up to the identification

$$Aut\left([x]\right) \cong G_x,$$

each of these ineffective isotropy groups is K. This extra information is stripped away when considering the stacky-quotient

$$M//\left(G/K\right),$$

that is to say, $M//\left(G/K\right)$ is the *effective part* of $M//G$.

Hence, having a kernel to the action (III.1) serves merely for "artificially inflating" each automorphism group. As an extreme example, suppose the action ρ is trivial. The stacky quotient $M//G$ is basically the same thing as M except each point x, instead of having a trivial automorphism group, has G as an automorphism group. These automorphisms are somehow "artificial", since the action ρ sees nothing of G. In this case, the entire automorphism group of each point is its ineffective isotropy group, and this is an example of a purely ineffective orbifold.

Since these arguments are local, the situation when $\mathscr{X}$ is an étale stack formed by gluing together stacks of the form $M_\alpha//G_\alpha$ for actions of finite groups, i.e. a general orbifold, is completely analogous. For a more general étale stack $\mathscr{X}$, for example a stack of the form $M//G$ where G is discrete but not finite, there is no such local-action of the automorphisms groups as in (III.2), but the situation can be mimicked at the level of germs:

There exists a manifold V and a (representable) local homeomorphism

$$V \to \mathscr{X}$$

such that for every point

$$x : * \to \mathscr{X},$$

i) the point x factors (up to isomorphism) as $* \xrightarrow{\tilde{x}} V \xrightarrow{p} \mathscr{X}$, and

ii) there is a canonical homomorphism $\tilde{\rho}_x : Aut\left(x\right) \to Diff_{\tilde{x}}\left(V\right),$

where $Diff_{\tilde{x}}\left(V\right)$ is the group of germs of locally defined diffeomorphisms of V that fix $\tilde{x}$. The kernel of each of these maps is again the "inflated" part of the automorphism group. In the case where $\mathscr{X}$ is of the form $M//G$ for a finite group G, or more generally, when $\mathscr{X}$ is an orbifold, for each x the kernel of $\tilde{\rho}_x$ is the same as the kernel of (III.2). In general, each $Ker\left(\tilde{\rho}_x\right)$ is called an *ineffective isotropy group*. Unlike in the case of a global quotient $M//G$, these groups need not be isomorphic for different points of the stack. However, these kernels may be killed off to obtain the so-called *effective part* of the étale stack.

There is another way of trying to artificially inflate the automorphism groups, and this is through gerbes. As a starting example, if M is a manifold, a gerbe over M is a stack $\mathscr{G}$ over M such that over each point x of M, the stalk $\mathscr{G}_x$ is equivalent to a group. From such a gerbe, one can construct an étale stack which looks just like M except each point x, now instead of having a trivial automorphism group, has (a group equivalent to) $\mathscr{G}_x$ as its automorphism group. This construction was eluded to in [27]. One can use this construction to show that étale stacks whose effective parts are manifolds are the same thing as manifolds equipped with a gerbe. In this chapter, I show that this result extends to general étale stacks, namely that any étale stack $\mathscr{X}$ encodes a small gerbe (in the sense I define in this chapter) over its effective part $\mathrm{Eff}(\mathscr{X})$, and moreover, every small gerbe over an effective étale stack $\mathscr{Y}$ arises uniquely from some étale stack $\mathscr{Z}$ whose effective part is equivalent to $\mathscr{Y}$. The construction of an étale stack $\mathscr{Z}$ out of an effective étale stack $\mathscr{Y}$ equipped with a small gerbe $\mathscr{G}$, is precisely the étalé realization of the gerbe $\mathscr{G}$, which is a 2-categorical analogue of the étalé space construction for sheaves which I develop in this chapter. In such a situation, there is a natural bijection between the points of $\mathscr{Z}$ and the points of $\mathscr{Y}$, the only difference being that points of $\mathscr{Z}$ have more automorphisms. For x a point $\mathscr{Z}$, its ineffective isotropy group, i.e. the kernel of

$$Aut\,(x) \to Diff_{\tilde{x}}\,(V)\,,$$

is equivalent to the stalk $\mathscr{G}_x$.

III.1.2 Organization and main results

Section III.2 starts by briefly recalling the basic definitions of étale topological and differentiable stacks. It is then explained how to associate to any stack a canonical topos of small sheaves in a functorial way. In case the stack in question is a topological or differentiable stack presented by a groupoid $\mathcal{G}$, this topos is equivalent to the classifying topos $\mathcal{B}\mathcal{G}$ as defined in [39]. It is then shown how the results of [54] imply that étale stacks are faithfully represented by their topos of small sheaves. Following [32], we associate to every (atlas for an) étale stack a canonical small site of definition for its topos of small sheaves. We define small stacks to be stacks over this site. We then give an abstract description of a generalized étalé space construction in this setting, which we call étalé realization.

As a demonstration of the abstract machinery developed in this section, we also prove a tangential (yet highly interesting) theorem to the effect that, in some sense, topological stacks subsume Grothendieck topoi, once we replace the role of topological spaces with that of locales:

Theorem III.1.1. *There is a 2-adjunction*

$$\mathfrak{Top} \underset{\mathscr{S}}{\overset{Sh}{\rightleftarrows}} \mathfrak{LocSt},$$

exhibiting the bicategory of topoi (with only invertible 2-cells) as a reflective subbicategory of localic stacks (stacks coming from localic groupoids).

Section III.3 aims at giving a concrete description of the abstract construction given in Section III.2. For this, we choose to represent small stacks by groupoid objects in the topos of small sheaves. We then show how a known generalization of the classical action groupoid construction can be used to give a concrete model for the étalé realization of small stacks. As a consequence, we prove:

Theorem III.1.2. *For any étale topological or differentiable stack $\mathscr{X}$, there is an adjoint-equivalence of 2-categories*

$$\mathrm{St}(\mathscr{X}) \underset{L}{\overset{\Gamma}{\rightleftarrows}} Et(\mathscr{X}),$$

between small stacks over $\mathscr{X}$ and the 2-category of étale stacks over $\mathscr{X}$ via a local homeomorphism.

Here L is the étalé realization functor, and Γ is the "stack of sections" functor. We also determine which local homeomorphisms over $\mathscr{X}$ correspond to sheaves:

Theorem III.1.3. *A local homeomorphism $f : \mathscr{Z} \to \mathscr{X}$ over an étale stack $\mathscr{X}$ is equivalent to the étalé realization of a small sheaf F over $\mathscr{X}$ if and only if it is a representable map.*

Section III.4 provides a concrete model for the "stack of sections" functor Γ in terms of groupoid objects in the topos of small sheaves.

In Section III.5, we introduce the concept of an effective étale stack and show how to associate to every étale stack $\mathscr{X}$ an effective étale stack $\mathrm{Eff}(\mathscr{X})$, which we call its effective part. Although this construction is not functorial for all maps, we show that it is functorial for any category of open maps which is étale invariant, a concept which we define. Examples of open étale invariant classes of maps include open maps, local homeomorphisms, and submersions.

The subject of Section III.6 is the classification of small gerbes. For $\mathscr{X}$ an effective étale stack, the answer is quite nice:

Theorem III.1.4. *For an effective étale stack $\mathscr{X}$, a local homeomorphism*

$$f : \mathscr{G} \to \mathscr{X}$$

is equivalent to the étalé realization of a small gerbe over $\mathscr{X}$ if and only if

$$\mathrm{Eff}(f) : \mathrm{Eff}(\mathscr{G}) \to \mathrm{Eff}(\mathscr{X}) \simeq \mathscr{X}$$

is an equivalence.

For a general étale stack, the theorem is as follows:

Theorem III.1.5. *For an étale stack $\mathscr{X}$, a local homeomorphism*

$$f : \mathscr{G} \to \mathscr{X}$$

is equivalent to the étalé realization of a small gerbe over $\mathscr{X}$ if and only if

i) $\operatorname{Eff}(f) : \operatorname{Eff}(\mathscr{G}) \to \operatorname{Eff}(\mathscr{X}) \simeq \mathscr{X}$ *is an equivalence, and*

ii) for every space T, the induced functor $\mathscr{G}(T) \to \mathscr{X}(T)$ is full.

We also prove in this section that the étalé realization of any small gerbe over an étale differentiable stack is, in particular, a differentiable gerbe in the sense of [9].

In Section III.7, we introduce the 2-category of gerbed effective étale stacks. The objects of this 2-category are effective étale stacks equipped with a small gerbe. We then show that when restricting to open étale invariant maps, this 2-category is equivalent to étale stacks. In particular, we prove:

Corollary III.1.1. *There is an equivalence of 2-categories between gerbed effective étale differentiable stacks and submersions, $\mathscr{Gerbed}\left(\mathfrak{EffEt}\right)_{subm}$, and the 2-category of étale differentiable stacks and submersions, $\mathfrak{Et}_{subm}$.*

III.2 Small Sheaves and Stacks over Étale Stacks

III.2.1 Conventions and notations concerning stacks

Throughout this chapter, S shall denote a fixed category whose objects we shall call "spaces". S shall always be assumed to be either (sober) topological spaces, or smooth manifolds, unless otherwise noted. We will employ a minimalist definition of smooth manifold in that manifolds will neither be assumed paracompact nor Hausdorff. This is done in order to consider the étalé space (espace étalé) of a sheaf over a manifold as a manifold itself. In this chapter, the term (local) homeomorphism will mean (local) *diffeo*morphism if S is the category of manifolds. Similarly, for terms such as *continuous*.

In this chapter, we will use the following definition of Lie groupoid:

Definition III.2.1. A **Lie groupoid** is a groupoid object in (possibly non-Hausdorff) smooth manifolds such that the source and target maps are submersions.

Remark. In Chapter I, Lie groupoids are required to have a Hausdorff object space, however, as every manifold is locally Hausdorff, any Lie groupoid in the sense we defined is Morita equivalent to one that meets this requirement. See definition I.2.20. (In this chapter, we will work with the open-cover Grothendieck topology, unless otherwise stated.)

Definition III.2.2. An S-groupoid $\mathcal{G}$ is **étale** if its source-map s (and therefore also its target map t) is a local homeomorphism.

Definition III.2.3. A topological or differentiable stack $\mathscr{X}$ is **étale** if it is equivalent to $[\mathcal{G}]$ for some étale S-groupoid $\mathcal{G}$.

Proposition III.2.1. *A stack $\mathscr{X}$ over S is étale if and only if it admits an étale atlas $p : X \to \mathscr{X}$, that is a representable epimorphism which is also étale, i.e. a local homeomorphism. (See Definition I.2.24.)*

Proof. This follows from the fact that if $\mathcal{G}$ is any S-groupoid, the following diagram is 2-Cartesian:

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, "t"] \arrow[d, "s"'] & \mathcal{G}_0 \arrow[d] \\ \mathcal{G}_0 \arrow[r] & {[\mathcal{G}]}, \end{tikzcd}$$

where the map $\mathcal{G}_0 \to [\mathcal{G}]$ is induced from the canonical map $\mathcal{G}_0 \to \mathcal{G}$. □

Definition III.2.4. By an **étale cover** of a space X, we mean a surjective local homeomorphism $U \to X$. In particular, for any open cover (U_α) of X, the canonical projection

$$\coprod_\alpha U_\alpha \to X$$

is an étale cover.

Definition III.2.5. Let $\mathcal{H}$ be an S-groupoid. If $\mathcal{U} = U \to \mathcal{H}_0$ is an étale cover of $\mathcal{H}_0$, then one can define the **Čech-groupoid** $\mathcal{H}_{\mathcal{U}}$. Its objects are U and the arrows fit in the pullback diagram

$$\begin{tikzcd} (\mathcal{H}_{\mathcal{U}})_1 \arrow[r] \arrow[d, "(s,t)"'] & \mathcal{H}_1 \arrow[d, "(s,t)"] \\ U \times U \arrow[r] & \mathcal{H}_0 \times \mathcal{H}_0, \end{tikzcd}$$

and the groupoid structure is induced from $\mathcal{H}$. There is a canonical map $\mathcal{H}_{\mathcal{U}} \to \mathcal{H}$ which is a Morita equivalence. Moreover,

$$\operatorname{Hom}\left([\mathcal{H}], [\mathcal{G}]\right) \simeq \underset{\mathcal{U} \in Cov(\mathcal{H}_0)}{\underrightarrow{\operatorname{holim}}} \operatorname{Hom}_{S-Gpd}\left(\mathcal{H}_{\mathcal{U}}, \mathcal{G}\right), \tag{III.3}$$

where the weak 2-colimit above is taken over a suitable 2-category of étale covers. For details see [24].

Remark. We could restrict to open covers, and a similar statement would be true. However, it will become convenient to work with étale covers later.

Applying equation (III.3) to the case where $[\mathcal{H}]$ is a space X, by the 2-Yoneda Lemma we have

$$[\mathcal{G}]\,(X) \simeq \underset{\mathcal{U} \in Cov(X)}{\underrightarrow{\operatorname{holim}}} \operatorname{Hom}_{S-Gpd}\left(X_{\mathcal{U}}, \mathcal{G}\right).$$

We end by a standard fact which we will find useful later:

Proposition III.2.2. *For any stack $\mathscr{X}$ on S, there is a canonical equivalence of 2-categories* $\operatorname{St}(S/\mathscr{X}) \simeq \operatorname{St}(S)/\mathscr{X}$*.*

The construction is as follows:
Given $\mathscr{Y} \to \mathscr{X}$ in $\operatorname{St}(S)/\mathscr{X}$, consider the stack

$$\tilde{\mathscr{Y}}(T \to \mathscr{X}) := \operatorname{Hom}_{\operatorname{St}(S)/\mathscr{X}}\left(T \to \mathscr{X}, \mathscr{Y} \to \mathscr{X}\right).$$

Given a stack $\mathscr{W}$ in $\operatorname{St}(S/\mathscr{X})$, consider it as a fibered category $\int \mathscr{W} \to S/\mathscr{X}$. Then since $S/\mathscr{X} \simeq \int \mathscr{X}$ (as categories), the composition $\int \mathscr{W} \to \int \mathscr{X} \to S$ is a category fibered in groupoids presenting a stack $\tilde{\mathscr{W}}$ over S, and since the diagram

$$\begin{tikzcd} \int \mathscr{W} \arrow[d] \arrow[dr] & \\ \int \mathscr{X} \arrow[r] & S \end{tikzcd}$$

commutes, $\int \mathscr{W} \to \int \mathscr{X}$ corresponds to a map of stacks $\tilde{\mathscr{W}} \to \mathscr{X}$.

We leave the rest to the reader.

III.2.2 Locales and frames

Our conventions on locales and frames closely follow [30]. Recall that the category of **frames** has as objects complete Heyting algebras, which are complete lattices of a certain kind, and morphism are given by functions which preserve finite meets and arbitrary joins. The category of **locales** is dual to that of frames. Locales are generalized spaces and find their home in the domain of so-called pointless topology. See for example [28].

Definition III.2.6. Given a topological space X, we denote its poset of open subsets by $\mathcal{O}(X)$. The poset $\mathcal{O}(X)$ together with intersection and union, forms a complete Heyting algebra, hence a locale.

Notice that a continuous map $f : X \to Y$ induces a map

$$\begin{array}{rcl} \mathcal{O}(Y) & \to & \mathcal{O}(X) \\ U & \mapsto & f^{-1}(U). \end{array}$$

It is easy to see that this is a map of frames, hence, is a map

$$\mathcal{O}(f) : \mathcal{O}(X) \to \mathcal{O}(Y)$$

in the category of locales. This makes $\mathcal{O}$ into a functor

$$\mathcal{O} : Top \to Locales.$$

In fact, this functor has a right adjoint

$$pt : Locales \to Top.$$

The adjoint-pair $\mathcal{O} \dashv pt$ restricts to an equivalence between sober topological spaces, and locales with enough points (both "sober" and "with enough points" have a precise mathematical meaning). This result is known as Stone duality. The class of sober spaces is quite large in practice. It includes many highly non-Hausdorff topological spaces such as the prime spectrum with the Zariski topology, $Spec(A)$, for a commutative ring A.

Note that the open-cover Grothendieck topology on topological spaces naturally extends to locales. We make the following definitions:

Definition III.2.7. A **localic groupoid** is a groupoid object in *Locales*. A **localic stack** is a stack $\mathscr{X}$ on the site of locales with the open cover topology, such that $\mathscr{X} \simeq [\mathcal{G}]$, for some localic groupoid $\mathcal{G}$. We denote the 2-category of localic stacks by $\mathfrak{LocSt}$.

III.2.3 Small sheaves as a Kan extension

Let $\mathfrak{Top}$ denote the bicategory of Grothendieck topoi, geometric morphisms, and *invertible* natural transformations, as in I.1.6. There is a canonical functor

$$S \to \mathfrak{Top},$$

which assigns each space X its topos of sheaves $Sh(X)$. By (weak) left Kan extension, we obtain a 2-adjoint pair $Sh \dashv \mathscr{S}$

$$Gpd^{S^{op}} \underset{Sh}{\overset{\mathscr{S}}{\rightleftarrows}} \mathfrak{Top},$$

where $Gpd^{S^{op}}$ denotes the bicategory of weak presheaves in groupoids. In fact, the essential image of $\mathscr{S}$ lies entirely within the bicategory of stacks over S, $\mathrm{St}(S)$, where S is equipped with the standard "open cover" Grothendieck topology [13]. So, by restriction, we obtain an adjoint pair

$$\mathrm{St}(S) \underset{Sh}{\overset{\mathscr{S}}{\rightleftarrows}} \mathfrak{Top} \tag{III.4}$$

Definition III.2.8. For $\mathscr{X}$ a stack over S, we define the topos of **small sheaves** over $\mathscr{X}$ to be the topos $Sh(\mathscr{X})$.

Suppose that $\mathscr{X} \simeq [\mathcal{G}]$ for a groupoid object $\mathcal{G}$ in spaces (S-groupoid for short). Then we may consider the nerve $N(\mathcal{G})$ as a simplicial S-object

$$\mathcal{G}_0 \leftleftarrows \mathcal{G}_1 \,\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\, \mathcal{G}_2 \cdots .$$

By composition with the Yoneda embedding, we obtain a simplicial stack

$$y \circ N(\mathcal{G}) : \Delta^{op} \to \mathrm{St}(S).$$

The weak colimit of this diagram is the stack $[\mathcal{G}]$. Since Sh is a left adjoint, it follows that $Sh([\mathcal{G}])$ is the weak colimit of the simplicial-topos

$$Sh(\mathcal{G}_0) \leftleftarrows Sh(\mathcal{G}_1) \,\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\, Sh(\mathcal{G}_2) \cdots .$$

From [39], it follows that $Sh([\mathcal{G}]) \simeq \mathcal{B}\mathcal{G}$, the classifying topos of $\mathcal{G}$. We will return to a more concrete description of the classifying topos later.

For the rest of this subsection, we will assume that S is sober topological spaces, or locales, unless otherwise noted.

The adjoint pair $Sh \dashv \mathscr{S}$ restricts to an equivalence between, on one hand, the subbicategory of $\mathrm{St}(S)$ for which the unit is an equivalence, and, on the other hand, the subbicategory of $\mathfrak{Top}$ on which the co-unit is an equivalence.

Proposition III.2.3. *If $\mathscr{X}$ is an étale stack, then the unit is an equivalence.*

Proof. Let T be a space, then

$$\mathscr{S}(Sh(\mathscr{X}))(T) = \operatorname{Hom}(Sh(T), Sh(\mathscr{X})),$$

and the latter is the groupoid of geometric morphisms from $Sh(T)$ to $\mathcal{BG}$, where $\mathcal{G}$ is some groupoid representing $\mathscr{X}$. From, [39] this in turn is equivalent to $\mathscr{X}(T)$. $\square$

Let $\mathfrak{Et}$ denote the full subbicategory of $\operatorname{St}(S)$ consisting of étale stacks. Then, since the unit restricted to $\mathfrak{Et}$ is an equivalence, Sh restricted to $\mathfrak{Et}$ is 2-categorically fully faithful. We now identify its essential image.

Definition III.2.9. A topos $\mathcal{E}$ is an **étendue** if there exists a well-supported object $E \in \mathcal{E}$ (i.e. $E \to 1$ is an epimorphism) such that the slice topos $\mathcal{E}/E$ is equivalent to $Sh(X)$ for some space X.

Theorem III.2.1. *A topos $\mathcal{E}$ is an étendue if and only if $\mathcal{E} \simeq \mathcal{BG}$ for some étale groupoid $\mathcal{G}$ [1].*

Corollary III.2.1. *Sh induces an equivalence between the bicategory of étale stacks and the bicategory of étendues.*

Remark. This result was original proven in [54].

This corollary should be interpreted as evidence that for étale groupoids $\mathcal{G}$, $Sh([\mathcal{G}]) = \mathcal{BG}$ is the correct notion for the topos of sheaves over $[\mathcal{G}]$ since just as for spaces, morphisms between étale stacks are the same as geometric morphisms between their topoi of sheaves.

Corollary III.2.2. *Let $\mathscr{X} \simeq [\mathcal{G}]$ and $\mathscr{Y} \simeq [\mathcal{H}]$ be two stacks with $\mathscr{Y}$ étale. Then*

$$\operatorname{Hom}(\mathscr{X}, \mathscr{Y}) \simeq \operatorname{Hom}(\mathcal{BG}, \mathcal{BH}).$$

The adjoint pair $Sh \dashv \mathscr{S}$ allows us also to prove another interesting result, which we shall now do, for completeness.

Definition III.2.10. An S-groupoid $\mathcal{G}$ is **étale-complete** if the diagram

$$\begin{tikzcd} Sh(\mathcal{G}_1) \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\swarrow \mu"] & Sh(\mathcal{G}_0) \arrow[d, "p"] \\ Sh(\mathcal{G}_0) \arrow[r, "p"'] & \mathcal{BG} \end{tikzcd}$$

is a (weak) pullback-diagram of topoi, where $\mathcal{BG}$ is the classifying topos of $\mathcal{G}$, p is induced from the inclusion $\mathcal{G}_0 \to \mathcal{G}$, and μ is induced by the obvious action of $\mathcal{G}$ on sheaves over $\mathcal{G}_0$. For details, see [39].

A stack $\mathscr{X}$ over S is **étale-complete** if it is equivalent to $[\mathcal{G}]$ for some étale-complete $\mathcal{G}$.

Remark. Every étale-groupoid is étale-complete [39].

Remark. Proposition III.2.3 and its proof remains valid if étale is replaced with étale-complete.

Let $\mathfrak{EtC}$ denote the full subbicategory consisting of étale-complete stacks. Sh restricted to $\mathfrak{EtC}$ is also 2-categorically fully faithful. For S sober-topological spaces, to the author's knowledge, there is no nice description for the essential image. However, for S locales, the answer is quite nice indeed:

Theorem III.2.2. *For S locales, Sh induces an equivalence between the bicategory of étale-complete stacks and the bicategory $\mathfrak{Top}$ of topoi. In particular,*

$$\mathscr{S} : \mathfrak{Top} \to St\,(S)$$

exhibits Grothendieck topoi as a reflective full subbicategory of stacks on locales.

Proof. It suffices to show that Sh is essentially surjective. Every topos is equivalent to $\mathcal{BG}$ for some localic groupoid $\mathcal{G}$ [30], and hence to $Sh\,(\mathscr{X})$ for some localic stack $\mathscr{X}$ over locales. The result now follows from the fact that every localic groupoid $\mathcal{G}$ has an étale-completion $\hat{\mathcal{G}}$ such that $\mathcal{BG} \simeq \mathcal{B}\hat{\mathcal{G}}$ [39]. ☐

Corollary III.2.3. *The adjunction $(III.4)$ restricts to a adjunction*

$$\mathfrak{Top} \underset{\mathscr{S}}{\overset{Sh}{\rightleftarrows}} \mathfrak{LocSt},$$

exhibiting the 2-category of topoi as a reflective subbicategory of localic stacks.

Remark. In light of the fact that every topos $\mathcal{E}$ with enough points is equivalent to $\mathcal{BG}$ for some topological groupoid $\mathcal{G}$ [14], one may be tempted to claim that étale-complete topological stacks are equivalent to topoi with enough points. However, the proof just given does not work for the topological case as a topological groupoid's étale-completion may not be a topological groupoid, but only a groupoid object in locales.

Remark. Most of what has been done in this subsection caries over for smooth manifolds if we use ringed-topoi rather than just topoi. In particular, the result of Pronk that étale differentiable stacks and smooth-étendues are equivalent can be proven along these lines.

III.2.4 The classifying topos of a groupoid

Recall the following definition from Section I.2.5:

Definition III.2.11. Given an S-groupoid $\mathcal{H}$, a (left) **$\mathcal{H}$-space** is a space E equipped with a **moment map** $\mu : E \to \mathcal{H}_0$ and an **action map**

$$\rho : \mathcal{H}_1 \times_{\mathcal{H}_0} E \to E,$$

where

$$\begin{tikzcd} \mathcal{H}_1 \times_{\mathcal{H}_0} E \arrow[r] \arrow[d] & E \arrow[d, "\mu"] \\ \mathcal{H}_1 \arrow[r, "s"] & \mathcal{H}_0 \end{tikzcd}$$

is the fibred product, such that the following conditions hold:

i) $(gh) \cdot e = g \cdot (h \cdot e)$ whenever e is an element of E and g and h elements of $\mathcal{H}_1$ with domains such that the composition makes sense,

ii) $\mathbb{1}_{\mu(e)} \cdot e = e$ for all $e \in E$, and

iii) $\mu (g \cdot e) = t (g)$ for all $g \in \mathcal{H}_1$ and $e \in E$.

A map of $\mathcal{H}$-spaces is simply an equivariant map, i.e., a map

$$(E, \mu, \rho) \to (E', \mu', \rho')$$

is map $f : (E, \mu,) \to (E', \mu')$ in $S/\mathcal{H}_0$ such that

$$f(he) = hf(e)$$

whenever this equation makes sense.

Definition III.2.12. An $\mathcal{H}$-space E is an **$\mathcal{H}$-equivariant sheaf** if the moment map μ is a local homeomorphism. The category of $\mathcal{H}$-equivariant sheaves and equivariant maps forms the **classifying topos** $\mathcal{BH}$ of $\mathcal{H}$.

III.2.5 The small-site of an étale stack

Definition III.2.13. Let $\mathcal{H}$ be an étale S-groupoid. Let $Site(\mathcal{H})$ be the following category: The objects are the open subsets of $\mathcal{H}_0$. An arrow $U \to V$ is a section σ of the source-map $s : \mathcal{H}_1 \to \mathcal{H}_0$ over U such that $t \circ \sigma : U \to V$ as a map in S. Composition is by the formula $\tau \circ \sigma(x) := \tau (t (\sigma(x)) .$

There is a canonical functor $i : \mathcal{O}(\mathcal{H}_0) \hookrightarrow Site(\mathcal{H})$ which sends an inclusion $U \hookrightarrow V$ in $\mathcal{O}(\mathcal{H}_0)$ to $\mathbb{1}|_U$, where $\mathbb{1}$ is the unit map of the groupoid, and $\mathcal{O}$ is as in Definition III.2.6.

This functor induces a Grothendieck pre-topology on $Site(\mathcal{H})$ by declaring covering families to be images under i of covering families of $\mathcal{O}(\mathcal{H}_0)$. The Grothendieck site $Site(\mathcal{H})$ equipped with the induced topology is called the **small site** of the groupoid $\mathcal{H}$.

Remark. Given an étale stack $\mathscr{X}$ with an étale atlas $X \to \mathscr{X}$, we can describe $Site\left(X \times_{\mathscr{X}} X \rightrightarrows X\right)$ in terms of this stack and atlas as follows. The objects are open subsets of X and the arrows are pairs (f, α), such that

$$\begin{tikzcd} U \arrow[rrrr, "f"] \arrow[dr, hook] \arrow[rrd, phantom, "\Leftarrow\alpha"] & & & & V . \arrow[dl, hook'] \\ & X \arrow[dr] & & X \arrow[dl] & \\ & & \mathscr{X} & & \end{tikzcd}$$

In other words, it is the full subcategory of $\mathrm{St}(S)/\mathscr{X} \simeq \mathrm{St}(S/\mathscr{X})$ (Proposition III.2.2) spanned by objects of the form $U \hookrightarrow X \to \mathscr{X}$, with $U \subseteq X$ open. To see this, let $X \times_{\mathscr{X}} X \rightrightarrows X = \mathcal{H}$. Given a section σ of s over U, we can associate to it the map

$$\begin{array}{rcl} \alpha(\sigma) : U & \to & \mathcal{H}_1 \\ x & \mapsto & \sigma(x)^{-1} . \end{array}$$

Then, letting

$$f := t \circ \sigma : U \to V,$$

$\alpha(\sigma) : U \to \mathcal{H}_1$ is a continuous natural transformation from

$$U^{id} \xrightarrow{f} V^{id} \hookrightarrow \mathcal{H}_0 \to \mathcal{H}$$

to

$$U \hookrightarrow X = \mathcal{H}_0 \to \mathcal{H}.$$

Definition III.2.14. Given an object $U \subset \mathcal{H}_0$ of $Site(\mathcal{H})$, the space $s^{-1}(U)$ comes equipped with a canonical left $\mathcal{H}$-action along the target map t. Since the target map is a local homeomorphism, this $\mathcal{H}$-space is in fact an equivariant sheaf. We denote it by m_U.

Extend this to a functor as follow:

Given $\sigma : U \to V$ in $Site(\mathcal{H})$, define a map $f : s^{-1}(U) \to s^{-1}(V)$ by sending $x \xrightarrow{h} y$ to $t(\sigma(x)) \xrightarrow{\sigma(x)^{-1}} x \xrightarrow{h} y$. It is easy to see this is an $\mathcal{H}$-equivariant map and that it induces a bijection

$$Hom_{Site(\mathcal{H})}(U, V) \cong Hom_{\mathcal{BH}}(m_U, m_V).$$

Hence we get a full and faithful functor $m : Site(\mathcal{H}) \to \mathcal{BH}$.

Proposition III.2.4. *The left Kan extension of m along the Yoneda embedding*

$$y : Site\,(\mathcal{H}) \to Sh((Site\,(\mathcal{H}))$$

is an equivalence between the topos of sheaves for the Grothendieck site $Site\,(\mathcal{H})$, and the classifying topos $\mathcal{BH}$ [32].

Definition III.2.15. By a **small stack** over an étale stack $\mathscr{X} \simeq [\mathcal{H}]$, we mean a stack $\mathscr{Z}$ over $Site\,(\mathcal{H})$. We denote the 2-category of small stacks over $\mathscr{X}$ by $\mathrm{St}\,(\mathscr{X})$.

Remark. This definition does not depend on the choice of presenting groupoid since, if $\mathcal{G}$ is another groupoid such that $[\mathcal{G}] \simeq \mathscr{X}$, then

$$Sh\,(Site\,(\mathcal{G})) \simeq \mathcal{BG} \simeq \mathcal{BH} \simeq Sh\,(Site\,(\mathcal{H}))$$

and hence $\mathrm{St}\,(Site\,(\mathcal{G})) \simeq \mathrm{St}\,(Site\,(\mathcal{H}))$ by the Comparison Lemma for stacks [1]. A more intrinsic definition would be that a small stack over $\mathscr{X}$ is a stack over the topos $Sh\,(\mathscr{X})$, which of course agrees [25]. Even better, since we are dealing with étale stacks, in light of Corollary III.2.1, we may instead work with the bicategory of étendues. Then, a small stack over an etendue $\mathcal{E}$ is precisely a stack over $\mathcal{E}$ in the sense of Giraud in [25] (that is a stack over $\mathcal{E}$ with respect to the canonical Grothendieck topology, which in this case is generated by jointly epimorphic families).

III.2.6 The étalé realization of a small stack

Recall that for a sheaf F over a space X, the étalé space (espace étalé) is a space $E \to X$ over X via a local homeomorphism (or étale map), such that the sheaf of sections of $E \to X$ is isomorphic to F. In fact, the étalé space can be constructed for any presheaf, and the corresponding sheaf of sections is isomorphic to its sheafification. As a set, E is the disjoint union of the stalks of F and the topology is induced by local sections.

Abstractly, this construction may be carried out as follows:

Consider the category of open subsets of X, $\mathcal{O}\,(X)$, where the arrows are inclusions, as in Definition III.2.6. This category, equipped with its natural Grothendieck topology, is of course the site over which "sheaves over X" are sheaves. There is a canonical functor $j : \mathcal{O}\,(X) \to S/X$ which sends an open $U \subseteq X$ to $U \hookrightarrow X$. Hence, there is an induced adjunction

$$\mathrm{Set}^{\mathcal{O}(X)^{op}} \underset{L}{\overset{\Gamma}{\rightleftarrows}} S/X\,.$$

Here, L takes a presheaf to its étalé space and Γ takes a space $T \to X$ over X to its sheaf of sections. The composite $\Gamma \circ L$ is isomorphic to the sheafification functor $a : \mathrm{Set}^{\mathcal{O}(X)} \to Sh\,(X)$, and the image of L lies completely in the subcategory $Et\,(X)$ of S/X spanned by spaces over X via a local homeomorphism. When restricted to $Sh\,(X)$ and $Et\,(X)$, the adjoint pair $L \dashv \Gamma$ is an equivalence of categories

$$Sh\,(X) \underset{L}{\overset{\Gamma}{\rightleftarrows}} Et\,(X)\,.$$

This construction can be done even more topos-theoretically as follows:

The canonical functor $j : \mathcal{O}\,(X) \to S/X$ produces three adjoint functors $j_! \dashv j^* \dashv j_*$

$$Sh\,(X) \rightleftarrows Sh\,(S/X)\,,$$

where the Grothendieck topology on S/X is induced from the open cover topology on S. For a sheaf F over X, $j_!\,(X) = y\,(L\,(F))$, where y denotes the Yoneda embedding $y : S/X \hookrightarrow Sh\,(S/X)$.

Hence,

$$y \circ L : \mathrm{Set}^{\mathcal{O}(X)^{op}} \to Sh\,(S/X)$$

can be identified with the left Kan extension of

$$\mathcal{O}\,(X) \xrightarrow{j} S/X \overset{y}{\hookrightarrow} Sh\,(S/X)$$

along Yoneda.

We now turn our attention to generalizing this construction to work when both X and F are stacks. Let $\mathcal{H}$ be an étale groupoid and let $\mathscr{X} \simeq [\mathcal{H}]$. In light of the remark after Definition III.2.13, there is a canonical fully faithful functor $j_{\mathcal{H}} : Site\,(\mathcal{H}) \to S/\mathscr{X}$ which sends $U \subseteq \mathcal{H}_0$ to $U \hookrightarrow \mathcal{H}_0 \to \mathscr{X}$. This produces three adjoint functors $j_! \dashv j^* \dashv j_*$

$$Gpd^{Site(\mathcal{H})^{op}} \rightleftarrows \mathrm{St}\,(S/\mathscr{X})\,.$$

We denote $j_!$ by L and j^* by Γ.

More explicitly, $j_!$ is the weak left Kan extension of $j_{\mathcal{H}}$ along Yoneda, and

$$\Gamma(\mathscr{Y})(U) = \mathrm{Hom}_{\mathrm{St}(S/\mathscr{X})}\,(y\,(U \hookrightarrow \mathcal{H}_0 \to \mathscr{X}\,)\,, \mathscr{Y}\,)\,.$$

Remark. Under the equivalence given in Proposition III.2.2, $\Gamma(f : \mathscr{Y} \to \mathscr{X}\,)$ assigns an open subset U of $\mathcal{H}_0$ the groupoid of "sections of f over U".

Definition III.2.16. Let $\mathscr{Z}$ be a weak presheaf in groupoids over $Site\,(\mathcal{H})$. Then $L\,(\mathscr{Z})$ is the **étalé realization** of $\mathscr{Z}$.

Proposition III.2.5. *Let $\mathscr{Y}$ be any stack in* $\mathrm{St}\,(S/\mathscr{X})$. *Then $\Gamma\,(\mathscr{X}\,)$ is a stack.*

Proof. This is immediate from the fact that $\mathscr{Y}$ satisfies descent. □

In fact, we can say more:

Theorem III.2.3. *The 2-functor $\Gamma \circ L$ is equivalent to the stackification 2-functor $a : Gpd^{Site(\mathcal{H})^{op}} \to \mathrm{St}\,(Site\,(\mathcal{H})) \simeq \mathrm{St}(\mathscr{X})$.*

Proof. Suppose $\mathscr{Z}$ is a weak presheaf in groupoids over $Site\,(\mathcal{H})$. Then

$$\Gamma\,(\mathscr{Z})\,(V) \simeq L\,(\mathscr{Z})\,(V \hookrightarrow \mathcal{H}_0 \to \mathscr{X})\,.$$

Let $G\,(\mathscr{Z})$ be the weak presheaf in groupoids over $S/\mathscr{X}$ given by

$$G\,(\mathscr{Z}) \simeq \underset{U \to \mathscr{Z}}{\underrightarrow{\mathrm{holim}}}\, y\,(U \hookrightarrow \mathcal{H}_0 \to \mathscr{X})\,.$$

Then $\Gamma L\,(\mathscr{Z})\,(V) \simeq a\,(G\,(\mathscr{Z}))\,(V \hookrightarrow \mathcal{H}_0 \to \mathscr{X})$, where a is stackification.

Note:

$$\begin{aligned}
G\,(\mathscr{Z})\,(W \hookrightarrow \mathcal{H}_0 \to \mathscr{X}) &\simeq \underset{U \to \mathscr{Z}}{\underrightarrow{\mathrm{holim}}}\; \mathrm{Hom}_{\mathrm{St}(S/\mathscr{X})}\,(y\,(W \hookrightarrow \mathcal{H}_0 \to \mathscr{X})\,, y\,(U \hookrightarrow \mathcal{H}_0 \to \mathscr{X}))\\
&\simeq \underset{U \to \mathscr{Z}}{\underrightarrow{\mathrm{holim}}}\; \mathrm{Hom}_{S/\mathscr{X}}\,(W \hookrightarrow \mathcal{H}_0 \to \mathscr{X}, U \hookrightarrow \mathcal{H}_0 \to \mathscr{X})\\
&\simeq \underset{U \to \mathscr{Z}}{\underrightarrow{\mathrm{holim}}}\; \mathrm{Hom}_{Site(\mathcal{H})}\,(W, U)\\
&\simeq \left(\underset{U \to \mathscr{Z}}{\underrightarrow{\mathrm{holim}}}\, y\,(U))\right)(W)\\
&\simeq \mathscr{Z}\,(W)\,.
\end{aligned}$$

Given any weak presheaf in groupoids $\mathscr{W}$ over a Grothendieck site $(\mathscr{C}, J)$, we define $\mathscr{W}^+$ by

$$\mathscr{W}^+\,(C) = \underset{(C_i \to C)_i}{\underrightarrow{\mathrm{holim}}}\; \underleftarrow{\mathrm{holim}} \left[\prod_i \mathscr{W}\,(C_i) \rightrightarrows \prod_{i,j} \mathscr{W}\,(C_{ij}) \rightrightarrows\!\!\!\!\!\to \prod_{i,j,k} \mathscr{W}\,(C_{ijk})\right].$$

Recall from Section I.1.7 that $a\,(\mathscr{W}) = \mathscr{W}^{+++}$. Now,

$$\begin{aligned}
G\,(\mathscr{Z})^+\,(j_{\mathcal{H}}\,(V)) &= \underset{(V_i \hookrightarrow V)_i}{\underrightarrow{\mathrm{holim}}}\; \underleftarrow{\mathrm{holim}} \left[\prod_i G\,(j_{\mathcal{H}}\,(V_i)) \rightrightarrows \prod_{i,j} G\,(j_{\mathcal{H}}\,(V_{ij})) \rightrightarrows\!\!\!\!\!\to \prod_{i,j,k} G\,(j_{\mathcal{H}}\,(V_{ijk}))\right]\\
&\simeq \underset{(V_i \hookrightarrow V)_i}{\underrightarrow{\mathrm{holim}}}\; \underleftarrow{\mathrm{holim}} \left[\prod_i \mathscr{Z}\,(V_i) \rightrightarrows \prod_{i,j} \mathscr{Z}\,(V_{ij}) \rightrightarrows\!\!\!\!\!\to \prod_{i,j,k} \mathscr{Z}\,(V_{ijk})\right]\\
&\simeq \mathscr{Z}^+\,(V)\,.
\end{aligned}$$

Hence

$$\begin{aligned}\Gamma L(\mathscr{Z})(V) &\simeq a(G(\mathscr{Z}))(V \hookrightarrow \mathcal{H}_0 \to \mathscr{X}) \\ &\simeq (G(\mathscr{Z}))^{+++}(V \hookrightarrow \mathcal{H}_0 \to \mathscr{X}) \\ &\simeq \mathscr{Z}^{+++}(V) \\ &\simeq a(\mathscr{Z})(V).\end{aligned}$$

□

Corollary III.2.4. *The adjunction $L \dashv \Gamma$ restricts to an adjunction*

$$\mathrm{St}(\mathscr{X}) \underset{\tilde{L}}{\overset{\tilde{\Gamma}}{\leftrightarrows}} \mathrm{St}(S/\mathscr{X}),$$

where $\tilde{L}$ and $\tilde{\Gamma}$ denote the restrictions. This furthermore restricts to an adjoint-equivalence

$$\mathrm{St}(\mathscr{X}) \underset{\bar{L}}{\overset{\bar{\Gamma}}{\leftrightarrows}} \mathscr{E}ss(L),$$

equivalence between $\mathrm{St}(\mathscr{X})$ *and its essential image under L.*

The first part of this Corollary is clear. In general, a 2-adjunction restricts to an equivalence between, on one hand, those objects for which the component of the unit is an equivalence, and on the other hand, those objects for which the component of the co-unit is an equivalence. Hence, it suffices to prove that the essential image of L is the same as the essential image of $\bar{L}$. In fact, we will prove more, namely:

Proposition III.2.6. *Suppose $\mathscr{Z}$ is a weak presheaf of groupoids over Site $(\mathcal{H})$. Then $L(\mathscr{Z}) \simeq L(a(\mathscr{Z}))$.*

Proof. $\tilde{L} \circ a$ and L are both weak colimit preserving and agree on representables. □

Remark. If $\mathscr{X}$ is equivalent to a space X, then this construction generalizes the étalé space construction from sheaves over X to stacks over X (in the ordinary sense). In the particular case when the stack over X is a sheaf of sets, then its étalé realization is its (Yoneda-embedded) étalé space.

III.3 A Concrete Description of Étalé Realization

The construction given for the étalé realization of a small stack over an étale stack, as of now, is rather abstract, since it is given as a weak left Kan extension. In order to work with this construction, we wish to give a more concrete description of it. To accomplish this, it is useful first to have a more concrete hold on how to represent these small stacks themselves.

For a general Grothendieck site $(\mathscr{C}, J)$, one way of representing stacks is by groupoid objects in sheaves. Given a groupoid object $\mathbb{G}$ in $Sh(\mathscr{C})$, it defines a strict presheaf of groupoids by assigning an object C of $\mathscr{C}$ the groupoid

$$\mathrm{Hom}_{Gpd(Sh(\mathscr{C}))}\left(y(C)^{id}, \mathbb{G}\right),$$

where $y(C)^{id}$ is the groupoid object in sheaves with objects $y(C)$ (Yoneda) and with only identity arrows. This strict presheaf is a sheaf of groupoids. In fact, there is an equivalence of 2-categories between groupoid objects in sheaves, and sheaves of groupoids. Moreover, every stack on $(\mathscr{C}, J)$ is equivalent to the stackification of such a strict presheaf arising from a groupoid object in sheaves. For details see Section I.1.8.

In our case, we have a nice description of the category of sheaves on $Site(\mathcal{H})$, namely, it is the classifying topos $\mathcal{BH}$ of equivariant sheaves. Hence, we can model small stacks over $[\mathcal{H}]$ by groupoid objects in $\mathcal{H}$-equivariant sheaves.

Note that the following is an (immediate) corollary of Lemma A.2.2:

Corollary III.3.1. *If*

$$j : Psh(Site(\mathcal{H}), Gpd) \to Gpd^{Site(\mathcal{H})^{op}}$$

is the "inclusion" of strict-presheaves of groupoids into weak ones, and

$$i : Sh(Site(\mathcal{H}), Gpd) \to Psh(Site(\mathcal{H}), Gpd)$$

is the inclusion of sheaves of groupoids into presheaves, then

$$\bar{L} \circ a \circ j \circ i : Sh(Site(\mathcal{H}), Gpd) \to \mathrm{St}(S/\mathscr{X})$$

is weak colimit preserving, where a denotes stackification and $\bar{L}$ is as in Corollary III.2.4.

Notice that any sheaf of groupoids is a weak colimit of representables. Hence, the composite $\bar{L} \circ a \circ j \circ i$ is uniquely determined (up to equivalence) by its values on representables, plus the fact that it is weak colimit preserving (i.e. it is also a weak left Kan extension along Yoneda). Our plan is to describe an explicit candidate for the realization functor $Sh(Site(\mathcal{H}), Gpd) \to \mathrm{St}(S/\mathscr{X})$, and show that it is weak colimit preserving and agrees with $\bar{L} \circ a \circ j \circ i$ on representables. For this, we will need a generalization of the action groupoid construction, which is the subject of the next subsection.

III.3.1 Generalized action groupoids

Definition III.3.1. Let $\mathcal{H}$ be any S-groupoid and let $\mathcal{K}$ be a groupoid object in $\mathcal{H}$-spaces. In particular we have two $\mathcal{H}$-spaces $(\mathcal{K}_0, \mu_0, \rho_0)$ and $(\mathcal{K}_1, \mu_1, \rho_1)$ which are the underlying objects and arrows of $\mathcal{K}$. Note that the source map

$$s : (\mathcal{K}_1, \mu_1, \rho_1) \to (\mathcal{K}_0, \mu_0, \rho_0)$$

and target map

$$t : (\mathcal{K}_1, \mu_1, \rho_1) \to (\mathcal{K}_0, \mu_0, \rho_0)$$

are maps $s, t : (\mathcal{K}_1, \mu_1,) \to (\mathcal{K}_0, \mu_0,)$ in $S/\mathcal{H}_0$, hence $\mu_0 \circ s = \mu_0 \circ t = \mu_1$. Similarly for other structure maps.

We define an S-groupoid $\mathcal{H} \ltimes \mathcal{K}$ as follows:

The space of **objects** of $\mathcal{H} \ltimes \mathcal{K}$ is $\mathcal{K}_0$. An **arrow** from x to y is a pair (h, k) with $h \in \mathcal{H}_1$ and $k \in \mathcal{K}_1$ such that $k : hx \to y$ (which implicitly means that $s(h) = \mu_0(x)$). We denote such an arrow pictorially as

$$x \overset{h}{\dashrightarrow} hx \xrightarrow{k} y.$$

In other words, $(\mathcal{H} \ltimes \mathcal{K})_1$ is the fibered product $\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{K}_1$:

$$\begin{tikzcd} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{K}_1 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{K}_1 \arrow[d, "\mu_1"] \\ \mathcal{H}_1 \arrow[r, "t"] & \mathcal{H}_0, \end{tikzcd}$$

and the **source** and **target** maps are given by

$$s\,(h, k) = h^{-1} s\,(k)$$

and

$$t\,(h, k) = t\,(k)\,.$$

We need to define **composition**. Suppose we have two composable arrows:

$$x \overset{h}{\dashrightarrow} hx \xrightarrow{k} t(k) \overset{h'}{\dashrightarrow} h't(k) \xrightarrow{k'} t(k').$$

Notice that $\mu_1\,(k) = \mu_0\,(t(k))$ so that h' can act on k. So we get an arrow $h' \cdot k : (h'h)\,x \to h't(k)$. We define the composition to be

$$x \overset{h'h}{\dashrightarrow} h'hx \xrightarrow{k'(h' \cdot k)} y.$$

In other words

$$(h', k') \circ (h, k) := (h'h, k' \circ (h' \cdot k))\,.$$

The **unit** map $\mathcal{K}_0 \to (\mathcal{H} \ltimes \mathcal{K})_1$ is given by

$$x \mapsto \left(\mathbb{1}_{\mu_0(x)}, \mathbb{1}_x\right),$$

and the **inverse** map is given by

$$(h,k)^{-1} := \left(h^{-1}, h^{-1} \cdot k^{-1}\right).$$

Notice that if $\mathcal{K}$ is actually an $\mathcal{H}$-space E considered as a groupoid object with only identity morphisms, then $\mathcal{H} \ltimes \mathcal{K}$ is the usual action groupoid $\mathcal{H} \ltimes E$. Hence, we call $\mathcal{H} \ltimes \mathcal{K}$ the **generalized action groupoid** of $\mathcal{K}$, or simply the action groupoid.

Remark. This construction is known. It appears, for example, in [44] under the name *semi-direct product.*

Notice that each action groupoid $\mathcal{H} \ltimes \mathcal{K}$ comes equipped with a canonical morphism $\theta_{\mathcal{K}} : \mathcal{H} \ltimes \mathcal{K} \to \mathcal{H}$ given by

$$(\theta_{\mathcal{K}})_0 = \mu_0 : \mathcal{K}_0 \to \mathcal{H}_0$$

and

$$(\theta_{\mathcal{K}})_1 = pr_1 : (\mathcal{H} \ltimes \mathcal{K})_1 = \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{K}_1 \to \mathcal{H}_1.$$

The following proposition is immediate:

Proposition III.3.1. *If $\mathcal{H}$ is étale and $\mathcal{K}$ is in fact a groupoid object in $\mathcal{H}$-equivariant sheaves, then $\mathcal{H} \ltimes \mathcal{K}$ is étale and the components of $\theta_{\mathcal{K}}$ are local homeomorphisms.*

Remark. Each groupoid object $\mathcal{K}$ in $\mathcal{H}$-spaces has an underlying S-groupoid $\underline{\mathcal{K}}$ and there is a canonical map $\tau_{\mathcal{K}} : \underline{\mathcal{K}} \to \mathcal{H} \ltimes \mathcal{K}$ given by the identity morphism on $\mathcal{K}_0$ and on arrows by

$$k \mapsto \left(\mathbb{1}_{\mu_1(k)}, k\right).$$

Definition III.3.2. Let $(S - Gpd)/\mathcal{H}$ denote the 2-category of S-groupoids over $\mathcal{H}$. It has **objects** homomorphisms $\varphi : \mathcal{G} \to \mathcal{H}$ of S-groupoids. A **morphism**

$$\left(\mathcal{G} \overset{\varphi}{\searrow} \mathcal{H}\right) \to \left(\mathcal{L} \overset{\psi}{\searrow} \mathcal{H}\right)$$

is a pair (f, α) with $f : \mathcal{G} \to \mathcal{L}$ and a 2-cell $\alpha : \psi \circ f \Rightarrow \varphi$. A **2-cell**

$$(f, \alpha) \Rightarrow (f', \alpha')$$

between two morphisms f and f' with domains and codomains

$$\left(\mathcal{G} \overset{\varphi}{\searrow} \mathcal{H}\right)$$

and

$$\left(\mathcal{L} \overset{\psi}{\searrow} \mathcal{H}\right)$$

is given by a 2-cell

$$\omega : f \Rightarrow f'$$

such that

$$\alpha' \psi \omega = \alpha.$$

We will show that the action groupoid construction

$$\mathcal{K} \mapsto \left((\mathcal{H} \ltimes \mathcal{K}) \overset{\theta_{\mathcal{K}}}{\searrow} \mathcal{H} \right)$$

extends to a 2-functor

$$\mathcal{H} \ltimes : Gpd\left(\mathcal{H} - spaces\right) \to \left(S - Gpd\right) / \mathcal{H}.$$

Suppose $\varphi : \mathcal{K} \to \mathcal{L}$ is a homomorphism of groupoid objects in $\mathcal{H}$-spaces. Then we can define $\mathcal{H} \ltimes (\varphi) : \mathcal{H} \ltimes \mathcal{K} \to \mathcal{H} \ltimes \mathcal{L}$ **on objects** as φ_0 and **on arrows** by

$$(h, k) \mapsto (h, \varphi(k)),$$

which strictly commutes over $\mathcal{H}$. Finally, for 2-**cells**, given an internal natural transformation

$$\alpha : \varphi \Rightarrow \psi$$

between two homomorphisms

$$\mathcal{K} \to \mathcal{L},$$

α is in particular a map of $\mathcal{H}$-spaces $\alpha : \mathcal{K}_0 \to \mathcal{L}_1$. It is easily checked that $(\tau_{\mathcal{L}})_1 \circ \alpha : \mathcal{K}_0 \to (\mathcal{H} \ltimes \mathcal{L})_1$ encodes a 2-cell

$$\mathcal{H} \ltimes (\alpha) : \mathcal{H} \ltimes (\varphi) \Rightarrow \mathcal{H} \ltimes (\psi),$$

where τ is as in the remark directly proceeding Proposition III.3.1 We leave it to the reader to check that this is a strict 2-functor.

Remark. This restricts to a 2-functor

$$\mathcal{H} \ltimes : Gpd\left(\mathcal{B}\mathcal{H}\right) \to \left(S^{et} - Gpd\right) / \mathcal{H},$$

where S^{et} denotes the category whose objects are spaces and arrows are all local homeomorphisms.

Let us now define a strict 2-functor in the other direction,

$$P : \left(S - Gpd\right) / \mathcal{H} \to Gpd\left(\mathcal{H} - spaces\right).$$

On objects:

Let $\varphi : \mathcal{G} \to \mathcal{H}$ be a map of S-groupoids. Consider the associated principal $\mathcal{H}$-bundle over $\mathcal{G}$. Its total space is $\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0$, where

$$\begin{tikzcd}
\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{G}_0 \arrow[d, "\varphi_0"] \\
\mathcal{H}_1 \arrow[r, "s"] & \mathcal{H}_0
\end{tikzcd}$$

is a pullback diagram. Together its projection $pr_2 : \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \to \mathcal{G}_0$, it is a right $\mathcal{G}$-space with action given by

$$(h, x)\, g := (h\varphi\,(g)\,, s(g))\,.$$

We define

$$\underline{P}\,(\varphi) := (\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0) \rtimes \mathcal{G},$$

that is, the right action groupoid of the underlying $\mathcal{G}$-space of the associated principal bundle of φ. Since the left $\mathcal{H}$-action and right $\mathcal{G}$-action on $\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0$ commute, this becomes a groupoid object in $\mathcal{H}$-spaces. Explicitly, the objects of $P\,(\varphi)$ are $\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0$ equipped with the obvious left $\mathcal{H}$-action along $s \circ pr_1$ given by

$$h'\,(h, x) = (h'h, x)\,.$$

The arrows are the fibered product

$$\begin{tikzcd}
\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_1 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{G}_1 \arrow[d, "\varphi_0 \circ t"] \\
\mathcal{H}_1 \arrow[r, "s"] & \mathcal{H}_0,
\end{tikzcd}$$

equipped with an analogously defined left $\mathcal{H}$-action along $s \circ pr_1$. The source and target maps are defined by

$$s\,(h, g) = (h\varphi(g), s(g))\,,$$

and

$$t\,(h, g) = (h, t(g))\,.$$

Composition is defined by $(h', g') \circ (h, g) = (h', g'g)$. The unit map is defined by

$$(h, x) \mapsto (h, \mathbb{1}_x)\,.$$

Inverses are given by

$$(h, g)^{-1} = \left(h, g^{-1}\right).$$

The following proposition is immediate:

Proposition III.3.2. *If $\mathcal{H}$ is étale and both φ_0, and φ_1 are local homeomorphisms (equivalently, φ_0 is a local homeomorphism), then $P\,(\varphi)$ is a groupoid object in $\mathcal{BH}$.*

On arrows:

Suppose we are given an arrow

$$(f,\alpha):\left(\mathcal{G}\overset{\varphi}{\searrow}\mathcal{H}\right)\to\left(\mathcal{L}\overset{\psi}{\searrow}\mathcal{H}\right).$$

We wish now to define an internal functor $P\left((f,\alpha)\right)$. On objects define it by:

$$P\left((f,\alpha)\right)(h,x)=\left(h\alpha(x),f(x)\right).$$

On arrows define it by

$$P\left((f,\alpha)\right)(h,g)=\left(h\varphi(g)\alpha\left(s(g)\right)\psi\left(f(g)\right)^{-1},f(g)\right).$$

On 2-cells

Suppose we are given a 2-cell $\omega:(f,\alpha)\Rightarrow(f',\alpha')$ between two maps

$$\left(\mathcal{G}\overset{\varphi}{\searrow}\mathcal{H}\right)\to\left(\mathcal{L}\overset{\psi}{\searrow}\mathcal{H}\right).$$

Define an internal natural transformation

$$P\left(\omega\right):P\left((f,\alpha)\right)\Rightarrow P\left((f',\alpha')\right)$$

by

$$P\left(\omega\right)(h,x)=\left(h\alpha(x),\omega(x)\right).$$

We leave it to the reader to check that P is indeed a strict 2-functor.

Lemma III.3.1. *There exists a natural transformation $\varepsilon:\mathcal{H}\ltimes P\Rightarrow id_{(S-Gpd)/\mathcal{H}}$ whose components are equivalences.*

Proof. Given $\varphi:\mathcal{G}\to\mathcal{H}$, consider the left-action of $\mathcal{H}\times\mathcal{G}$ on

$$\mathcal{H}_1\times_{\mathcal{H}_0}\mathcal{G}_0=P\left(\varphi\right)_0$$

defined by

$$(l,g)\cdot(h,x):=\left(lh\varphi(g)^{-1},t(g)\right).$$

Consider

$$\theta_{P(\varphi)}:\left(\mathcal{H}\times\mathcal{G}\right)\ltimes\left(\mathcal{H}_1\times_{\mathcal{H}_0}\mathcal{G}_0\right)\to\left(\mathcal{H}\times\mathcal{G}\right)$$

where $\theta_{P(\varphi)}$ is the canonical morphism.

By direct inspection, we see that $\mathcal{H}\ltimes P\left(\varphi\right)$ is canonically isomorphic to

$$\tilde{\theta}_{P(\varphi)}:=pr_1\circ\theta_{P(\varphi)}.$$

Consider the map

$$\tilde{\epsilon}_{\varphi} := pr_2 \circ \theta_{P(\varphi)} : (\mathcal{H} \times \mathcal{G}) \ltimes (\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0) \to \mathcal{G}.$$

Let $\xi_{\varphi} : \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \to \mathcal{H}_1$ be defined by $\xi_{\varphi} = pr_1$. Then ξ_{φ} is a natural isomorphism from $\varphi \circ \tilde{\epsilon}_{\varphi}$ to $\tilde{\theta}_{P(\varphi)}$. Hence $(\tilde{\epsilon}_{\varphi}, \xi_{\varphi})$ is a morphism in $(S - Gpd)/\mathcal{H}$ from $\tilde{\theta}_{P(\varphi)}$ to φ. It is easy to check that

$$\epsilon : \mathcal{H} \ltimes P \circ \Rightarrow id_{(S-Gpd)/\mathcal{H}}$$

defined by

$$\varepsilon(\varphi) = (\tilde{\varepsilon}_{\varphi}, \xi_{\varphi}),$$

is a strict natural transformations of 2-functors. It remains to see that its components consist of equivalences.

Define $\chi_{\varphi} : \mathcal{G} \to (\mathcal{H} \times \mathcal{G}) \ltimes (\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0)$ on objects by

$$\chi_{\varphi}(x) = \left(\mathbb{1}_{\varphi(x)}, x\right),$$

and on arrows by

$$\chi_{\varphi}(g) = \left(\left(\mathbb{1}_{\varphi(s(g))}, s(g)\right), (\varphi(g), g)\right).$$

Then

$$\tilde{\varepsilon}_{\varphi} \circ \chi_{\varphi} = id_{\mathcal{G}}.$$

Note that $\tilde{\theta}_{P(\varphi)} \circ \xi_{\varphi} = \varphi$ so that ξ_{φ} is a morphism in $(S - Gpd)/\mathcal{H}$.

Define

$$\lambda_{\varphi} : \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G} \to (\mathcal{H} \times \mathcal{G}) \ltimes (\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0)_1$$

by

$$\lambda_{\varphi}(h, x) = \left(\left(\mathbb{1}_{\varphi(x)}, x\right), (h, \mathbb{1}_x)\right).$$

Then λ_{φ} encodes a 2-cell $id_{\mathcal{H} \ltimes P(\varphi)} \Rightarrow \chi_{\varphi} \circ \varepsilon_{\varphi}$. □

Corollary III.3.2. *The 2-functors*

$$\mathcal{H}\ltimes : Gpd(\mathcal{H} - spaces) \to (S - Gpd)/\mathcal{H}$$

and

$$\mathcal{H}\ltimes : Gpd(\mathcal{BH}) \to \left(S^{et} - Gpd\right)/\mathcal{H}$$

are bicategorically essentially surjective.

III.3.2 Action groupoids are étalé realizations

Theorem III.3.2. *Let $\mathcal{H}$ be an étale groupoid and $\mathscr{X}$ its associated étale stack, $[\mathcal{H}]$. Let*

$$Y : \left(S^{et} - Gpd\right) / \mathcal{H} \to \mathrm{St}\left(S\right) / \mathscr{X}$$

be the 2-functor which sends a groupoid $\varphi : \mathcal{G} \to \mathcal{H}$ over $\mathcal{H}$ to

$$[\varphi] : [\mathcal{G}] \to [\mathcal{H}] = \mathscr{X}.$$

Then the composite $Y \circ \mathcal{H} \ltimes : Gpd\left(\mathcal{BH}\right) \to \mathrm{St}\left(S\right) / \mathscr{X}$ preserves weak colimits.

The proof of this theorem is quite involved, so it is delayed to the appendix. See Theorem A.4.8.

Given the above theorem and Corollary III.3.1, if we can show that $Y \circ \mathcal{H} \ltimes$ agrees with $\bar{L} \circ a \circ j \circ i$ on representables, then we are guaranteed that this is a concrete description of the étalé realization 2-functor.

Theorem III.3.3. *For $U \subset \mathcal{H}_0$ an open subset, the stacks $y\left(U \hookrightarrow \mathcal{H}_0 \to \mathscr{X}\right)$ and*

$$Y\left(\theta_{m_U}\right) = [\theta_{m_U}]$$

are canonically equivalent in $\mathrm{St}\left(S\right) / \mathscr{X}$, where m_U is the equivariant sheaf associated to the representable $U \in Site\left(\mathcal{H}\right)_0$ (Definition III.2.14), and

$$\theta_{m_U} : \mathcal{H} \ltimes m_U \to \mathcal{H}.$$

Proof. $\mathcal{H} \ltimes m_U$ has objects $s^{-1}\left(U\right)$ and arrows are of the form

$$(h, \gamma) : \gamma \to h \circ \gamma.$$

Define an internal functor

$$f_U : \mathcal{H} \ltimes m_U \to U^{id}$$

on objects as

$$s^{-1}\left(U\right) \xrightarrow{s} U$$

and on arrows by

$$(h, \gamma) \mapsto s\left(\gamma\right).$$

Define another internal functor

$$g_U : U^{id} \to \mathcal{H} \ltimes m_U$$

on objects as

$$x \mapsto \mathbb{1}_x$$

and on arrows as

$$x \mapsto \left(\mathbb{1}_x, \mathbb{1}_x\right).$$

Clearly
$$f_U \circ g_U = id_{U^{id}}.$$
Moreover, there is a canonical internal natural transformation
$$\lambda_U : g_U \circ f_U \Rightarrow id_{\mathcal{H} \ltimes m_U},$$
given by
$$\lambda_U(\gamma) = \left(\gamma, \mathbb{1}_{s(\gamma)}\right).$$
Denote by $a_U : U^{id} \to \mathcal{H}$, the composite
$$U \hookrightarrow \mathcal{H}_0 \to \mathcal{H}.$$
Notice that g_U extends to a morphism from $a_U : U^{id} \to \mathcal{H}$ to $\mathcal{H} \ltimes m_U$ in $(S^{et} - Gpd)/\mathcal{H}$ as $\theta_{m_U} \circ g_U = a_u$. Note that the formula
$$\alpha_U(\gamma) = \gamma$$
defines an internal natural transformation
$$\alpha_U : a_U \circ f_U \Rightarrow \theta_{m_U}.$$
Hence (f_U, α_U) is morphism in $(S^{et} - Gpd)/\mathcal{H}$ from θ_{m_U} to a_U. It is easy to check that λ_U is in fact a 2-cell in $(S^{et} - Gpd)/\mathcal{H}$. Hence a_u and θ_{m_U} are canonically equivalent, so the same is true of their images under Y. □

Corollary III.3.3. *$\bar{L} \circ a \circ j \circ i$ and $Y \circ \mathcal{H}\ltimes$ are equivalent.*

Remark. In particular, this implies that if $\mathscr{Z}$ is a small stack over $\mathscr{X}$ represented by a groupoid object $\mathcal{K}$ in $\mathcal{BH}$, then $L(\mathscr{Z}) \simeq Y(\mathcal{H} \ltimes \mathcal{K})$.

Definition III.3.3. A morphism $\mathscr{Y} \to \mathscr{X}$ of étale stacks is said to be a **local homeomorphism** if it can be represented by a map $\varphi : \mathcal{G} \to \mathcal{H}$ of S-groupoids such that φ_0 (and hence φ_1) is a local homeomorphisms of spaces. Denote the full sub-2-category of $St(S)/\mathscr{X}$ spanned by local homeomorphisms over $\mathscr{X}$ by $Et(\mathscr{X})$.

Remark. In identifying étale stacks with etendues, this notion is equivalent to the map being a local homeomorphism of topoi in the sense of [29].

In light of Corollaries III.3.2 and III.3.3 and Proposition III.2.6, the essential image of L is precisely the local homeomorphisms over $\mathscr{X}$. Moreover, with Corollary III.2.4, this implies:

Corollary III.3.4.
$$\mathrm{St}(\mathscr{X}) \underset{\bar{L}}{\overset{\bar{\Gamma}}{\rightleftarrows}} Et(\mathscr{X}),$$
is an adjoint-equivalence between $\mathrm{St}(\mathscr{X})$ *and local homeomorphisms over* $\mathscr{X}$.

Remark. Note that there is a small error on the top of page 44 of [38]; the construction P_1, which assigns a stack $\mathscr{Z}$ over a space X an étale groupoid over X via a local homeomorphism, is not functorial with respect to all maps of stacks. It is only functorial with respect to *strict* natural transformations of stacks, but in general, one must consider also pseudo-natural transformations. The above corollary may be seen as a corrected version of this construction, in the case that $\mathscr{X}$ is a space X. Note that this error also makes Theorem 94 of [38] incorrect. The corrected version of Theorem 94 is explained in Section III.7 of this chapter.

III.3.3 The inverse image functor

Suppose $f : \mathscr{Y} \to \mathscr{X}$ is a morphism of étale stacks. This induces a geometric morphism of 2-topoi $\mathrm{St}\,(\mathscr{Y}) \to \mathrm{St}\,(\mathscr{X})$, where by this we mean a pair of adjoint 2-functors $f^* \dashv f_*$, such that f^* preserves finite (weak) limits [35]. To see this, note that there is a canonical trifunctor

$$\mathfrak{Top} \to 2 - \mathfrak{Top},$$

from topoi to 2-topoi, which sends a topos $\mathcal{E}$ to the 2-topos of stacks over $\mathcal{E}$ with the canonical topology. Since,

$$Sh : \mathrm{St}\,(S) \to \mathfrak{Top}$$

is a 2-functor, so we get an induced geometric morphism

$$Sh\,(f) : Sh\,(\mathscr{Y}) \to Sh\,(\mathscr{X})\,,$$

which in turn gives rise to a geometric morphism

$$F : \mathrm{St}\,(\mathscr{Y}) \to \mathrm{St}\,(\mathscr{X})\,,$$

after applying the trifunctor $\mathfrak{Top} \to 2-\mathfrak{Top}$. We denote the direct and inverse image 2-functors by f_* and f^*.

We also get an induced geometric morphism between the 2-topoi of *large* stacks,

$$F : \mathrm{St}\,(S/\mathscr{Y}) \to \mathrm{St}\,(S/\mathscr{X})\,.$$

This arises as the adjoint pair of slice 2-categories

$$\mathrm{St}\,(S)\,/\mathscr{Y} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{St}\,(S)\,/\,\mathscr{X}\,,$$

induced by f. The inverse image 2-functor f^* is given by pullbacks:
If $\mathscr{Z} \to \mathscr{X}$ is in $\mathrm{St}\,(S)\,/\,\mathscr{X}$, then $f^*\,(\mathscr{Z} \to \mathscr{X})$ is given by $\mathscr{Y} \times_{\mathscr{X}} \mathscr{Z} \to \mathscr{Y}$.

Theorem III.3.4. *The following diagram 2-commutes:*

$$\begin{tikzcd} \mathrm{St}\left(\mathscr{X}\right) \arrow[r, "\bar{L}"] \arrow[d, "f^*"'] & \mathrm{St}\left(S\right)/\mathscr{X} \arrow[d, "f^*"] \\ \mathrm{St}\left(\mathscr{Y}\right) \arrow[r, "\bar{L}"] & \mathrm{St}\left(S\right)/\mathscr{Y}, \end{tikzcd}$$

where $\bar{L}$ is as in Corollary III.2.4.

Proof. As both composites $f^* \circ \bar{L}$ and $\bar{L} \circ f^*$ are weak colimit preserving, it suffices to show that they agree on representables. We fix an étale S-groupoid $\mathcal{H}$ such that $[\mathcal{H}] \simeq \mathscr{X}$ and *choose* a particular $\mathcal{G}$ such that

$$[\mathcal{G}] \simeq \mathscr{Y}$$

and $f = [\varphi]$ with $\varphi : \mathcal{G} \to \mathcal{H}$ an internal functor. Choose a representable sheaf $m_U \in \mathcal{B}\mathcal{H}$. From [39], for any equivariant sheaf

$$\mathcal{H} \circlearrowright E \xrightarrow{\mu} \mathcal{H}_0,$$

$\varphi^*\left(E\right)$ as a sheaf over $\mathcal{G}_0$ is given by

$$\mathcal{G}_0 \times_{\mathcal{H}_0} E \to \mathcal{G}_0$$

and has the $\mathcal{G}$-action

$$g \cdot (x, e) = \left(t\left(g\right), \varphi\left(g\right) \cdot e\right).$$

Hence $\bar{L}\left(m_u\right)$ is given by $Y\left(\mathcal{G} \ltimes \left(\mathcal{G}_0 \times_{\mathcal{H}_0} s^{-1}\left(U\right)\right)\right).$ Explicitly, the arrows may be described by pairs $(g, h) \in \mathcal{G}_1 \times s^{-1}\left(U\right)$ such that

$$s\varphi\left(g\right) = t\left(h\right).$$

The other composite,

$$f^*\bar{L}\left(m_u\right)$$

is given by

$$[\mathcal{G}] \times_{[\mathcal{H}]} \left[\mathcal{H} \ltimes s^{-1}\left(U\right)\right] \to [\mathcal{G}].$$

Since the extended Yoneda 2-functor preserves all weak limits, and stackification preserves finite ones, this pullback may be computed in S-groupoids.

Its objects are triples

$$(z, h, \alpha) \in \mathcal{G}_0 \times s^{-1}\left(U\right) \times \mathcal{H}_1$$

such that

$$\varphi_0\left(z\right) \xrightarrow{\alpha} t\left(h\right).$$

Its arrows are quadruples

$$(g, h, h', \alpha) \in \mathcal{G}_1 \times \mathcal{H}_1 \times s^{-1}(U) \times \mathcal{H}_1$$

such that

$$s(\varphi(g)) = s(\alpha)$$

and

$$t(\alpha) = s(h') = t(h).$$

Such a quadruple is an arrow from $(s(g), h, \alpha)$ to $\left(t(g), h'h, h'\alpha\varphi(g)^{-1}\right)$. The projections are defined by

$$\begin{array}{rcl} pr_1 : \mathcal{G} \times_{\mathcal{H}} \left(\mathcal{H} \ltimes s^{-1}(U)\right) & \to & \mathcal{G} \\ (z, h, \alpha) & \mapsto & z \\ (g, h', h, \alpha) & \mapsto & g \end{array}$$

and

$$\begin{array}{rcl} pr_2 : \mathcal{G} \times_{\mathcal{H}} \left(\mathcal{H} \ltimes s^{-1}(U)\right) & \to & \mathcal{H} \ltimes s^{-1}(U) \\ (z, h, \alpha) & \mapsto & h \\ (g, h', h, \alpha) & \mapsto & (h', h). \end{array}$$

We define an internal functor

$$\zeta : \mathcal{G} \times_{\mathcal{H}} \left(\mathcal{H} \ltimes s^{-1}(U)\right) \to \mathcal{G} \ltimes \left(\mathcal{G}_0 \times_{\mathcal{H}_0} s^{-1}(U)\right)$$

on objects by

$$(z, h, \alpha) \mapsto \left(z, \alpha^{-1}h\right)$$

and on arrows by

$$(g, h', h, \alpha) \mapsto \left(g, \alpha^{-1}h\right).$$

We define another internal functor

$$\psi : \mathcal{G} \ltimes \left(\mathcal{G}_0 \times_{\mathcal{H}_0} s^{-1}(U)\right) \to \mathcal{G} \times_{\mathcal{H}} \left(\mathcal{H} \ltimes s^{-1}(U)\right)$$

on objects as

$$(z, h) \mapsto \left(z, \mathbb{1}_{s(h)}, h^{-1}\right)$$

and on arrows as

$$(g, h) = \left(g, \mathbb{1}_{s(h)}, \mathbb{1}_{s(h)}, h^{-1}\right).$$

Note that ψ is a left inverse for ζ. We define an internal natural isomorphism

$$\omega : \psi \circ \zeta \Rightarrow id_{\mathcal{G} \times_{\mathcal{H}} (\mathcal{H} \ltimes s^{-1}(U))}$$

by

$$\omega(z,h,\alpha) = \left(\mathbb{1}_z, h^{-1}, h, \alpha\right) : (z,h,\alpha) \to \left(z, \mathbb{1}_{s(h)}, h^{-1}\alpha\right) = \psi\zeta(z,h,\alpha).$$

As both ζ and ψ commute strictly over $\mathcal{G}$, this establishes our claim. $\square$

Definition III.3.4. For $\mathscr{Z}$ a small stack over an étale stack $\mathscr{X}$, and

$$x : * \to \mathscr{X}$$

a point of $\mathscr{X}$, the **stalk** of $\mathscr{Z}$ at x is the groupoid $x^*(\mathscr{Z})$, where we have made the identification $\mathrm{St}(*) \simeq Gpd$. We denote this stalk by $\mathscr{Z}_x$.

As we have just seen, this stalk may be computed as the fiber of

$$\bar{L}(\mathscr{Z}) \to \mathscr{X}$$

over x, i.e. the weak pullback $* \times_{\mathscr{X}} \bar{L}(\mathscr{Z})$, which is a constant stack with value $x^*(\mathscr{Z})$. This stalk can also be computed analogously to stalks of sheaves:

Lemma III.3.5. *Let $x \in X$ be a point of a space, and let $\mathscr{Z}$ be a small stack over X. Then the stalk at x of $\mathscr{Z}$ can be computed by*

$$\mathscr{Z}_x \simeq \underset{x \in U}{\underrightarrow{\mathrm{holim}}}\, \mathscr{Z}(U),$$

where the weak colimit is taken over the open neighborhoods of x regarded as a full subcategory of $\mathcal{O}(X)$.

Proof. The 2-functor

$$\begin{aligned} \mathrm{St}(X) &\to Gpd \\ \mathscr{Z} &\mapsto \underset{x \in U}{\underrightarrow{\mathrm{holim}}}\, \mathscr{Z}(U), \end{aligned}$$

is clearly weak colimit preserving. If $\mathscr{Z} = V \subseteq X$ is a representable sheaf, i.e., an open subset of X, then

$$\mathscr{Z}_x \simeq \underset{x \in U}{\underrightarrow{\mathrm{holim}}}\, \mathrm{Hom}(U,V) \simeq \underset{x \in U}{\underrightarrow{\lim}}\, \mathrm{Hom}(U,V),$$

and the latter expression is equivalent to the singleton set if $x \in V$ and the empty set otherwise. This set is the same as the fiber of V over x, i.e. the stalk $V_x \cong x^*(V)$. So

$$\mathscr{Z} \mapsto \underset{x \in U}{\underrightarrow{\mathrm{holim}}}\, \mathscr{Z}(U)$$

is weak colimit preserving and agrees with x^* on representables, hence is equivalent to x^*. $\square$

Corollary III.3.5. *Let $x : * \to \mathscr{X}$ be a point of an étale stack $\mathscr{X} \simeq [\mathcal{H}]$, with $\mathcal{H}$ an étale groupoid. Pick a point $\tilde{x} \in \mathcal{H}_0$ such that $x \cong p \circ \tilde{x}$ where*

$$p : \mathcal{H}_0 \to \mathscr{X}$$

is the atlas associated to $\mathcal{H}$. Let $\mathscr{Z}$ be a small stack over $\mathscr{X}$. Then the stalk at x of $\mathscr{Z}$ can be computed by

$$\mathscr{Z}_x \simeq \underset{\tilde{x} \in U}{\underrightarrow{\operatorname{holim}}}\, \mathscr{Z}(U),$$

where the weak colimit is taken over the open neighborhoods of $\tilde{x}$ in $\mathcal{H}_0$ regarded as a full subcategory of $\mathcal{O}(\mathcal{H}_0)$.

Proof. Since $x \cong p \circ \tilde{x}$, it follows that

$$Sh(x) \simeq Sh(p) \circ Sh(\tilde{x}) : Sh(*) \to Sh(\mathscr{X}),$$

and hence

$$x^* \simeq \tilde{x}^* \circ p^*.$$

By definition, for U an open subset of $\mathcal{H}_0$,

$$p^*(\mathscr{Z})(U) \simeq \mathscr{Z}(U).$$

Hence,

$$\begin{aligned} \mathscr{Z}_x &= x^* \mathscr{Z} \\ &\simeq \tilde{x}^*(p^* \mathscr{Z}) \\ &\simeq (p^* \mathscr{Z})_{\tilde{x}} \\ &\simeq \underset{\tilde{x} \in U}{\underrightarrow{\operatorname{holim}}}\, (p^* \mathscr{Z})(U) \\ &\simeq \underset{\tilde{x} \in U}{\underrightarrow{\operatorname{holim}}}\, \mathscr{Z}(U). \end{aligned}$$

□

III.3.4 A classification of sheaves

From Corollary III.3.4, we know that for an étale stack $\mathscr{X}$, the 2-category of local homeomorphisms over $\mathscr{X}$ is equivalent to the 2-category of small stacks over $\mathscr{X}$. A natural question is which objects in $Et(\mathscr{X})$ are actually *sheaves* over $\mathscr{X}$, as opposed to stacks, i.e., what are the 0-truncated objects?

Theorem III.3.6. *A local homeomorphism $f : \mathscr{Z} \to \mathscr{X}$ over an étale stack $\mathscr{X}$ is a equivalent to $\bar{L}(F)$ for a small sheaf F over $\mathscr{X}$ if and only if it is a representable map.*

Proof. Suppose F is a small sheaf over $\mathscr{X} \simeq [\mathcal{H}]$ with $\mathcal{H}$ an étale S-groupoid. Denote by

$$\underline{\bar{L}(F)} \to \mathscr{X}$$

the map $\bar{L}(F)$. We wish to show that

$$\underline{\bar{L}(F)} \to \mathscr{X}$$

is representable. It suffices to show that the 2-pullback

$$\begin{tikzcd} \mathcal{H}_0 \times_{\mathscr{X}} \underline{\bar{L}(F)} \arrow[r] \arrow[d] & \underline{\bar{L}(F)}, \arrow[d] \\ \mathcal{H}_0 \arrow[r, "a"] & \mathscr{X} \end{tikzcd}$$

is (equivalent to) a space, where $a : \mathcal{H}_0 \to \mathscr{X}$ is the atlas associated to $\mathcal{H}$. By Theorem III.3.4, this pullback is equivalent to the total space of the étalé space of the sheaf $a^*(F)$ over $\mathcal{H}_0$. Conversely, suppose $\mathscr{Z} \to \mathscr{X}$ is a representable local homeomorphism equivalent to $\bar{L}(\mathscr{W})$ for some small stack $\mathscr{W}$. Then the pullback

$$\mathcal{H}_0 \times_{\mathscr{X}} \underline{\bar{L}(F)}$$

is equivalent to a space. This implies that $a^*(W)$ is a sheaf of sets over $\mathcal{H}_0$. By definition $a^*(\mathscr{W})$ assigns to each open subset U of $\mathcal{H}_0$ the groupoid $\mathscr{W}(m_U)$. It follows that $\mathscr{W}$ must be a sheaf. $\square$

Corollary III.3.6. *For an étale stack $\mathscr{X}$, the category of small sheaves over $\mathscr{X}$ is equivalent to the 2-category of representable local homeomorphisms over $\mathscr{X}$.*

Remark. This implies that the 2-category of representable local homeomorphisms over $\mathscr{X}$ is equivalent to its 1-truncation.

Remark. This gives a purely intrinsic definition of the topos of sheaves $Sh(\mathscr{X})$. In particular, a posteriori, we could define a small stack over $\mathscr{X}$ to be a stack over this topos. We note for completeness that a site of definition of this topos is the category of local homeomorphisms $T \to \mathscr{X}$ from a space T, with the induced open cover topology. This is equivalent to the category of principal $\mathcal{H}$-bundles whose moment maps are local homeomorphisms.

III.4 A Groupoid Description of the Stack of Sections

Now that we have a concrete description of $\bar{L}$ in terms of groupoids, it is natural to desire a similar description for $\bar{\Gamma}$ (where $\bar{L}$ and $\bar{\Gamma}$ are as in Corollary III.2.4).

Lemma III.4.1. *Suppose that $\varphi : T \to \mathcal{H}$ is a local homeomorphism from a space T, with $\mathcal{H}$ an étale groupoid. Then $\bar{\Gamma}([\varphi])$ is the equivariant sheaf $P(\varphi) \in \mathcal{BH}$, where P is as in Section III.3.1.*

Proof. Let m_U be a representable sheaf in $\mathcal{BH}$. Then

$$\begin{aligned}\Gamma([\varphi])(U) &\simeq \operatorname{Hom}\left(\bar{L}(m_U), [\varphi]\right) \\ &\simeq \operatorname{Hom}\left(\left[\mathcal{H} \ltimes s^{-1}(U)\right], [\varphi]\right).\end{aligned}$$

Since T is a sheaf, the later is in turn equivalent to

$$\operatorname{Hom}_{Gpd/\mathcal{H}}\left(\mathcal{H} \ltimes s^{-1}(U), \varphi\right).$$

This follows from the canonical equivalence

$$\operatorname{Hom}\left(\tilde{y}\left(\mathcal{H} \ltimes s^{-1}(U)\right), T\right) \simeq \operatorname{Hom}\left(\left[\mathcal{H} \ltimes s^{-1}(U)\right], T\right).$$

In fact, this is a set, since T has no arrows, so there are no natural transformations. An element of this set is the data of a groupoid homomorphism

$$\psi : \mathcal{H} \ltimes s^{-1}(U) \to T$$

together with an internal natural transformation

$$\alpha : \varphi \circ \psi \Rightarrow \theta_{m_U}.$$

Since T is a space, ψ_1 is determined by ψ_0 by the formula

$$\psi_1((h, \gamma)) = \psi_0(\gamma) = \psi_0(h\gamma).$$

Notice that this also imposes conditions on ψ_0, namely that it is constant on orbits. The internal natural transformation is a map of spaces

$$\alpha : s^{-1}(U) \to \mathcal{H}_1$$

such that for all $\gamma \in s^{-1}(U)$,

$$\alpha(\gamma) : \varphi\psi_0(\gamma) \to t(\gamma).$$

Because of the constraints on ψ, the naturality condition is equivalent to

$$\alpha(h\gamma) = h\alpha(\gamma).$$

This data defines a map

$$m_U \to P(\varphi)$$

by

$$\begin{array}{rcl} s^{-1}(U) & \to & \mathcal{H}_1 \times_{\mathcal{H}_0} T \\ \gamma & \mapsto & (\alpha(\gamma), \psi_0(\gamma)). \end{array}$$

Conversely, any map $f : m_U \to P(\varphi)$ defines a morphism

$$\hat{f} : \mathcal{H} \ltimes s^{-1}(U) \to T$$

on objects by $pr_2 \circ f$ (and hence determines it on arrows), and since f is $\mathcal{H}$-equivariant, and the $\mathcal{H}$-action on $\mathcal{H}_1 \times_{\mathcal{H}_0} T$ does not affect T, this map is constant on orbits. The map f induces an internal natural transformation

$$\alpha_f : \varphi \circ \hat{f} \to \theta_{m_U}$$

by $\alpha_f = pr_1 \circ f$. This establishes a bijection

$$\mathrm{Hom}_{Gpd/\mathcal{H}}\left(\mathcal{H} \ltimes s^{-1}(U), \varphi\right) \cong \mathrm{Hom}_{\mathcal{BH}}(m_U, P(\varphi)).$$

Hence

$$\Gamma([\varphi])(U) \simeq \mathrm{Hom}_{\mathcal{BH}}(m_U, P(\varphi)),$$

so we are done by the Yoneda Lemma. □

Theorem III.4.2. *Suppose that $\varphi : \mathcal{G} \to \mathcal{H}$ is a homomorphism of étale S-groupoids with φ_0 a local homeomorphism. Then $\bar{\Gamma}([\varphi])$ is equivalent to the stack associated to the groupoid object $P(\varphi)$ in $\mathcal{BH}$.*

Proof. Let $a : \mathcal{G}_0 \to [\mathcal{G}]$ denote the atlas of the stack $[\mathcal{G}]$. There is a canonical map

$$p : \bar{\Gamma}([\varphi] \circ a) \to \bar{\Gamma}([\varphi]),$$

and since a is an epimorphism, it follows that p is an epimorphism as well. Since p is an epimorphism from a sheaf to a stack, it follows that

$$\bar{\Gamma}([\varphi] \circ a) \times_{\bar{\Gamma}([\varphi])} \bar{\Gamma}([\varphi] \circ a) \rightrightarrows \bar{\Gamma}([\varphi] \circ a),$$

is a groupoid object in sheaves (i.e. the classifying topos $\mathcal{BH}$) whose stackification is equivalent to $\bar{\Gamma}([\varphi])$. We will show that this groupoid is isomorphic to $P(\varphi)$. This isomorphism is clear on objects from the previous lemma.

Since pullbacks are computed point-wise, as a sheaf,

$$\bar{\Gamma}([\varphi] \circ a) \times_{\bar{\Gamma}([\varphi])} \bar{\Gamma}([\varphi] \circ a)$$

assigns $U \in Site(\mathcal{H})$ the pullback groupoid

$$\bar{\Gamma}([\varphi] \circ a)(U) \times_{\bar{\Gamma}([\varphi])(U)} \bar{\Gamma}([\varphi] \circ a)(U),$$

which is indeed (equivalent to) a set. It is the set of pairs of objects in $\Gamma\left([\varphi]\circ a\right)(U)$ together with a morphism in $\bar{\Gamma}\left([\varphi]\right)(U)$ between their images under $p\left(U\right)$.

Since for all S-groupoids, the induced map

$$\operatorname{Hom}\left(\mathcal{L},\mathcal{K}\right)\to\operatorname{Hom}\left([\mathcal{L}],[\mathcal{K}]\right)$$

is full and faithful, we may describe this set in terms of maps of groupoids. It has the following description:

An element of $\bar{\Gamma}\left([\varphi]\circ a\right)(U)\times_{\bar{\Gamma}([\varphi])(U)}\bar{\Gamma}\left([\varphi]\circ a\right)(U)$, can be represented by two pairs (σ_0,α_0) and (σ_1,α_1), such that for $i=0,1$,

$$\begin{tikzcd}
\mathcal{H}\ltimes s^{-1}(U) \arrow[r, "\sigma_i"] \arrow[ddrrr, bend right, "\theta_{m_U}"'] & \mathcal{G}_0 \arrow[dr, "\tilde{a}"] \arrow[d, phantom, "\Swarrow \alpha_i"] & & \\
 & \, & \mathcal{G} \arrow[dr, "\varphi"] & \\
 & & & \mathcal{H},
\end{tikzcd}$$

where $\tilde{a}:\mathcal{G}_0\to\mathcal{G}$ is the obvious map such that $[\tilde{a}]=a$, together with a 2-cell

$$\beta:\tilde{a}\circ\sigma_0\Rightarrow\tilde{a}\circ\sigma_1,$$

such that the following diagram commutes:

(III.5)
$$\begin{tikzcd}
\varphi\circ\tilde{a}\circ\sigma_0 \arrow[rr, Rightarrow, "\varphi\beta"] \arrow[dr, Rightarrow, "\alpha_0"'] & & \varphi\circ\tilde{a}\circ\sigma_1\ . \arrow[dl, Rightarrow, "\alpha_1"] \\
 & \theta_{m_U} &
\end{tikzcd}$$

Each pair (σ_i,α_i) represents a diagram of stacks

$$\begin{tikzcd}
\left[\mathcal{H}\ltimes s^{-1}(U)\right] \arrow[r, "{[\sigma_i]}"] \arrow[ddrrr, bend right, "{[\theta_{m_U}]}"'] & \mathcal{G}_0 \arrow[dr, "a"] \arrow[d, phantom, "\Swarrow {[\alpha_i]}"] & & \\
 & \, & {[\mathcal{G}]} \arrow[dr, "{[\varphi]}"] & \\
 & & & {[\mathcal{H}]}\,,
\end{tikzcd}$$

i.e. the element $([\sigma_i],[\alpha_i])$ of the set $\bar{\Gamma}\left([\varphi]\circ a\right)(U)$. The groupoid structure on

$$\bar{\Gamma}\left([\varphi]\circ a\right)\times_{\bar{\Gamma}([\varphi])}\bar{\Gamma}\left([\varphi]\circ a\right)\rightrightarrows\bar{\Gamma}\left([\varphi]\circ a\right),$$

is such that the data

$$((\sigma_0, \alpha_0), (\sigma_1, \alpha_1), \beta)$$

is an arrow from $([\sigma_0], [\alpha_0])$ to $([\sigma_0], [\alpha_0])$.

Recall that the arrows of $P(\varphi)$ are the equivariant sheaf described as the fibered product

$$\begin{tikzcd} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_1 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{G}_1 \arrow[d, "\varphi_0 \circ t"] \\ \mathcal{H}_1 \arrow[r, "s"] & \mathcal{H}_0, \end{tikzcd}$$

equipped with the left $\mathcal{H}$-action along $s \circ pr_1$ given by

$$h \cdot (\gamma, g) = (h\gamma, g),$$

and that the source and target maps are given by

$$s(h, g) = (h\varphi(g), s(g)),$$

and

$$t(h, g) = (h, t(g)).$$

Viewing the arrows of $P(\varphi)$ as a sheaf, they assign U the set

$$\mathrm{Hom}\left(m_U, P(\varphi)_1\right).$$

Let

$$\pi(U) : \bar{\Gamma}([\varphi] \circ a)(U) \times_{\bar{\Gamma}([\varphi])(U)} \bar{\Gamma}([\varphi] \circ a)(U) \to \mathrm{Hom}\left(m_U, P(\varphi)_1\right)$$

be the map that sends

$$\zeta := ((\sigma_0, \alpha_0), (\sigma_1, \alpha_1), \beta)$$

to the morphism

$$\begin{array}{rcl} \theta(\zeta) : m_U & \to & \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_1 \\ \gamma & \mapsto & (\alpha_1(\gamma), \beta(\gamma)). \end{array}$$

It is easy to check that this morphism is $\mathcal{H}$-equivariant, hence is a map in $\mathcal{BH}$. We will show that under the identification

$$\bar{\Gamma}([\varphi] \circ a)(U) \simeq P(\varphi \circ \tilde{a})(U) = \mathrm{Hom}\left(m_U, P(\varphi \circ \tilde{a})\right),$$

$\pi(U)$ respects source and targets. Indeed, suppose we start with a triple

$$\zeta := ((\sigma_0, \alpha_0), (\sigma_1, \alpha_1), \beta).$$

By Lemma III.4.1, each (σ_i, α_i) corresponds to an element of

$$\bar{\Gamma}\left(\left[\varphi\right]\circ a\right)\left(U\right),$$

which in turn corresponds to a morphism

$$\begin{array}{rcl} m_U = s^{-1}\left(U\right) & \to & \mathcal{H}_1 \times_{\mathcal{H}_0} T \\ \gamma & \mapsto & \left(\alpha_i\left(\gamma\right), \sigma_i\left(\gamma\right)\right) \end{array}$$

(III.6)

in $\mathcal{BH}$. Now $\pi\left(U\right)\left(\zeta\right)$ is a map from $d_0\pi\left(U\right)\left(\zeta\right)$ to $d_1\pi\left(U\right)\left(\zeta\right)$, where we have used simplicial notation for the source and target. For each i, we have a map

$$m_U \xrightarrow{\pi(U)(\zeta)} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_1 \xrightarrow{d_i} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0,$$

which we may interpret as an element of

$$P\left(\varphi\right)_0\left(U\right) = P\left(\varphi\circ\tilde{a}\right)\left(U\right).$$

From equation III.6 and the definition of the source and target map, it follows that

$$\begin{array}{rcl} s\pi\left(U\right)\left(\zeta\right) & = & \gamma \mapsto s\left(\alpha_1\left(\gamma\right), \beta\left(\gamma\right)\right) \\ & = & \gamma \mapsto \left(\alpha_1\varphi\beta\left(\gamma\right), s\beta\left(\gamma\right)\right) \\ & = & \gamma \mapsto \left(\alpha_0\left(\gamma\right), \sigma_0\left(\gamma\right)\right), \end{array}$$

and

$$\begin{array}{rcl} t\pi\left(U\right)\left(\zeta\right) & = & \gamma \mapsto t\left(\alpha_1\left(\gamma\right), \beta\left(\gamma\right)\right) \\ & = & \gamma \mapsto \left(\alpha_1\left(\gamma\right), t\beta\left(\gamma\right)\right) \\ & = & \gamma \mapsto \left(\alpha_1\left(\gamma\right), \sigma_1\left(\gamma\right)\right). \end{array}$$

Hence $\pi\left(U\right)$ respects the source and target. We will now show it is an isomorphism. Suppose we are given an arbitrary equivariant map

$$\theta : m_U \to \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_1.$$

Denote its components by

$$\theta\left(\gamma\right) = \left(h\left(\gamma\right), g\left(\gamma\right)\right).$$

Since θ is $\mathcal{H}$-equivariant, it follows that h is $\mathcal{H}$-equivariant and g is $\mathcal{H}$-invariant. Now

$$\begin{array}{rcl} s\circ\theta : m_u & \to & P(\varphi)_0 \\ s\theta(\gamma) & = & (h(\gamma)\varphi(g(\gamma)), s(g(\gamma))) \end{array}$$

and

$$\begin{array}{rcl} t\circ\theta : m_u & \to & P(\varphi)_0 \\ t\theta(\gamma) & = & (h(\gamma), t(g(\gamma))). \end{array}$$

Each of these maps correspond to an element in

$$P(\varphi)_0(U) \simeq \bar{\Gamma}([\varphi]\circ a)(U).$$

By Lemma III.4.1, we know that $s\circ\theta$ corresponds to the morphism of groupoids

$$\widehat{s\circ\theta} : \mathcal{H} \ltimes s^{-1}(U) = s^{-1}(U) \to \mathcal{G}_0$$

given on objects as

$$\gamma \mapsto s(g(\gamma)),$$

together with a 2-cell

$$\alpha_{s\theta} : [\varphi]\circ a \circ \widehat{s\circ\theta} \Rightarrow \theta_{m_U},$$

given by

$$\alpha_{s\theta} = pr_1 \circ s \circ \theta.$$

Explicitly we have:

$$\alpha_{s\theta}(\gamma) = h(\gamma)\varphi(g(\gamma)).$$

Similarly, we know that $t\circ\theta$ corresponds to the morphism

$$\widehat{t\circ\theta} : \mathcal{H} \ltimes s^{-1}(U) = s^{-1}(U) \to \mathcal{G}_0$$

given on objects as

$$\gamma \mapsto t(g(\gamma)),$$

together with a 2-cell

$$\alpha_{t\theta} : [\varphi]\circ a \circ \widehat{t\circ\theta} \Rightarrow \theta_{m_U},$$

given by

$$\alpha_{t\theta} = pr_1 \circ t \circ \theta,$$

and we have:

$$\alpha_{t\theta}(\gamma) = h(\gamma).$$

The map
$$\beta(\theta) := pr_2 \circ \theta : s^{-1}(U) \to \mathcal{G}_1$$
which assigns $\gamma \mapsto g(\gamma)$ encodes a natural transformation
$$\beta(\theta) : \widehat{s \circ \theta} \Rightarrow \widehat{s \circ \theta}.$$

Moreover, we have that

$$\begin{aligned} \alpha_{t\theta}\varphi\beta(\gamma) &= h(\gamma) \circ \varphi(g(\gamma)) \\ &= \alpha s \theta(\gamma), \end{aligned}$$

which implies the diagram III.5 commutes.

Define a map
$$\Xi(U) : \mathrm{Hom}\left(m_U, P(\varphi)_1\right) \to \bar{\Gamma}([\varphi] \circ a)(U) \times_{\bar{\Gamma}([\varphi])(U)} \bar{\Gamma}([\varphi] \circ a)(U)$$
which assigns the morphism $\theta : m_U \to P(\varphi)_1$ the triple
$$\left(\left(\widehat{s \circ \theta}, \alpha_{s\theta}\right), \left(\widehat{t \circ \theta}, \alpha_{t\theta}\right), \beta(\theta)\right).$$

This map is clearly inverse to π. We leave it to the reader to check that $\pi(U)$ respects composition. It then follows that the groupoids in sheaves
$$\bar{\Gamma}([\varphi] \circ a) \times_{\bar{\Gamma}([\varphi])} \bar{\Gamma}([\varphi] \circ a)$$
and $P(\varphi)$ are isomorphic. □

III.5 Effective Stacks

III.5.1 Basic definitions

In this section, we recall a special class of étale stacks, called effective étale stacks. This is a summary of results well known in the groupoid literature, expressed in a more stack-oriented language. We claim no originality for the ideas. We start with defining effectiveness for orbifolds, as this definition is more intuitive. This will also make the general definition for an arbitrary étale stack more clear.

Definition III.5.1. A differentiable stack $\mathscr{X}$ is called an **orbifold** if it is étale and the diagonal map $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is proper. If $\mathscr{X}$ is instead a topological stack, we call $\mathscr{X}$ a **topological orbifold**. To simplify things, we will refer to both differentiable and topological orbifolds, simply as orbifolds.

Remark. We should explain what we mean in saying that the diagonal map is proper. In the differentiable setting, this map is not representable, even for (positive dimensional) manifolds. We say that a map of $f : \mathscr{X} \to \mathscr{Z}$ between differentiable stacks is proper if and only if for any *representable* map $M \to \mathscr{Z}$ from a manifold, the induced map $M \times_{\mathscr{Z}} \mathscr{X} \to M$ is a proper map of manifolds. Equivalently, as properness is a topological property, and the diagonal map of any topological stack is representable [49], (and proper maps are invariant under restriction and local on the target) stating that the diagonal of a differentiable stack is proper in the above sense is equivalent to saying that the diagonal of the underlying topological stack is a representable proper map. Yet another characterization is viewing $\mathscr{X}$ and $\mathscr{X} \times \mathscr{X}$ as etendue and asking the map to be a proper map of topoi in the sense of [29].

Definition III.5.2. An S-groupoid is an **orbifold groupoid** if it is étale and **proper**, i.e. the map

$$(s,t) : \mathcal{H}_1 \to \mathcal{H}_0 \times \mathcal{H}_0$$

is proper.

Proposition III.5.1. *$\mathscr{X}$ is an orbifold if and only if there exists an orbifold groupoid $\mathcal{H}$ such that $\mathscr{X} \simeq [\mathcal{H}]$.*

Proof. For any étale $\mathcal{H}$ such that $[\mathcal{H}] \simeq \mathscr{X}$,

$$\begin{tikzcd} \mathcal{H}_1 \arrow[r] \arrow[d, "(s{,}t)"'] & \mathscr{X} \arrow[d, "\Delta"] \\ \mathcal{H}_0 \times \mathcal{H}_0 \arrow[r, "a \times a"] & \mathscr{X} \times \mathscr{X} \end{tikzcd}$$

is a weak pullback diagram, where $a : \mathcal{H}_0 \to \mathscr{X}$ is the atlas associated to $\mathcal{H}$. $\square$

Recall the following definition:

Definition III.5.3. If G is a S-group acting on a space X, the action is **effective** (or faithful), if $\bigcap\limits_{x \in X} G_x = e$, i.e., for all non-identity elements $g \in G$, there exists a point $x \in X$ such that $g \cdot x \neq x$. Equivalently, the induced homomorphism

$$\rho : G \to \mathit{Diff}\,(X),$$

where $\mathit{Diff}\,(X)$ is the group of diffeomorphisms (homeomorphisms) of X is a monomorphism. (These two definitions are equivalence since

$$Ker\,(\rho) = \bigcap_{x \in X} G_x = e.)$$

If ρ above has a non-trivial kernel K, then there is an inclusion of K into each isotropy group of the action, or equivalently into each automorphism group of the quotient stack (the stack associated to the action groupoid). In this case K is "tagged-along" as extra data in each automorphism group. Each of these copies of K is the kernel of the induced homomorphism

$$(\rho)_x : Aut([x]) \to Diff(M)_x,$$

where $Diff(M)_x$ is the group of diffeomorphisms of M which fix x, and $[x]$ is the image of x in the quotient stack. In the *differentiable* setting, when G is *finite*, these kernel are called the *ineffective isotropy groups* of the associated étale stack. In this case, the *effective part* of this stack is the stacky-quotient of X by the induced action of G/K. This latter stack has only trivial ineffective isotropy groups.

Remark. If G is not finite, this notion of ineffective isotropy group may not agree with Definition III.5.5, since non-identity elements can induce the germ of the identity around a point. It the topological setting, this problem can occur even when G is finite.

Proposition III.5.2. *Suppose $\mathscr{X}$ is an orbifold and $x : * \to \mathscr{X}$ is a point. Then there exists a local homeomorphism $p : V_x \to \mathscr{X}$ from a space V_x such that:*

i) the point x factors (up to isomorphism) as $* \xrightarrow{\tilde{x}} V_x \xrightarrow{p} \mathscr{X}$

ii) the automorphism group $Aut(x)$ acts on V_x.

Proof. The crux of this proof comes from [55]. Recall that for a point x of a topological or differentiable stack $\mathscr{X}$, $Aut(x)$ fits into the 2-Cartesian diagram [49]

$$\begin{tikzcd} Aut(x) \arrow[r] \arrow[d] & * \arrow[d, "x"] \\ * \arrow[r, "x"] & \mathscr{X}, \end{tikzcd}$$

and is a topological group or a Lie group. If $\mathscr{X} \simeq [\mathcal{H}]$ for an S-groupoid $\mathcal{H}$, there is a point $\tilde{x} \in \mathcal{H}_0$ such that $x \cong a \circ \tilde{x}$, where $a : \mathcal{H}_0 \to \mathscr{X}$ is the atlas associated to the groupoid $\mathcal{H}$, and moreover, $\mathcal{H}_{\tilde{x}} \cong Aut(\tilde{x})$, where $\mathcal{H}_{\tilde{x}} = s^{-1}(\tilde{x}) \cap t^{-1}(\tilde{x})$ is the S-group of automorphisms of $\tilde{x}$. (In particular, this implies that if $\mathscr{X}$ is étale, then $Aut(x)$ is discrete for all x.) Suppose now that $\mathscr{X}$ and $\mathcal{H}$ are étale. Then for each $h \in \mathcal{H}_{\tilde{x}}$, there exists an open neighborhood U_h such that the two maps

$$s : U_h \to s(U_h)$$

$$t : U_h \to t(U_h)$$

are homeomorphisms. Now, suppose that $\mathscr{X}$ is in fact an orbifold (so that $\mathcal{H}$ is an orbifold groupoid). Then, it follows that $\mathcal{H}_{\tilde{x}}$ is finite. Given f and g in $\mathcal{H}_{\tilde{x}}$, we can find a small enough neighborhood W of $\tilde{x}$ in $\mathcal{H}_0$ such that for all z in W,

$$t \circ s^{-1}|_{U_g}(z) \in s(U_f),$$
$$t \circ s^{-1}|_{U_g}\left(t \circ s^{-1}|_{U_f}(z)\right) \in W,$$

and

$$s^{-1}|_{U_g}(z) \circ s^{-1}|_{U_f}(z) \in U_{gf}. \tag{III.7}$$

In this case, by plugging in $z = \tilde{x}$ in (III.7), we see that (III.7) as a function of z must be the same as

$$s^{-1}|_{U_{gf}}.$$

Therefore, on W, the following equation holds

$$t \circ s^{-1}|_{U_g}\left(t \circ s^{-1}|_{U_f}(z)\right) = t \circ s^{-1}|_{U_{gf}}. \tag{III.8}$$

Since $\mathcal{H}_{\tilde{x}}$ is finite, we may shrink W so that equation (III.8) holds for all composable arrows in $\mathcal{H}_{\tilde{x}}$. Let

$$V_x := \bigcap_{h \in \mathcal{H}_{\tilde{x}}} \left(t \circ s^{-1}|_{U_f}(W)\right).$$

Then, for all $h \in \mathcal{H}_{\tilde{x}}$,

$$t \circ s^{-1}|_{U_h}(V_x) = V_x.$$

So $t \circ s^{-1}|_{U_h}$ is a homeomorphism from V_x to itself for all x, and since equation (III.8) holds, this determines an action of $\mathcal{H}_{\tilde{x}} \cong Aut(x)$ on V_x. Finally, define p to be the atlas a composed with the inclusion $V_x \hookrightarrow \mathcal{H}_0$ $\square$

Definition III.5.4. An orbifold $\mathscr{X}$ is an **effective** orbifold, if the actions of $Aut(x)$ on V_x as in the previous lemma can be chosen to be effective.

The finiteness of the stabilizer groups played a crucial role in the proof of Proposition III.5.2. Without this finiteness, one cannot arrange (in general) for even a single arrow in an étale groupoid to induce a self-diffeomorphism of an open subset of the arrow space. Additionally, even if each arrow had such an action, there is no guarantee that the (infinite) intersection running over all arrows in the stabilizing group of these neighborhoods will be open. Hence, for a general étale groupoid, the best we can get is a germ of a locally defined diffeomorphism. It is using these germs that we shall extend the definition of effectiveness to arbitrary étale groupoids and stacks.

Given a space X and a point $x \in X$, let $Diff_x(X)$ denote the group of germs of (locally defined) diffeomorphisms (homeomorphisms if X is a topological space) that fix x.

Proposition III.5.3. *Let $\mathscr{X}$ be an étale stack and pick an étale atlas*

$$V \to \mathscr{X}.$$

Then for each point $x : * \to \mathscr{X}$,

i) the point x factors (up to isomorphism) as $* \xrightarrow{\tilde{x}} V \xrightarrow{p} \mathscr{X}$, and

ii) there is a canonical homomorphism $Aut\,(x) \to Diff_{\tilde{x}}\,(V)$.

Proof. Following the proof of Proposition III.5.2, let $V = \mathcal{H}_0$ and let the homomorphism send each $h \in \mathcal{H}_x$ to the germ of $t \circ s^{-1}|_{U_h}$, which is a locally defined diffeomorphism of V fixing $\tilde{x}$. □

Corollary III.5.1. *For $\mathcal{H}$ an étale S-groupoid, for each $x \in \mathcal{H}_0$, there exists a canonical homomorphism of groups $\mathcal{H}_x \to Diff_x\,(\mathcal{H}_0)$.*

Definition III.5.5. Let x be a point of an étale stack $\mathscr{X}$. The **ineffective isotropy group** of x is the kernel of the induced homomorphism

$$Aut\,(x) \to Diff_{\tilde{x}}\,(V)\,.$$

Similarly for $\mathcal{H}$ an étale groupoid.

Definition III.5.6. An étale stack $\mathscr{X}$ is **effective** if the ineffective isotropy group of each of its points is trivial. Similarly for $\mathcal{H}$ an étale groupoid.

Proposition III.5.4. *An orbifold $\mathscr{X}$ is an effective orbifold if and only if it is effective when considered as an étale stack.*

Proof. This follows from [44], Lemma 2.11. □

Definition III.5.7. Let X be a space. Consider the presheaf

$$Emb : \mathcal{O}\,(X)^{op} \to \mathrm{Set},$$

which assigns an open subset U the set of embeddings of U into X. Denote by $\mathscr{H}\,(X)_1$ the total space of the étalé space of the associated sheaf. Denote the map to X by s. The stalk at x is the set of germs of locally defined diffeomorphisms (which no longer need to fix x). If $germ_x\,(f)$ is one such germ, the element $f\,(x) \in X$ is well-defined. We assemble this into a map

$$t : \mathscr{H}\,(X)_1 \to X.$$

This extends to a natural structure of an étale S-groupoid $\mathscr{H}\,(X)$ with objects X, called the **Haefliger groupoid** of X.

Remark. In literature, the Haefliger groupoid is usually denoted by $\Gamma\,(X)$, but, we wish to avoid the clash of notation with the stack of sections 2-functor.

Proposition III.5.5. *For $\mathcal{H}$ an étale S-groupoid, there is a canonical map*

$$\tilde{\iota}_{\mathcal{H}} : \mathcal{H} \to \mathscr{H}(\mathcal{H}_0).$$

Proof. For each $h \in \mathcal{H}_1$, choose a neighborhood U such that s and t restrict to embeddings. Then h induces a homeomorphism

$$s(h) \in s(U) \to t(U) \ni t(h),$$

namely $t \circ s^{-1}|_U$. Define $\tilde{\iota}_{\mathcal{H}}$ by having it be the identity on objects and having it send an arrow h to the germ at $s(h)$ of $t \circ s^{-1}|_U$. This germ clearly does not depend on the choice of U. □

The following proposition is immediate:

Proposition III.5.6. *An étale S-groupoid $\mathcal{H}$ is effective if and only if $\tilde{\iota}_{\mathcal{H}}$ is faithful.*

Definition III.5.8. Let $\mathcal{H}$ be an étale S-groupoid. The **effective part** of $\mathcal{H}$ is the image in $\mathscr{H}(\mathcal{H}_0)$ of $\tilde{\iota}_{\mathcal{H}}$. It is denoted by $\mathrm{Eff}(\mathcal{H})$. This is an open subgroupoid, so it is clearly effective and étale. We will denote the canonical map $\mathcal{H} \to \mathrm{Eff}\,\mathcal{H}$ by $\iota_{\mathcal{H}}$.

Remark. $\mathcal{H}$ is effective if and only if $\iota_{\mathcal{H}}$ is an isomorphism.

III.5.2 Étale invariance

Unfortunately, the assignment $\mathcal{H} \mapsto \mathrm{Eff}(\mathcal{H})$ is not functorial with respect to all maps, that is, a morphism of étale S-groupoids need not induce a morphism between their effective parts. However, there are classes of maps for which this assignment is indeed functorial. In this subsection, we shall explore this functoriality.

Definition III.5.9. Let P be a property of a map of spaces which forms a subcategory of S. We say that P is **étale invariant** if the following two properties are satisfied:

i) P is stable under pre-composition with local homeomorphisms

ii) P is stable under pullbacks along local homeomorphisms.

If in addition, every morphism in P is open, we say that P is a class of **open étale invariant** maps. Examples of such open étale invariant maps are open maps, local homeomorphisms, or, in the smooth setting, submersions. We say a map $\psi : \mathcal{G} \to \mathcal{H}$ of étale S-groupoids has property P if both ψ_0 and ψ_1 do. We denote the corresponding 2-category of S-groupoids as $(S - Gpd)^{et}_P$. We say a morphism

$$\varphi : \mathscr{Y} \to \mathscr{X}$$

has property P is there exists a homomorphism of étale S-groupoids

$$\psi : \mathcal{G} \to \mathcal{H}$$

with property P, such that

$$\varphi \cong [\psi] .$$

Warning: Do not confuse notation with $(S^{et} - Gpd)$, the 2-category of étale S-groupoids and only local homeomorphisms. This only agrees with $(S - Gpd)^{et}_P$ when P is local homeomorphisms.

Remark. This agrees with our previous definition of a local homeomorphism of étale stacks in the case P is local homeomorphisms. When P is open maps, under the correspondence between étale stacks and étendues, this agrees with the notion of an open map of topoi in the sense of [29]. When P is submersions, this is equivalent to the definition of a submersion of smooth étendues given in [41].

Remark. Notice that being étale invariant implies being invariant under restriction and local on the target, as in Definition I.2.23.

Proposition III.5.7. *Let P be a property of a map of spaces which forms a subcategory of S. P is étale invariant if and only if the following conditions are satisfied:*

i) every local homeomorphism is in P

ii) for any commutative diagram

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ X \arrow[r, "f"] & Z, \end{tikzcd}$$

with both g and g' local homeomorphisms, if f has property P, then so does f'.

Proof. Suppose that P is étale invariant. Then, as P is a subcategory, it contains all the identity arrows, and since it is stable under pre-composition with local homeomorphisms, this implies that every local homeomorphism is in P. Now suppose that $f \in P$, and

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ X \arrow[r, "f"] & Z, \end{tikzcd}$$

is commutative with both g and g' local homeomorphisms. Then as P is stable under pullbacks along local homeomorphisms, the induced map $X \times_Z Y \to Y$

has property P. Moreover, as local homeomorphisms are invariant under change of base (Definition I.2.23), the induced map $X \times_Z Y \to X$ is a local homeomorphism. It follows that the induced map $W \to X \times_Z Y$ is a local homeomorphism, and since f' can be factored as

$$W \to X \times_Z Y \to Y,$$

and P is stable under pre-composition with local homeomorphisms, it follows that f' has property P. Conversely, suppose that the conditions of the proposition are satisfied. Condition *ii*) clearly implies that P is stable under pullbacks along local homeomorphisms. Suppose that $e : W \to X$ is a local homeomorphism and $f : X \to Z$ is in P. Then as

$$\begin{tikzcd} W \arrow[r, "f\circ e"] \arrow[d, "e"'] & Z \arrow[d, "id_Z"] \\ X \arrow[r, "f"] & Z, \end{tikzcd}$$

commutes, it follows that the $f \circ e$ has property P. □

Lemma III.5.1. *For any open étale invariant P, the assignment*

$$\mathcal{H} \mapsto \mathrm{Eff}\,(\mathcal{H})$$

extends to a 2-functor

$$\mathrm{Eff}_P : (S - Gpd)^{et}_P \to \mathrm{Eff}\,(S - Gpd)^{et}_P$$

from étale S-groupoids and P-morphisms to effective étale S-groupoids and P-morphisms.

Proof. Suppose $\varphi : \mathcal{G} \to \mathcal{H}$ has property P. Since Eff does not affect objects, we define

$$\mathrm{Eff}\,(\varphi)_0 = \varphi_0.$$

Given $g \in \mathcal{G}_1$, denote its image in $\mathrm{Eff}\,(\mathcal{G})_1$ by $[g]$. Define

$$\mathrm{Eff}\,(\varphi)_1\,([g]) = [\varphi\,(g)]\,.$$

We need to show that this is well defined. Suppose that $[g] = [g']$. Let V_g and $V_{g'}$ be neighborhoods of g and g' respectively, on which both s and t restrict to embeddings. Denote by x the source of g and g'. Then there exists a neighborhood W of x over which

$$t \circ s^{-1}|_g$$

and

$$t \circ s^{-1}|_{g'}$$

agree. Since φ_1 has property P, it is open, so $\varphi_1(V_g)$ is a neighborhood of $\varphi_1(g)$, and similarly for g'. By making V_g and V_g' smaller if necessary, we may assume that s and t restrict to embeddings on $\varphi_1(V_g)$ and $\varphi_1\left(V_g'\right)$. Since φ is a groupoid homomorphism, it follows that

$$t \circ s^{-1}|_{\varphi_1(V_g)}$$

and

$$\varphi_0 \circ t \circ s^{-1}|_{V_g}$$

agree on W, and similarly for g'. Hence, if g and g' induce the same germ of a locally defined homeomorphism, so do $\varphi_1(g)$ and $\varphi_1(g')$. It is easy to check that $\mathrm{Eff}(\varphi)$ as defined is a homomorphism of S-groupoids. In particular, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Eff}(\mathcal{G})_1 \arrow[r, "\mathrm{Eff}(\varphi)_1"] \arrow[d, "s"'] & \mathrm{Eff}(\mathcal{H}) \arrow[d, "s"] \\
\mathcal{G}_0 \arrow[r, "\varphi_0"] & \mathcal{H}_0.
\end{tikzcd}$$

Since P is étale invariant and the source maps are local homeomorphisms, it implies that $\mathrm{Eff}(\varphi)_1$ has property P. The rest is proven similarly. □

Theorem III.5.2. *Let $j_P : \mathrm{Eff}(S-Gpd)^{et}_P \hookrightarrow (S-Gpd)^{et}_P$ be the inclusion. Then Eff_P is left adjoint to j_P.*

Proof. There is a canonical natural isomorphism

$$\mathrm{Eff}_P \circ j_P \Rightarrow id_{\mathrm{Eff}(S-Gpd)^{et}_P}$$

since any effective étale groupoid is canonically isomorphic to its effective part. Furthermore, the maps $\iota_{\mathcal{H}}$ assemble into a natural transformation

$$\iota : id_{(S-Gpd)^{et}_P} \Rightarrow j_P \circ \mathrm{Eff}_P\,.$$

It is easy to check that these define a 2-adjunction. □

Theorem III.5.3. Eff_P *sends Morita equivalences to Morita equivalences.*

Proof. Suppose $\varphi : \mathcal{G} \to \mathcal{H}$ is a Morita equivalence. Since $\mathcal{G}$ and $\mathcal{H}$ are étale, this implies φ is a local homeomorphism. Hence, in the pullback diagram

$$\begin{tikzcd}
\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{G}_0 \arrow[d, "\varphi_0"] \\
\mathcal{H}_1 \arrow[r, "s"] & \mathcal{H}_0,
\end{tikzcd}$$

pr_1 is a local homeomorphism, and therefore the map

$$t \circ pr_1 : \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \to \mathcal{H}_0$$

is as well. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \arrow[r, "t \circ pr_1"] \arrow[d] & \mathcal{H}_0 \\
\mathrm{Eff}\,(\mathcal{H})_1 \times_{\mathcal{H}_0} \mathcal{G}_0. \arrow[ur, "t \circ pr_1"'] &
\end{tikzcd}$$

The map

$$\mathrm{Eff}\,(\mathcal{H})_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \to \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0$$

is the pullback of a local homeomorphism, hence one itself, and the upper arrow $t \circ pr_1$ is a local homeomorphism. This implies

$$t \circ pr_1 : \mathrm{Eff}\,(\mathcal{H})_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \to \mathcal{H}_0$$

is a local homeomorphism as well. In particular, it admits local sections, and if S is manifolds, is a surjective submersion. Therefore $\mathrm{Eff}\,(\varphi)$ is essentially surjective. Now suppose that

$$[h] : \varphi(x) \to \varphi(y).$$

Then

$$h : \varphi(x) \to \varphi(y).$$

So there is a unique $g : x \to y$ such that $\varphi(g) = h$. Now suppose

$$[h] = [h'].$$

We can again choose a unique g' such that $\varphi(g') = h'$. We need to show that

$$[g] = [g'].$$

Let V_g and $V_{g'}$ be neighborhoods of g and g' respectively chosen so small that s and t of $\mathcal{G}_1$ restrict to embeddings on them, s and t of $\mathcal{H}_1$ restrict to embeddings on $\varphi_1(V_g)$ and $\varphi_1\left(V_g'\right)$, φ_0 restricts to an embedding on $s(V_g)$, which is possible since φ_0 is a local homeomorphism, and

$$t \circ s^{-1}|_{\varphi_1(V_g)}$$

and

$$s^{-1}|_{\varphi_1\left(V_g'\right)}$$

agree on $\varphi_0(s(V_g))$, which is possible since

$$[\varphi(g)] = [\varphi(g')].$$

Then by the proof of Lemma III.5.1,

$$t \circ s^{-1}|_{\varphi_1(V_g)}$$

and

$$\varphi_0 \circ t \circ s^{-1}|_{V_g}$$

agree on $s(V_g)$, and similarly for g'. Hence

$$\varphi_0 \circ t \circ s^{-1}|_{V_g}$$

and

$$\varphi_0 \circ t \circ s^{-1}|_{V_g'}$$

agree on W, but φ_0 is an embedding when restricted to W, hence

$$t \circ s^{-1}|_{V_g}$$

and

$$t \circ s^{-1}|_{V_g'}$$

agree on W so $[g] = [g']$. □

Lemma III.5.4. *Let $\mathcal{U}$ be an étale cover of $\mathcal{H}_0$, with $\mathcal{H}$ an étale S-groupoid. Then there is a canonical isomorphism between* $\mathrm{Eff}(\mathcal{H}_{\mathcal{U}})$ *and* $(\mathrm{Eff}(\mathcal{H}))_{\mathcal{U}}$ *(See Definition III.2.5).*

Proof. Both of these groupoids have the same object space. It suffices to show that their arrow spaces are isomorphic (and that this determines an internal functor). Suppose the cover $\mathcal{U}$ is given by a local homeomorphism $e : U \to \mathcal{H}_0$. An arrow in $\mathcal{H}_{\mathcal{U}}$ is a triple

$$(h, p, q)$$

with

$$h : e(p) \to e(p).$$

An arrow in $(\mathrm{Eff}(\mathcal{H}))_{\mathcal{U}}$ is a triple

$$([h], p, q)$$

such that $[h]$ is the image of an arrow $h \in \mathcal{H}_1$ under $\iota_{\mathcal{H}}$ such that

$$h : e(p) \to e(p).$$

Define a map

$$\begin{aligned} (\mathcal{H}_{\mathcal{U}})_1 &\to ((\mathrm{Eff}(\mathcal{H}))_{\mathcal{U}})_1 \\ (h, p, q) &\mapsto ([h], p, q). \end{aligned} \tag{III.9}$$

This map is clearly surjective.

We make the following claim:

$$[h] = [h']$$

if and only if

$$[(h, p, q)] = [(h', p, q)]$$

Suppose that

$$[h] = [h'] .$$

Pick a neighborhood U_h of h in $\mathcal{H}_1$ such that both s and t are injective over it, and U'_h an analogous neighborhood of h'. Let W be a neighborhood of $s(h) = s(h')$ over which

$$t \circ s|_{U_h}^{-1} = t \circ s|_{U'_h}^{-1}. \tag{III.10}$$

Pick neighborhoods V_p and V_q of p and q respectively so small that e is injective over them, and for all $a \in V_p$,

$$e(a) \in W$$

and

$$t \circ s|_{U_h}^{-1}(e(a)) \in e(V_q) .$$

As the arrow space $(\mathcal{H}_{\mathcal{U}})_1$ fits into the pullback diagram

$$\begin{tikzcd}
(\mathcal{H}_{\mathcal{U}})_1 \arrow[r] \arrow[d, "{(s,t)}"'] & \mathcal{H}_1 \arrow[d, "{(s,t)}"] \\
U \times U \arrow[r] & \mathcal{H}_0 \times \mathcal{H}_0,
\end{tikzcd}$$

$(V_p \times V_q \times U_h) \cap (\mathcal{H}_{\mathcal{U}})_1$ is a neighborhood of (h, p, q) over which both the source and target maps are injective. The set $(V_p \times V_q \times U'_h) \cap (\mathcal{H}_{\mathcal{U}})_1$ is an analogous neighborhood of (h', p, q). The local inverse of s through (h, p, q) is then given by

$$a \mapsto \left(s|_{U_h}^{-1}(e(a)), a, e|_{V_p}^{-1}\left(t \circ s|_{U_h}^{-1}(e(a))\right)\right) .$$

Hence, the germ associated to (h, p, q) is the germ of

$$a \mapsto e|_{V_p}^{-1}\left(t \circ s|_{U_h}^{-1}(e(a))\right) .$$

Similarly the germ associated to (h', p, q) is the germ of

$$a \mapsto e|_{V_p}^{-1}\left(t \circ s|_{U'_h}^{-1}(e(a))\right) .$$

From equation (III.10), it follows that these maps are identical. Moreover, supposing instead that

$$[(h, p, q)] = [(h', p, q)] ,$$

by the above argument, it follows that $[h]=[h']$ since e is injective over V_q.

Hence the assignment (III.9) depends only on the image of (h,p,q) in $\mathrm{Eff}\,(\mathcal{H}_{\mathcal{U}})$. So there is an induced well defined and surjective map

$$(\mathrm{Eff}\,(\mathcal{H}_{\mathcal{U}}))_1 \to ((\mathrm{Eff}\,(\mathcal{H}))_{\mathcal{U}})_1\,. \tag{III.11}$$

Since, $[h]=[h']$ implies $[(h,p,q)]=[(h',p,q)]$, it follows that this map is also injective, hence bijective. It is easy to check that it is moreover a homeomorphism. It clearly defines a groupoid homomorphism $\square$

Corollary III.5.2. *There is an induced* 2*-adjunction*

$$\mathfrak{EffEt}_P \underset{j_P}{\overset{\mathrm{Eff}_P}{\rightleftarrows}} \mathfrak{Et}_P,$$

between étale stacks with P*-morphisms and effective étale stacks with* P*-morphisms, where* j_P *is the canonical inclusion.*

Proof. Let $\mathcal{U}$ be an étale cover of $\mathcal{H}_0$, with $\mathcal{H}$ an étale S-groupoid. From the previous lemma, there is a canonical isomorphism between $\mathrm{Eff}\,(\mathcal{H}_{\mathcal{U}})$ and $(\mathrm{Eff}\,(\mathcal{H}))_{\mathcal{U}}$. Let $\mathcal{G}$ be an effective étale S-groupoid. Then

$$\begin{aligned}
\mathrm{Hom}\left([\mathrm{Eff}\,(\mathcal{H})]\,,[\mathcal{G}]\right) &\simeq \underset{\underset{\mathcal{U}}{\longrightarrow}}{\mathrm{holim}}\ \mathrm{Hom}\left((\mathrm{Eff}\,(\mathcal{H}))_{\mathcal{U}}\,,\mathcal{G}\right)\\
&\simeq \underset{\underset{\mathcal{U}}{\longrightarrow}}{\mathrm{holim}}\ \mathrm{Hom}\left(\mathrm{Eff}\,(\mathcal{H}_{\mathcal{U}})\,,\mathcal{G}\right)\\
&\simeq \underset{\underset{\mathcal{U}}{\longrightarrow}}{\mathrm{holim}}\ \mathrm{Hom}\left(\mathcal{H}_{\mathcal{U}}, j_P\mathcal{G}\right)\\
&\simeq \mathrm{Hom}\left([\mathcal{H}]\,, j_P\,[\mathcal{G}]\right).
\end{aligned}$$

$\square$

Note that this implies that $\mathfrak{EffEt}_P$ is a localization of $\mathfrak{Et}_P$ with respect to those morphisms whose image under Eff_P become equivalences. When P is local homeomorphisms, denote $P=et$. We make the following definition for later:

Definition III.5.10. A morphism $\varphi:\mathscr{Y}\to\mathscr{X}$ between étale stacks is called an **effective local equivalence** if φ is a local homeomorphism and $\mathrm{Eff}_{et}\,(\varphi)$ is an equivalence.

III.6 Small Gerbes

III.6.1 Gerbes

Gerbes are a special type of stack. Gerbes were first introduced by Jean Giraud in [25]. Intuitively, gerbes are to stacks what groups are to groupoids. In some sense, a gerbe is "locally" a sheaf of groups. The most concise definition of a gerbe is:

Definition III.6.1. A **gerbe** over a Grothendieck site $(\mathscr{C}, J)$ is a stack $\mathscr{G}$ over $\mathscr{C}$ such that

i) the unique map $\mathscr{G} \to *$ to the terminal sheaf is an epimorphism, and

ii) the diagonal map $\mathscr{G} \to \mathscr{G} \times \mathscr{G}$ is an epimorphism.

The first condition means that for any object $C \in \mathscr{C}_0$, the unique map $C \to *$ *locally* factors through $\mathscr{G} \to *$, up to isomorphism. Spelling this out means that there exists a cover $(f_\alpha : C_\alpha \to C)$ of C such that each groupoid $\mathscr{G}(C_\alpha)$ is non-empty. This condition is often phrased by saying $\mathscr{G}$ is *locally non-empty.*

The second condition means that for all C, any map $C \to \mathscr{G} \times \mathscr{G}$ *locally* factors through the diagonal $\mathscr{G} \to \mathscr{G} \times \mathscr{G}$ up to isomorphism. Spelling this out, any map $C \to \mathscr{G} \times \mathscr{G}$, by Yoneda, corresponds to objects x and y of the groupoid $\mathscr{G}(C)$. The fact that this map locally factors through the diagonal means that, given two such objects x and y, there exists a cover $(g_\beta : C_\beta \to C)$ of C such that for all β,

$$\mathscr{G}(f_\beta)(x) \cong \mathscr{G}(f_\beta)(y)$$

in $\mathscr{G}(C_\beta)$. This condition is often phrased by saying $\mathscr{G}$ is *locally connected.*

If it were not for the locality of these properties, then this would mean that each $\mathscr{G}(C)$ would be a non-empty and connected groupoid, hence, equivalent to a group.

Definition III.6.2. The full sub-2-categeory of St $(\mathscr{C})$ on all gerbes, is called the 2-category of gerbes and is denoted by $\mathcal{G}erbe(\mathscr{C})$.

Definition III.6.3. Let $(\mathscr{C}, J)$ be a Grothendieck site, then a **bouquet** over $(\mathscr{C}, J)$ is a groupoid object in sheaves, $\mathfrak{G}$, such that

i) the canonical map $\mathfrak{G}_0 \to *$ to the terminal sheaf is an epimorphism, and

ii) the canonical map $(s, t) : \mathfrak{G}_1 \to \mathfrak{G}_0 \times \mathfrak{G}_0$ is an epimorphism.

Notice the similarity of this definition with that of Definition III.6.1.

Theorem III.6.1. *A stack $\mathscr{X}$ over $(\mathscr{C}, J)$ is a gerbe if and only if it is equivalent to the stack associated to a bouquet $\mathfrak{G} \in Gpd(Sh(\mathscr{C}))$. [22]*

III.6.2 Small gerbes over an étale stack

Definition III.6.4. A **small gerbe** over an étale stack $\mathscr{X}$ is a small stack $\mathscr{G}$ over $\mathscr{X}$ which is a gerbe. To be more concrete, a small gerbe over $[\mathcal{H}]$ is a gerbe over the site $Site(\mathcal{H})$.

Remark. Under the correspondence between étale stacks and etendues, we could also define a small gerbe over $\mathscr{X}$ as a gerbe over the topos $Sh(\mathscr{X})$.

Lemma III.6.2. *Let $\mathscr{X}$ be an étale stack and let $f : \mathscr{Z} \to \mathscr{X}$ be a local homeomorphism. Then f is an epimorphism in $\mathrm{St}(S)$ if and only if f is an epimorphism when considered as a map in $Et(\mathscr{X})$ from $\mathscr{Z} \to \mathscr{X}$ to the terminal object $\mathscr{X} \to \mathscr{X}$, where $Et(\mathscr{X})$ is the 2-category of local homeomorphisms over $\mathscr{X}$.*

Proof. Fix an étale S-groupoid $\mathcal{H}$ such that $\mathscr{X} \simeq [\mathcal{H}]$. If f is an epimorphism in $\mathrm{St}(S)$, then any map $T \to \mathscr{X}$ from a space T locally factors through f up to isomorphism. In particular, this holds for every local homeomorphism $T \to \mathscr{X}$ from a space. Hence, f is an epimorphism in $Et(\mathscr{X})$. Conversely, suppose that f is an epimorphism in $Et(\mathscr{X})$. Then the atlas $a : \mathcal{H}_0 \to \mathscr{X}$ locally factors through f up to isomorphism. However, every map $T \to \mathscr{X}$ from a space locally factors through a up to isomorphism as well. It follows that f is an epimorphism in $\mathrm{St}(S)$. □

Corollary III.6.1. *Let $f : \mathscr{Z} \to \mathscr{X}$ be a local homeomorphism of étale stacks. Then the stack in $\mathrm{St}(\mathscr{X})$ represented by f is a gerbe over $\mathscr{X}$ if and only if*

i) f is an epimorphism in $\mathrm{St}(S)$, and

ii) the induced map $\mathscr{Z} \to \mathscr{Z} \times_{\mathscr{X}} \mathscr{Z}$ is an epimorphism in $\mathrm{St}(S)$.

In other words, when identifying f with an object of $\mathrm{St}(S/\mathscr{X})$, it is a gerbe.

Proof. It suffices to show that if $q : \mathscr{Z} \to \mathscr{Z} \times_{\mathscr{X}} \mathscr{Z}$ is an epimorphism in $Et(\mathscr{X})$, then it is an epimorphism in $\mathrm{St}(S)$. Choose an étale atlas

$$Y \to \mathscr{Z} \times_{\mathscr{X}} \mathscr{Z}$$

for $\mathscr{Z} \times_{\mathscr{X}} \mathscr{Z}$. Then this atlas locally factors through q up to isomorphism. However, any map $T \to \mathscr{Z} \times_{\mathscr{X}} \mathscr{Z}$ locally factors through Y up to isomorphism. □

Remark. In the differentiable setting, this implies that the étalé realization of a small gerbe $\mathscr{G}$ over an étale stack $\mathscr{X}$, in particular, is a differentiable gerbe over $\mathscr{X}$ in the sense of [9], Definition 4.7.

Definition III.6.5. Let $\mathcal{H}$ be an étale S-groupoid. By a bouquet over $\mathcal{H}$, we mean a bouquet over $Site(\mathcal{H})$. Explicitly, this is a groupoid object $\mathfrak{G}$ in $\mathcal{BH}$ such that

i) $\mu_0 : \mathfrak{G}_0 \to \mathcal{H}_0$ is surjective, and

ii) $(s,t) : \mathfrak{G}_1 \to \mathfrak{G}_0 \times_{\mathcal{H}_0} \mathfrak{G}_0$ is surjective.

In light of Theorem III.6.1 and the adjoint-equivalence

$$\mathrm{St}(\mathscr{X}) \underset{\bar{L}}{\overset{\bar{\Gamma}}{\rightleftarrows}} Et(\mathscr{X}),$$

of Corollary III.3.4, we have the following corollary:

Corollary III.6.2. *For an étale stack $\mathscr{X} \simeq [\mathcal{H}]$, the 2-category of small gerbes over $\mathscr{X}$, $\mathfrak{Gerbe}(\mathscr{X})$, is equivalent to the full sub-2-category of those local homeomorphisms $\mathscr{Z} \to \mathscr{X}$ in $Et(\mathscr{X})$ which are of the form $\bar{L}(\mathfrak{G})$ for a bouquet $\mathfrak{G}$ over $\mathcal{H}$.*

III.6.3 Characterizing gerbes by their stalks

In this subsection, we will show that small gerbes over an étale stack have a simple characterization in terms of their stalks:

Theorem III.6.3. *Let $\mathscr{X}$ be an étale stack. A small stack $\mathscr{Z}$ over $\mathscr{X}$ is a small gerbe if and only if for every point*

$$x : * \to \mathscr{X},$$

the stalk $\mathscr{Z}_x$ of $\mathscr{Z}$ at x is equivalent to a group.

Proof. Fix $\mathcal{H}$ an étale groupoid such that $\mathscr{X} \simeq [\mathcal{H}]$ and fix $\tilde{x} \in \mathcal{H}_0$ a point such that $x \cong p \circ \tilde{x}$, where $p : \mathcal{H}_0 \to \mathscr{X}$ is the atlas associated to $\mathcal{H}$. Suppose that $\mathscr{G}$ is a small gerbe over $\mathscr{X}$. Then, since $\mathscr{G}$ is locally non-empty,

$$\mathscr{Z}_x \simeq \underset{\tilde{x} \in U}{\underrightarrow{\mathrm{holim}}}\, \mathscr{Z}(U),$$

is a non-empty groupoid. Furthermore, since $\mathscr{G}$ is locally connected, it follows that

$$\underset{\tilde{x} \in U}{\underrightarrow{\mathrm{holim}}}\, \mathscr{Z}(U)$$

is also connected, hence, equivalent to a group.

Conversely, suppose that $\mathscr{Z}$ is a small stack and that

$$\mathscr{Z}_x \simeq \underset{\tilde{x} \in U}{\underrightarrow{\mathrm{holim}}}\, \mathscr{Z}(U)$$

is equivalent to a group. This means it is a non-empty and connected groupoid. It follows that $\mathscr{Z}$ is locally non-empty and locally connected, hence a gerbe. □

The significance of this theorem is the following:

Suppose we are given an effective étale stack $\mathscr{X}$ and a small gerbe $\mathscr{G}$ over it. By taking stalks, we get an assignment to each point x of $\mathscr{X}$ a group $\mathscr{G}_x$. From this data, we can build a new étale stack by taking the étale realization of $\mathscr{G}$. Denote this new étale stack by $\mathscr{Y}$. As it will turn out, if $\mathscr{G}$ is non-trivial, $\mathscr{Y}$ will not be effective, but it will have $\mathscr{X}$ as its effective part and, for each point x of $\mathscr{X}$, the stalk $\mathscr{G}_x$ will be equivalent to the ineffective isotropy group of x in $\mathscr{Y}$. In particular, if $\mathscr{X}$ is a space X, $\mathscr{Y}$ will be an étale stack which "looks like X" except that each point $x \in X$, instead of having a trivial automorphism group, will have a group equivalent to $\mathscr{G}_x$ as an automorphism group. In this case, every automorphism group will consist entirely of purely ineffective automorphisms.

III.6.4 Gerbes are full effective local equivalences

In this subsection, we will characterize which small stacks $\mathscr{Z}$ over an étale stack $\mathscr{X}$ are gerbes in terms of their étale realization. In particular, we will show that when $\mathscr{X}$ is effective, gerbes over $\mathscr{X}$ are the same as étale stacks $\mathscr{Y}$ whose effective part are equivalent to $\mathscr{X}$.

Let $\mathcal{H}$ be an étale S-groupoid and let $\mathcal{K}$, with

$$\mu_i : \mathcal{K}_i \to \mathcal{H}_0$$

for $i = 0, 1$, be a groupoid object in $\mathcal{BH}$. Then the map

$$\theta_{\mathcal{K}} : \mathcal{H} \ltimes \mathcal{K} \to \mathcal{H}$$

factors through the canonical map

$$p_{\mathcal{K}} : \mathcal{H}_{\mu_0} \to \mathcal{H}.$$

Recall (Definition III.2.5) that $\mathcal{H}_{\mu_0}$ has $\mathcal{K}_0$ as object space, and an arrow from x to y is an arrow

$$h : \mu_0\left(x\right) \to \mu_0\left(y\right)$$

in $\mathcal{H}$, and we write such an arrow as (h, x, y). Define $\theta'_{\mathcal{K}}$ on objects to be the identity, and on arrows by sending an arrow (h, k) in $\mathcal{H} \ltimes \mathcal{K}$ to

$$\left(h, s\left(k\right), t\left(k\right)\right).$$

Then $\theta_{\mathcal{K}} = p_{\mathcal{K}} \circ \theta'_{\mathcal{K}}$.

Lemma III.6.4. *In the situation above, let $\mathcal{H}$ be effective. Then for two arrows (h_i, k_i) $i = 1, 2$, in $\mathcal{H} \ltimes \mathcal{K}$,*

$$\theta'_{\mathcal{K}}\left(h_1, k_1\right) = \theta'_{\mathcal{K}}\left(h_2, k_2\right) \tag{III.12}$$

if and only if

$$\left[\left(h_1, k_1\right)\right] = \left[\left(h_1, k_1\right)\right], \tag{III.13}$$

where the bracket denotes the image in $\mathrm{Eff}\left(\mathcal{H} \ltimes \mathcal{K}\right)$.

Proof. Suppose that III.12 holds, with $k_i : h_i \cdot x_i \to y_i$. Then

$$h_1 = h_2 =: h,$$

$$x_1 = x_2 := x,$$

and

$$y_1 = y_2 =: y.$$

Let V be a neighborhood of hx in $\mathcal{K}_0$ over which μ_0 restricts to an embedding. Let U be a neighborhood of h in $\mathcal{H}_1$ over which s and t restrict to embeddings, and W_i be analogous neighborhoods of k_1 and k_2 in $\mathcal{K}_1$. For all i, let

$$\mathcal{O}_i := \left(\left(W_i \cap \mu_1^{-1}(U)\right) \times U\right) \cap (\mathcal{H} \ltimes \mathcal{K})_1 \subset (\mathcal{H} \ltimes \mathcal{K})_1 .$$

Let

$$M := \bigcap_{i=1,2} \left(t\left(\mathcal{O}_i \cap s \circ t|_{W_i}^{-1}(V)\right)\right) \subset \mathcal{K}_0,$$

and

$$f_i : M \to \mathcal{K}_1$$

be given by $f_i := \left(t|_{W_i}^{-1}\right)|_M$. Then the target map of $\mathcal{H} \ltimes \mathcal{K}$ restricts to an embedding over $\mathcal{O}_i$, and letting

$$\sigma_i := \left(t|_{\mathcal{O}_i}^{-1}\right)|_M,$$

we have

$$\left[(h_i, k_i)^{-1}\right] = s \circ \sigma_i.$$

Moreover, for each $x \in \mathcal{O}_i$,

$$\sigma_i(x) = \left(t|_U^{-1}(\mu_1(f_i(x))), f_i(x)\right).$$

Since for all i,

$$\mu_1(f_i(x)) = \mu_0(t \circ f_i(x)) = \mu_0(x),$$

this simplifies to

$$\sigma_i(x) = \left(t|_U^{-1}(\mu_0(x)), f_i(x)\right).$$

So, for all i,

$$s \circ \sigma_i(x) = \left(t|_U^{-1}(\mu_0(x))\right)^{-1} \cdot s(f_i(x)).$$

But

$$\mu_0(x) = \mu_0(s \circ f_1(x)) = \mu_0(s \circ f_2(x)),$$

and $s \circ f_i(x) \in V$ for all i, so,

$$s \circ f_1(x) = s \circ f_2(x).$$

This implies

$$s \circ \sigma_1(x) = s \circ \sigma_2(x),$$

i.e.

$$\left[(h_1, k_1)^{-1}\right] = \left[(h_2, k_2)^{-1}\right].$$

Hence

$$[(h_1, k_1)] = [(h_2, k_2)].$$

Conversely, suppose

$$[(h_1, k_1)] = [(h_2, k_2)].$$

Then

$$\left(t|_{U_1}^{-1}(\mu_0(x))\right)^{-1} \cdot s(f_1(x)) = \left(t|_{U_2}^{-1}(\mu_0(x))\right)^{-1} \cdot s(f_2(x)) \tag{III.14}$$

on some neighborhood Ω of x, which we may assume maps homeomorphically onto its image under μ_0. Applying μ_0 to (III.14) yields

$$s \circ t|_{U_1}^{-1}(y) = s \circ t|_{U_2}^{-1}(y),$$

for all $y \in \mu_0(\Omega)$. This implies that h_1 and h_2 have the same germ. As $\mathcal{H}$ is effective, this implies $h_1 = h_2$. This in turn implies that $s \circ f_1$ and $s \circ f_2$ agree on Ω, hence k_1 and k_2 have the same germ. In particular, they have the same source and target, hence

$$\theta'_{\mathcal{K}}(h_1, k_1) = \theta'_{\mathcal{K}}(h_2, k_2).$$

□

Theorem III.6.5. *Let $\mathcal{H}$ be an effective étale S-groupoid and let $\mathfrak{G}$ be a bouquet over $\mathcal{H}$. Then*

$$\mathrm{Eff}(\mathcal{H} \ltimes \mathfrak{G}) \cong \mathcal{H}_{\mu_0}.$$

Proof. Define a map $\varphi : \mathcal{H}_{\mu_0} \to \mathrm{Eff}(\mathcal{H} \ltimes \mathfrak{G})$ as follows. On objects define it as the identity. Notice that the arrows of $\mathcal{H}_{\mu_0}$ are triples (h, x, y) such that

$$h : \mu_0(x) \to \mu_0(y).$$

For such a triple,

$$(hx, y) \in \mathfrak{G}_0 \times_{\mathcal{H}_0} \mathfrak{G}_0.$$

Recall that

$$(s, t) : \mathfrak{G}_1 \to \mathfrak{G}_0 \times_{\mathcal{H}_0} \mathfrak{G}_0$$

is surjective. For each $(hx, y) \in \mathfrak{G}_1 \to \mathfrak{G}_0 \times_{\mathcal{H}_0} \mathfrak{G}_0$, choose a $\gamma \in \mathfrak{G}_0$ such that

$$\gamma : hx \to y,$$

and define

$$\kappa(h, x, y) := [(h, \gamma)].$$

From the previous lemma, κ does not depend on our choice. Moreover, (s,t) admits continuous local sections, so it follows that κ is continuous. Suppose that

$$\kappa(h,x,y)=\kappa(h',x',y').$$

Then

$$\theta(h,\gamma)=\theta(h',\gamma')$$

which in turn implies

$$(h,x,y)=(h',x',y').$$

Hence κ is injective. Now let $[(h,\gamma)]\in\mathrm{Eff}(\mathcal{H}\ltimes\mathfrak{G})_1$ be arbitrary. Then

$$[(h,\gamma)]=\kappa(h,s(\gamma),t(\gamma)),$$

so κ is bijective, hence an isomorphism. Moreover, identifying $\mathcal{H}_{\mu_0}$ with its effective part,

$$\kappa^{-1}=\mathrm{Eff}(\theta'_{\mathfrak{G}}),$$

and in particular, is continuous. □

Corollary III.6.3. *For $\mathfrak{G}$ a bouquet over an effective étale S-groupoid $\mathcal{H}$,*

$$L(\mathfrak{G})=\mathscr{G}\to\mathscr{X}$$

is an effective local equivalence over $\mathscr{X}\simeq[\mathcal{H}]$.

Proof. As $\theta_{\mathfrak{G}}=p_{\mathfrak{G}}\circ\theta'_{\mathfrak{G}}$ and $p_{\mathfrak{G}}$ is a Morita equivalence, it suffices to show that $\mathrm{Eff}(\theta'_{\mathfrak{G}})$ is an equivalence, but this is clear as κ is its inverse, by construction. □

Corollary III.6.4. *If $\mathscr{G}=(\rho:\underline{\mathscr{G}}\to\mathscr{X})\in\mathit{Gerbe}(\mathscr{X})$ is a small gerbe over an effective étale stack $\mathscr{X}$, $\underline{\mathscr{G}}\to\mathscr{X}$ is an effective local equivalence.*

Theorem III.6.6. *Consider the map of étale S-groupoids*

$$\iota_{\mathcal{G}}:\mathcal{G}\to\mathrm{Eff}(\mathcal{G})=:\mathcal{H}.$$

Then

$$\bar{\Gamma}([\iota_{\mathcal{G}}])\in\mathrm{St}([\mathcal{H}])$$

is a gerbe.

Proof. By Theorem III.4.2, it suffices to show that $P(\iota_{\mathcal{G}})$ is a bouquet over $\mathcal{H}$. The map

$$t\circ pr_1:\mathcal{H}_1\times_{\mathcal{H}_0}\mathcal{G}_0\to\mathcal{H}_0$$

is clearly surjective, as we may identify it with the map

$$t:\mathcal{H}_1\to\mathcal{H}_0.$$

It suffices to show that the map

$$\begin{array}{rcl} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_1 & \to & \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1 \\ (h,g) & \mapsto & (h[g],h) \end{array}$$

is surjective, where $\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1$ is the pullback

$$\begin{tikzcd} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1 \arrow[r] \arrow[d] & \mathcal{H}_1 \arrow[d, "t"] \\ \mathcal{H}_1 \arrow[r, "t"] & \mathcal{H}_0. \end{tikzcd}$$

Given l and l' in $\mathcal{H}_1$ with common target, choose g such that $[g] = l'^{-1}l$. Then (h,g) gets sent to (l,l'). □

Corollary III.6.5. *$\mathscr{G} = (\rho : \underline{\mathscr{G}} \to \mathscr{X}) \in Et(\mathscr{X})$ is a small gerbe over an effective étale stack $\mathscr{X}$ if and only if $\rho : \underline{\mathscr{G}} \to \mathscr{X}$ is an effective local equivalence.*

Corollary III.6.6. *If $\mathscr{X}$ is an orbifold, it encodes a small gerbe over its effective part* $\mathrm{Eff}(\mathscr{X})$ *via*

$$\iota_{\mathscr{X}} : \mathscr{X} \to \mathrm{Eff}(\mathscr{X}),$$

where ι is the unit of the adjunction in Theorem III.5.2.

Theorem III.6.7. *Let $\mathscr{X}$ be an effective étale stack and $\mathscr{G}$ a small gerbe over it. Denote by $\mathscr{Y}$ the underlying étale stack of the étale realization of $\mathscr{G}$. Then, under the natural bijection between the points of $\mathscr{X}$ and the points of $\mathscr{Y}$, for each point x, the stalk $\mathscr{G}_x$ is equivalent to the ineffective isotropy group of x in $\mathscr{Y}$, as defined in Definition III.5.5.*

Proof. Represent $\mathscr{X}$ by an étale groupoid $\mathcal{H}$ and $\mathscr{G}$ by a bouquet $\mathfrak{G}$ over $\mathcal{H}$. Denote the objects of the bouquet by

$$\mu_0 : \mathfrak{G}_0 \to \mathcal{H}_0.$$

Then the étale realization of $\mathscr{G}$ is induced by the map of groupoids

$$\theta'_{\mathfrak{G}} : \mathcal{H} \ltimes \mathfrak{G} \to \mathcal{H}_{\mu_0},$$

where $\mathcal{H}_{\mu_0}$ is the Čech groupoid with respect to the étale cover μ_0 and $\theta'_{\mathfrak{G}}$ is as defined in the beginning of this subsection. To the étale cover μ_0, there is an associated atlas

$$p' : \mathfrak{G}_0 \to \mathscr{X}.$$

Let x be a point of $\mathscr{X}$. Then there exists a point $\tilde{x} \in \mathfrak{G}_0$ such that $x \cong p' \circ \tilde{x}$. On one hand, from Section III.3.3, it follows that the stalk $\mathscr{G}_x$ is equivalent to the weak pullback in S-groupoids

$$\begin{tikzcd}
* \times_{\mathcal{H}_{\mu_0}} (\mathcal{H} \ltimes \mathfrak{G}) \arrow[rr] \arrow[d] & & \mathcal{H} \ltimes \mathfrak{G} \arrow[d, "\theta'_{\mathfrak{G}}"] \\
* \arrow[r, "\tilde{x}"] & \mathfrak{G}_0 \arrow[r, "p'"] & \mathcal{H}_{\mu_0},
\end{tikzcd}$$

which is necessarily a discrete groupoid.

An **object** of this groupoid can be described by pairs of the form (z, h) with $z \in \mathfrak{G}_0$ and $h \in \mathcal{H}_1$ such that

$$h : \mu_0(\tilde{x}) \to \mu_0(z).$$

An **arrow** from (z, h) to (z', h'), can be described simply as an arrow

$$\gamma : z \to z'$$

in $\mathfrak{G}_1$.

Since this groupoid is equivalent to a group, it must be equivalent to the isotropy group of any object. Consider the object $\left(\tilde{x}, \mathbb{1}_{\mu_0(\tilde{x})}\right)$. Then its isotropy group is canonically isomorphic to $\mathfrak{G}_{\tilde{x}}$, the isotropy group of $\tilde{x}$ in $\mathfrak{G}$.

On the other hand, the ineffective isotropy group of x is isomorphic to the kernel of the homomorphism

$$(\mathcal{H} \ltimes \mathfrak{G})_{\tilde{x}} \to \mathit{Diff}_{\tilde{x}}(\mathfrak{G}_0)$$

induced from the canonical map

$$(\mathcal{H} \ltimes \mathfrak{G}) \to \mathscr{H}(\mathfrak{G}_0).$$

From Lemma III.6.4, it follows that the this is the same as the kernel of the map

$$\begin{array}{rcl} (\mathcal{H} \ltimes \mathfrak{G})_{\tilde{x}} & \to & \mathcal{H}_{\mu_0(\tilde{x})} \\ (h, l) & \mapsto & h \end{array}$$

which is induced from $\theta'_{\mathfrak{G}}$ and the canonical identification

$$(\mathcal{H}_{\mu_0})_{\tilde{x}} \cong \mathcal{H}_{\mu_0(\tilde{x})}.$$

This kernel is clearly isomorphic to $\mathfrak{G}_{\tilde{x}}$ as well. □

Hence, we can use the data of a small gerbe over an effective étale stack to add ineffective isotropy groups to its points, as claimed. The rest of this subsection will be devoted to characterizing gerbes over general étale stacks which need not be effective.

Definition III.6.6. A map of stacks $\mathscr{X} \to \mathscr{Y}$ is **full** if for every space T, the induced map of groupoids $\mathscr{X}(T) \to \mathscr{Y}(T)$ if full as a functor.

Proposition III.6.1. *Let $\rho : \underline{\mathscr{G}} \to \mathscr{X}$ be a small gerbe over an étale stack. Then ρ is a full epimorphism.*

Proof. The fact that it is an epimorphism is clear. To see that it is full, we may assume it is of the form $\bar{L}(\mathfrak{G})$ for a bouquet $\mathfrak{G}$. This means it is the stackification of the map

$$\theta_{\mathfrak{G}} : \mathcal{H} \ltimes \mathfrak{G} \to \mathcal{H},$$

where $\mathscr{X} \simeq [\mathcal{H}]$. Such a map is clearly full. □

Theorem III.6.8. *Let $\rho : \underline{\mathscr{G}} \to \mathscr{Y}$ be a local homeomorphism of étale stacks. If ρ is a small gerbe over $\mathscr{Y}$, then*

$$\iota_{\mathscr{Y}} \circ \rho : \underline{\mathscr{G}} \to \mathrm{Eff}(\mathscr{Y})$$

is a small gerbe over $\mathrm{Eff}(\mathscr{Y})$*. Conversely, $\rho : \underline{\mathscr{G}} \to \mathscr{Y}$ is a small gerbe over $\mathscr{Y}$ if and only if*

$$\iota_{\mathscr{Y}} \circ \rho : \underline{\mathscr{G}} \to \mathrm{Eff}(\mathscr{Y})$$

is a small gerbe over $\mathrm{Eff}(\mathscr{Y})$ *and ρ is full.*

Proof. Suppose that $\rho : \underline{\mathscr{G}} \to \mathscr{Y}$ is a small gerbe over $\mathscr{Y}$. In particular, this implies ρ is an epimorphism. From Theorem III.6.6, $\iota_{\mathscr{Y}}$ is a gerbe, hence also an epimorphism. This implies $\iota_{\mathscr{Y}} \circ \rho$ is an epimorphism. Let $\mathscr{X} := \mathrm{Eff}(\mathscr{Y})$. The following diagram is a 2-pullback:

$$\begin{tikzcd}
\underline{\mathscr{G}} \times_{\mathscr{Y}} \underline{\mathscr{G}} \arrow[r] \arrow[d] & \mathscr{Y} \arrow[d] \\
\underline{\mathscr{G}} \times_{\mathscr{X}} \underline{\mathscr{G}} \arrow[r] & \mathscr{Y} \times_{\mathscr{X}} \mathscr{Y}.
\end{tikzcd}$$

Since $\iota_{\mathscr{Y}}$ is a gerbe, the map $\mathscr{Y} \to \mathscr{Y} \times_{\mathscr{X}} \mathscr{Y}$ is an epimorphism, hence so is

$$\underline{\mathscr{G}} \times_{\mathscr{Y}} \underline{\mathscr{G}} \to \underline{\mathscr{G}} \times_{\mathscr{X}} \underline{\mathscr{G}}.$$

The composite,

$$\underline{\mathscr{G}} \to \underline{\mathscr{G}} \times_{\mathscr{Y}} \underline{\mathscr{G}} \to \underline{\mathscr{G}} \times_{\mathscr{X}} \underline{\mathscr{G}}$$

is an epimorphism, since $\mathscr{G}$ is a gerbe over $\mathscr{Y}$. Hence $\mathscr{G}$ is a gerbe over $\mathscr{X}$.

Conversely, suppose that $\iota_{\mathscr{Y}} \circ \rho$ is a gerbe over $\mathscr{X}$ and that ρ is full. In particular, it is an epimorphism. Let $\varphi : \mathcal{G} \to \mathcal{K}$ be a map of S-groupoids such that

$$[\varphi] \cong \rho.$$

Let $\mathcal{H} = \mathrm{Eff}(\mathcal{K})$. Then the map

$$t \circ pr_1 : \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \to \mathcal{H}_0$$

is a surjective local homeomorphism. To show that ρ is an epimorphism, we want to show that the induced map

$$t \circ pr_1 : \mathcal{K}_1 \times_{\mathcal{K}_0} \mathcal{G}_0 \to \mathcal{K}_0 = \mathcal{H}_0$$

is a surjective local homeomorphism. It is automatically a local homeomorphism as pr_1 is the pullback of one and t is one. It suffices to show that it is surjective. However, it can be factors as

$$\mathcal{K}_1 \times_{\mathcal{K}_0} \mathcal{G}_0 \to \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \to \mathcal{H}_0.$$

To show that $\mathscr{G}$ is in fact a gerbe over $\mathscr{Y}$, we need to show that $\underline{\mathscr{G}} \to \underline{\mathscr{G}} \times_{\mathscr{Y}} \underline{\mathscr{G}}$ is an epimorphism. In terms of groupoids, this is showing that the map

$$t \circ pr_1 : (\mathcal{G} \times_{\mathcal{K}} \mathcal{G})_1 \times_{(\mathcal{G} \times_{\mathcal{K}} \mathcal{G})_0} \mathcal{G}_0 \to (\mathcal{G} \times_{\mathcal{K}} \mathcal{G})_0$$

is a surjective local homeomorphism, where $(\mathcal{G} \times_{\mathcal{K}} \mathcal{G})$ is a weak pullback of S-groupoids. To see that it is a local homeomorphism, note that we have the following commutative diagram:

$$\begin{tikzcd}
(\mathcal{G} \times_{\mathcal{H}} \mathcal{G})_1 \times_{(\mathcal{G} \times_{\mathcal{H}} \mathcal{G})_0} \mathcal{G}_0 \arrow[r, "et"] \arrow[d, "et"'] & (\mathcal{G} \times_{\mathcal{H}} \mathcal{G})_0 \arrow[d, "et"] \\
(\mathcal{G} \times_{\mathcal{K}} \mathcal{G})_1 \times_{(\mathcal{G} \times_{\mathcal{K}} \mathcal{G})_0} \mathcal{G}_0 \arrow[r] & (\mathcal{G} \times_{\mathcal{K}} \mathcal{G})_0 ,
\end{tikzcd}$$

where the maps marked as et are local homeomorphisms. Since $\underline{\mathscr{G}} \to \mathscr{X}$ is a gerbe, we know the map

$$(\mathcal{G} \times_{\mathcal{H}} \mathcal{G})_1 \times_{(\mathcal{G} \times_{\mathcal{H}} \mathcal{G})_0} \mathcal{G}_0 \to (\mathcal{G} \times_{\mathcal{H}} \mathcal{G})_0$$

is a surjective local homeomorphism. This implies that for every

$$[\gamma] : \varphi(x_1) \to \varphi(x_2)$$

in $(\mathcal{G} \times_{\mathcal{H}} \mathcal{G})_0$, there exists g_1 and g_2 in $\mathcal{G}_1$ such that

$$[\gamma] \circ [\varphi(g_1)] = [\varphi(g_2)] .$$

Suppose instead we are given

$$\gamma : \varphi(x_1) \to \varphi(x_2)$$

in $(\mathcal{G} \times_{\mathcal{K}} \mathcal{G})_0$. Then as ρ is full, so is φ, hence

$$\gamma = \varphi(a)$$

for some $a \in \mathcal{G}_1$. Now, there exists g_1 and g_2 in $\mathcal{G}_1$ such that

$$[\gamma] \circ [\varphi(g_1)] = [\varphi(g_2)] .$$

Let $g_2' := (a \circ g_1)^{-1}$. Then (g_1, g_2) is an arrow in $(\mathcal{G} \times_{\mathcal{K}} \mathcal{G})$ from

$$\left(x, x, \mathbb{1}_{\varphi(x)}\right)$$

to

$$(x_1, x_2, \gamma).$$

Hence

$$(\mathcal{G} \times_{\mathcal{K}} \mathcal{G})_1 \times_{(\mathcal{G} \times_{\mathcal{K}} \mathcal{G})_0} \mathcal{G}_0 \to (\mathcal{G} \times_{\mathcal{K}} \mathcal{G})_0$$

is a surjection. □

Corollary III.6.7. *Let* $\mathscr{G} = (\rho : \underline{\mathscr{G}} \to \mathscr{X})$ *be a local homeomorphism of étale stacks. Then* $\mathscr{G}$ *is a small gerbe over* $\mathscr{X}$ *if and only if* ρ *is a full, effective local equivalence.*

Proof. Suppose that $\mathscr{G}$ is a gerbe. From Theorem III.6.6, $\iota_{\mathscr{X}} : \mathscr{X} \to \operatorname{Eff}(\mathscr{X})$ is a small gerbe over $\operatorname{Eff}(\mathscr{X})$. Hence the composite $\iota_{\mathscr{X}} \circ \rho$ is a gerbe over $\operatorname{Eff}(\mathscr{X})$ by Theorem III.6.8. By Corollary III.6.4, this implies that it is an effective local equivalence, i.e.

$$\operatorname{Eff}(\iota_{\mathscr{X}} \circ \rho) = \operatorname{Eff}(\iota_{\mathscr{X}}) \circ \operatorname{Eff}(\rho)$$

is an equivalence. But $\operatorname{Eff}(\iota_{\mathscr{X}})$ is an isomorphism, hence $\operatorname{Eff}(\rho)$ is an equivalence. So ρ is an effective local equivalence. It is full by Proposition III.6.1.

Conversely, suppose that ρ is full and an effective local equivalence. It follows that $\iota_{\mathscr{X}} \circ \rho$ is an effective local equivalence over $\operatorname{Eff}(\mathscr{X})$, hence a gerbe by Corollary III.6.5. The result now follows from Theorem III.6.8. □

III.7 The 2-Category of Gerbed Effective Étale Stacks

In this section, we will treat topological and differentiable stacks as fibered categories (categories fibered in groupoids over S). The Grothendieck construction provides an equivalence of 2-categories between this description, and the one in terms of groupoid valued weak 2-functors, see Section I.1.7. For a fibered category $\mathscr{X}$, we shall denote the structure map which makes it a fibered category over S by $p_{\mathscr{X}}$.

For a diagram of fibered categories:

$$\begin{tikzcd} & \mathscr{Z} \arrow[d, "\rho"] \\ \mathscr{X} \arrow[r, "g"] & \mathscr{Y}, \end{tikzcd}$$

we choose the explicit weak pullback described as follows. The objects of $g^*\mathscr{Y}$ are triples (x, z, r) in $\mathscr{X}_0 \times \mathscr{Z}_0 \times \times \mathscr{Y}_1$ such that

$$p_{\mathscr{X}}(x) = p_{\mathscr{Z}}(z) = T$$

and

$$r : g(x) \to \rho(z),$$

with $r \in \mathscr{Y}(T)$. An arrow between a triple (z_1, x_1, r_1) and a triple (z_2, x_2, r_2) is a pair $(u, v) \in \mathscr{Z}_1 \times \mathscr{X}_1$ such that

$$p_{\mathscr{Z}}(u) = p_{\mathscr{X}}(v),$$

making the following diagram commute:

$$\begin{tikzcd} g(x_1) \arrow[r, "g(u)"] \arrow[d, "r_1"'] & g(x_2) \arrow[d, "r_2"] \\ \rho(z_1) \arrow[r, "\rho(v)"] & \rho(z_2). \end{tikzcd}$$

It has structure map

$$p_{g^*\mathscr{Y}}(x, z, r) = p_{\mathscr{Z}}(z) = p_{\mathscr{X}}(x),$$

$$p_{g^*\mathscr{Y}}(u, v) = p_{\mathscr{Z}}(u) = p_{\mathscr{X}}(v).$$

We denote the canonical projections as $pr_1 : g^*\mathscr{Y} \to \mathscr{Z}$ and $pr_2 : g^*\mathscr{Y} \to \mathscr{Y}$. We define $g^*\rho$ as the map $pr_1 : g^*\mathscr{Y} \to \mathscr{Z}$.

Given $\alpha : f \Rightarrow g$ with $g : \mathscr{X} \to \mathscr{Y}$, there is a canonical map

$$\alpha^* : g^*\rho \to f^*\rho$$

given on objects as

$$(z, x, r) \mapsto (z, x, r \circ \alpha(z)),$$

and given as the identity on arrows. This strictly commutes over $\mathscr{X}$. We denote the associated map in $St(S)/\mathscr{X}$ as $\alpha^*\rho$.

Given a composable sequence of arrows,

$$\mathscr{W} \xrightarrow{f} \mathscr{X} \xrightarrow{g} \mathscr{Y},$$

there is a canonical isomorphism $\chi_{g,f} : f^*g^*\mathscr{Z} \to (gf)^*\mathscr{Z}$ given on objects as

$$(w, (x, z, r), q) \mapsto (w, z, r \circ g(q)),$$

and on arrows as

$$(u, (a, b)) \mapsto (u, b).$$

This strictly commutes over $\mathscr{W}$. We denote the associated map in $St(S)/\mathscr{W}$ by the same name.

In a similar spirit, given $\tau : \mathscr{W} \to \mathscr{Y}$ with $m : \rho \to \tau$ in $\mathrm{St}(S)/\mathscr{Y}$, and

$$f : \mathscr{X} \to \mathscr{Y},$$

there is a canonical map

$$f^*m : f^*\rho \to f^*\tau$$

in $\mathrm{St}(S)/\mathscr{X}$, and given $\phi : m \Rightarrow n$, with $g : \rho \to \tau$, there is a canonical 2-cell

$$f^*\phi : f^*m \Rightarrow f^*n.$$

We invite the reader to work out the details.

Finally, we note that if

$$\begin{aligned} f &: \mathscr{X} \to \mathscr{Y}, \\ \rho &: \mathscr{Z} \to \mathscr{Y}, \\ \lambda &: \mathscr{W} \to \mathscr{X}, \\ \zeta &: \mathscr{W} \to \mathscr{Z}, \end{aligned}$$

and

$$\omega : \rho \circ \zeta \Rightarrow f \circ \lambda,$$

there is a canonical map

$$\begin{array}{rcl} (\lambda, \zeta, \omega) : \mathscr{W} & \to & f^*\mathscr{Z} \\ w & \mapsto & \left(\lambda(w), \zeta(w), \omega(w)^{-1}\right) \\ l & \mapsto & (\lambda(l), \zeta(l)). \end{array}$$

This data provides us with coherent choices of pullbacks. We will now use this data to construct a 2-category we will call the 2-category of **gerbed effective étale stacks**. We will denote it by $\mathscr{G}erbed\,(\mathfrak{EffEt})$.

Its **objects** are pairs $(\mathscr{X}, \sigma)$ with $\mathscr{X}$ an effective étale stack and $\sigma \to \mathscr{X}$ an effective local equivalence. Of course, this is the same data as a small gerbe over $\mathscr{X}$.

An **arrow** from $(\mathscr{X}, \sigma)$ to $(\mathscr{Y}, \tau)$ is a pair (f, m) where $f : \mathscr{X} \to \mathscr{Y}$ and $m : \sigma \to f^*\tau$ in $\mathrm{St}(S)/\mathscr{X}$. Note that this is equivalent data to a map in $\mathrm{St}(\mathscr{X})$ from σ to $f^*\tau$ viewed as gerbes.

A 2-**cell** between such an (f, m) and a

$$(g, n) : (\mathscr{X}, \sigma) \to (\mathscr{Y}, \tau),$$

is a pair (α, ϕ) with $\alpha : f \Rightarrow g$ a 2-cell in $\mathfrak{EffEt}$, and ϕ a 2-cell in $\mathrm{St}(S)/\mathscr{X}$ such that

$$
\begin{tikzcd}
\sigma \arrow[r, "n"] \arrow[dr, "m"'] & g^*\tau \arrow[d, "\alpha^*\tau"] \arrow[dl, phantom, "\Leftarrow\phi"] \\
& f^*\tau.
\end{tikzcd}
$$

Composition of 1**-morphisms** is given as follows:
If

$$
(\mathscr{X}, \sigma) \overset{(f,m)}{\longrightarrow} (\mathscr{Y}, \tau) \overset{(g,n)}{\longrightarrow} (\mathscr{Z}, \rho),
$$

is a pair of composable 1-morphisms, define their composition as $(gf, n * m)$, where $n * m$ is defined as the composite

$$
\sigma \xrightarrow{\quad m \quad} f^*\tau \xrightarrow{\quad f^*(n) \quad} f^*g^*\rho \xrightarrow{\quad \chi_{g,f} \quad} (gf)^*\,\rho.
$$

Vertical composition of 2**-cells** is defined in the obvious way.
Suppose

$$
(\alpha, \phi) : (f, m) \Rightarrow (k, p)
$$

and

$$
(\beta, \psi) : (g, n) \Rightarrow (l, p),
$$

with

$$
(f, m) : (\mathscr{X}, \sigma) \to (\mathscr{Y}, \tau)
$$

and

$$
(g, n) : (\mathscr{Y}, \tau) \to (\mathscr{Z}, \rho).
$$

Denote the horizontal composition of β with α by $\beta \circ \alpha$. Then we define the **horizontal composition of 2-cells**

$$
(\beta, \psi) \circ (\alpha, \phi) : (g, n) \circ (f, m) \Rightarrow (l, p) \circ (k, o),
$$

by

$$
(\beta, \psi) \circ (\alpha, \phi) = (\beta \circ \alpha, \psi * \phi),
$$

where $\psi * \phi$ is defined by the pasting diagram:

$$
\begin{tikzcd}
& & k^*l^*\rho \arrow[r, "\chi_{l,k}"] \arrow[d, "(k^*\circ\beta^*)(\rho)"] & (lk)^*(\rho) \arrow[d, "(\beta\circ k)^*(\rho)"] \\
& k^*\tau \arrow[ur, "k^*(\rho)"] \arrow[r, "k^*(n)"'] \arrow[d, "\alpha^*(\tau)"] \arrow[ur, phantom, shift right=3, "\Leftarrow k^*(\psi)"] & k^*g^*\rho \arrow[r, "\chi_{g,k}"'] \arrow[d, "(\alpha^*\circ g^*)(\rho)"] & (gk)^*(\rho) \arrow[d, "(g\alpha)^*(\rho)"] \\
\sigma \arrow[ur, "p"] \arrow[r, "m"'] \arrow[ur, phantom, shift right=3, "\Leftarrow\phi"] & f^*\tau \arrow[r, "f^*(n)"'] & f^*g^*\rho \arrow[r, "\chi_{g,f}"'] & (gf^*)^*(\rho).
\end{tikzcd}
$$

Remark. What we have actually done is applied the Grothendieck construction for bicategories [6] to the trifunctor which associates to each effective étale stack, the 2-category of effective local equivalences over $\mathscr{X}$ (which we know to be equivalent to the 2-category $\mathscr{Gerbe}(\mathscr{X})$ of small gerbes over $\mathscr{X}$).

If P is an étale invariant subcategory of spaces, we can similarly define the 2-category $\mathscr{Gerbed}(\mathfrak{EffEt})_P$ in which each underlying 1-morphism in $\mathfrak{EffEt}$ must lie in $\mathfrak{EffEt}_P$.

Theorem III.7.1. *Suppose P is an open étale invariant subcategory of spaces. Then the 2-category $\mathscr{Gerbed}(\mathfrak{EffEt})_P$ of gerbed effective étale stacks and P-morphisms is equivalent to the 2-category $\mathfrak{Et}_P$ of étale stacks and P-morphisms (See Corollary III.5.2).*

Proof. Define a 2-functor $\Theta : \mathfrak{Et}_P \to \mathscr{Gerbed}(\mathfrak{EffEt})_P$.

On objects:

$$\Theta(\mathscr{X}) = (\mathrm{Eff}(\mathscr{X}), \iota_{\mathscr{X}}),$$

where ι is the unit of the adjunction in Theorem III.5.2. This associates $\mathscr{X}$ to the gerbe it induces over $\mathrm{Eff}(\mathscr{X})$.

On arrows: Suppose $\varphi : \mathscr{X} \to \mathscr{Y}$ is a map in $\mathfrak{Et}_P$. Notice that the diagram

$$\begin{tikzcd} \mathscr{X} \arrow[r, "\varphi"] \arrow[d, "\iota_{\mathscr{X}}"'] & \mathscr{Y} \arrow[d, "\iota_{\mathscr{Y}}"] \\ \mathrm{Eff}(\mathscr{X}) \arrow[r, "\mathrm{Eff}(\varphi)"] & \mathrm{Eff}(\mathscr{Y}), \end{tikzcd}$$

commutes on the nose, so there is an associated map

$$(\iota_{\mathscr{X}}, \varphi, id) : \mathscr{X} \to \mathrm{Eff}(\varphi)^* \mathscr{Y}.$$

Define $\Theta(\varphi) = (\mathrm{Eff}(\varphi), (\iota_{\mathscr{X}}, \varphi, id))$.

On 2-cells: Suppose that $\varphi' : \mathscr{X} \to \mathscr{Y}$ and

$$\alpha : \varphi \Rightarrow \varphi'.$$

Then define

$$\Theta(\alpha) = (\mathrm{Eff}(\alpha), \tilde{\alpha}),$$

where

$$\tilde{\alpha} : \mathscr{X}_0 \to (\mathrm{Eff}(\varphi)^* \rho)_1$$

is defined by the equation

$$\tilde{\alpha}(x) = \left(id, \alpha(x)^{-1}\right).$$

We leave it to the reader to check that Θ is 2-functor.

Define another 2-functor

$$\Xi : \mathscr{Gerbed}(\mathfrak{EffEt})_P \to \mathfrak{Et}_P.$$

On objects: If $\sigma : \mathscr{G} \to \mathscr{X}$ is an effective local equivalence, denote $\mathscr{G}$ by $\underline{\sigma}$. Let

$$\Xi\left(\mathscr{X}, \sigma\right) := \underline{\sigma}.$$

On arrows: Suppose $(f, m) : (\mathscr{X}, \sigma) \to (\mathscr{Y}, \tau)$. Denote the underlying map of m by

$$\underline{m} : \underline{\sigma} \to \underline{f^*\tau}.$$

Define

$$\Xi\left(f, m\right) := pr_2 \circ \underline{m} : \underline{\sigma} \to \underline{\tau},$$

where

$$pr_2 : f^*\tau \to \tau$$

is the canonical projection.
On 2-cells: Given

$$\left(g, n\right) : \left(\mathscr{X}, \sigma\right) \to \left(\mathscr{Y}, \tau\right)$$

and

$$\left(\alpha, \phi\right) : \left(f, m\right) \to \left(g, n\right),$$

define $\Xi\left(\left(\alpha, \phi\right)\right)$ by the following pasting diagram:

$$\begin{tikzcd}
\underline{\sigma} \arrow[rr, "\underline{m}"] \arrow[drr, "\underline{n}"'] \arrow[dr, phantom, "\Rightarrow \phi^{-1}"] & & f^*\mathscr{Y} \arrow[rr, "pr_2"] & & \mathscr{Y}. \\
& & g^*\mathscr{Y} \arrow[u, "\underline{\alpha^*(\tau)}"'] \arrow[urr, "pr_2"'] & &
\end{tikzcd}$$

By direction inspection, one can see that

$$\Xi \circ \Theta = id_{\mathfrak{Eff}\mathfrak{Et}_P}.$$

There is furthermore a canonical natural isomorphism

$$\Theta \circ \Xi \Rightarrow id_{\mathscr{G}erbed\left(\mathfrak{Eff}\mathfrak{Et}\right)_P}.$$

On objects

$$\Theta \circ \Xi\left(\left(\mathscr{X}, \sigma\right)\right) = \left(\mathrm{Eff}\left(\underline{\sigma}\right), \iota_{\underline{\sigma}}\right).$$

By Corollary III.6.4 and Theorem III.6.6, this is canonically isomorphic to $(\mathscr{X}, \sigma)$. Moreover, if

$$\left(f, m\right) : \left(\mathscr{X}, \sigma\right) \to \left(\mathscr{Y}, \tau\right),$$

then

$$\Theta\Xi\left(\left(f, m\right)\right) = \left(\mathrm{Eff}\left(pr_2 \circ \underline{m}\right), \left(\iota_{\underline{\sigma}}, pr_2 \circ \underline{m}, id\right)\right).$$

Consider the following diagram:

$$\begin{tikzcd}
\mathrm{Eff}\,(\underline{\sigma}) \arrow[r, "\mathrm{Eff}(\underline{m})"] \arrow[dr, "\mathrm{Eff}(\sigma)"'] & \mathrm{Eff}\,(pr_2) \arrow[r, "\mathrm{Eff}(f^*\tau)"] \arrow[d, "\mathrm{Eff}(f^*\underline{\tau})"] & \mathrm{Eff}\,(\underline{\tau}) \arrow[d, "\mathrm{Eff}(\tau)"] \\
 & \mathrm{Eff}\,(\mathscr{X}) \arrow[r, "\mathrm{Eff}(f)"] & \mathrm{Eff}\,(\mathscr{Y}) \\
 & \mathscr{X} \arrow[u, "\iota_{\mathscr{X}}"] \arrow[r, "f"] & \mathscr{Y}. \arrow[u, "\iota_{\mathscr{Y}}"']
\end{tikzcd}$$

Since σ and $f^*\tau$ are effective local equivalences, the triangle consists of all equivalences. The lower square likewise consists of all equivalences as $\mathscr{X}$ and $\mathscr{Y}$ are effective. We leave the rest of the details to the reader. □

Corollary III.7.1. *There is an equivalence of* 2*-categories between gerbed effective étale differentiable stacks and submersions,* $\mathscr{G}erbed\,(\mathfrak{EffEt})_{subm}$*, and the* 2*-category of étale differentiable stacks and submersions,* $\mathfrak{Et}_{subm}$.

Remark. Some variations of this are possible. For example, if we restrict to étale stacks whose effective parts are (equivalent to) spaces, so-called **purely ineffective** étale stacks, then the functor Eff extends to all maps. The proof of Theorem III.7.1 extends to this setting to show that purely ineffective étale stacks are equivalent to the 2-category of gerbed spaces, a result claimed in [27]. This theorem is a corrected version of theorem 94 of [38] (which is unfortunately incorrect since there is an error on the top of page 44, see the remark after Corollary III.3.4). Moreover, by results of [27], this restricts to an equivalence between purely ineffective orbifolds and gerbed manifolds whose gerbe has a locally constant band with finite stabilizers.

Appendix A

Weak Colimits and Pseudo-Colimits

In this Appendix, we will explain the concepts of weak colimits and pseudo-colimits, as well as prove many technical lemmas involving these concepts. This Appendix can also be used as a reference for weak limits and pseudo-limits by dualizing everything.

A.1 Definitions

Definition A.1.1. Let F and G be weak functors $\mathscr{C} \to \mathscr{D}$ of 2-categories (as in Definition I.1.24). A **weak natural transformation** $\alpha : F \Rightarrow G$ consists of an assignment to each object $c \in \mathscr{C}_0$ a 1-morphism

$$\alpha(c) : F(c) \to G(c),$$

and to each arrow $f : c \to c'$ an invertible 2-cell

$$\alpha(f) \in \mathscr{D}_2$$

$$\begin{tikzcd} F(c) \arrow[r, "\alpha(c)"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\ F(c') \arrow[r, "\alpha(c')"'] \arrow[ur, phantom, "\alpha(f)"] & G(c'), \end{tikzcd}$$

such that for composable arrows, the obvious pentagon (as in Definition I.1.24) commutes.

Definition A.1.2. A **modification** $\Omega : \alpha \Rrightarrow \beta$ between two weak natural transformations $\alpha, \beta : F \Rightarrow G$, is an assignment to each object $c \in \mathscr{C}_0$ a 2-cell

$$\Omega(c) : \alpha(c) \Rightarrow \beta(c)$$

such that for each $f : c \to d$ of $\mathscr{C}$ the square

$$\begin{tikzcd}
\beta(d)F(f) \arrow[r, Rightarrow, "\alpha(d)F(f)"] \arrow[d, Rightarrow, "\beta(f)"'] & \alpha(d)F(f) \arrow[d, Rightarrow, "\alpha(f)"] \\
G(f)\beta(c) \arrow[r, Rightarrow, "G(f)\alpha(c)"] & G(f)\alpha(c)
\end{tikzcd}$$

commutes, as in Definition I.1.25.

Definition A.1.3. A **cocone** with vertex $d \in \mathscr{D}_0$ for a weak functor

$$F : \mathscr{C} \to \mathscr{D}$$

is a weak natural transformation $\alpha : F \Rightarrow \Delta_d$, where

$$\Delta_d : \mathscr{C} \to \mathscr{D}$$

is the constant functor with vertex d. Cocones for F with vertex d, with modifications for arrows, form a category $\mathit{Cocone}\,(F, d)$.

Notice that this makes $\mathit{Cocone}\,(F, \cdot)$ into a strict co-presheaf of categories on $\mathscr{D}$. Furthermore, given any object $d \in \mathscr{D}_0$, it defines a strict co-presheaf of categories on $\mathscr{D}$ by co-Yoneda:

$$y\,(d)\,(d') := \mathrm{Hom}\,(d, d')\,.$$

Given any cocone $\alpha : F \Rightarrow \Delta_d$, we get an induced *strict* transformation

$$\hat{\alpha} : y\,(d) \Rightarrow \mathit{Cocone}\,(F, \cdot)$$

given by sending each morphism $f : d \to d'$ to $F \xRightarrow{\alpha} \Delta_d \xRightarrow{\Delta_f} \Delta_{d'}$.

Definition A.1.4. A cocone $\alpha : F \Rightarrow \Delta_d$ is said to be weak colimiting if the components of the natural transformation $\hat{\alpha}$ are all equivalences of categories. In such a scenario, we say d (with its cocone) is a **weak colimit** for F, which we denote by $\underrightarrow{\mathrm{holim}}\, F = d$. If α satisfies the stronger condition that the components $\hat{\alpha}$ are *isomorphisms* of categories, we say that d is a **pseudo-colimit** for F. We will adopt the notation $\underrightarrow{\mathrm{plim}}\, F = d$ for pseudo-colimits.

Remark. The property of being a weak colimit is invariant under equivalences, i.e. if $\underrightarrow{\mathrm{holim}}\, F = d$ and there exists $f : d \to d'$, $g : d' \to d$ and invertible $\alpha : gf \Rightarrow id_d$ and $\beta : id'_d \Rightarrow fg$, then d' is also a weak colimit for F. However, the property of being a pseudo-colimit is only invariant under strict isomorphism. But, if $\underrightarrow{\mathrm{plim}}\, F = d$ and d' is equivalent to d, then d' is a weak colimit for F.

In general, even if a weak colimit for a functor F exists, it does not imply that a pseudo-colimit exists. However, there are many 2-categories which are pseudo-co-complete. An important example is the 2-category Gpd of (essentially small) groupoids. An explicit formula for calculating pseudo-colimits of groupoids is given in the Appendix of [24].

Remark. Suppose that $F \dashv G$ is a weak adjunction. Then F preserves weak colimits and G preserves weak limits. It is not true that F need preserve pseudo-colimits (dually for G). However, this is true if the equivalences of categories

$$\operatorname{Hom}(F(c), d) \xrightarrow{\sim} \operatorname{Hom}(c, G(d))$$

arising from the adjunction are in fact *isomorphisms* of categories, i.e., if the adjunction is actually a pseudo-adjunction. Note that any left pseudo-adjoint preserves all weak colimits, since, in a particular, it is a weak left adjoint (and dually for right pseudo-adjoint).

An important example of such a pseudo-adjunction arises from a functor

$$j : \mathscr{C} \to \mathscr{D}$$

between two categories. Such a functor induces pseudo-functors

$$j_* : Psh\,(\mathscr{C}, Gpd) \to Psh\,(\mathscr{D}, Gpd)$$
$$j^* : Psh\,(\mathscr{D}, Gpd) \to Psh\,(\mathscr{C}, Gpd)$$
$$j_! : Psh\,(\mathscr{C}, Gpd) \to Psh\,(\mathscr{D}, Gpd)$$

between presheaves of groupoids, with $j_! \dashv j^* \dashv j_*$ being two pseudo-adjunctions. This is due to the strict version of the Yoneda lemma for strict 2-functors. Whenever any of these adjoint pairs descend to sheaves of groupoids (when $\mathscr{C}$ and $\mathscr{D}$ are equipped with Grothendieck topologies), the resulting adjunctions are of course still pseudo-adjunctions.

A.2 Pseudo-colimits of strict presheaves

We now aim to prove that for any category $\mathscr{C}$, the "inclusion" 2-functor

$$j : Psh\,(\mathscr{C}, Gpd) \to Gpd^{\mathscr{C}^{op}}$$

from strict presheaves of groupoids into weak presheaves of groupoids is pseudo-colimit preserving.

First we make a few observations. Since Gpd is pseudo-co-complete, the same is true for both $Psh\,(\mathscr{C}, Gpd)$ and $Gpd^{\mathscr{C}^{op}}$, because pseudo-colimits may be computed "point-wise", i.e. by the formula

$$\left(\underrightarrow{\operatorname{plim}}\, F\right)(C) = \underrightarrow{\operatorname{plim}}\, (F(C))\,.$$

This observation alone makes it believable that j preserves pseudo-colimits, seeing as how they agree point-wise.

Theorem A.2.1. *$j : Psh(\mathscr{C}, Gpd) \to Gpd^{\mathscr{C}^{op}}$ preserves pseudo-colimits.*

Proof. Let $F : J \to Psh(\mathscr{C}, Gpd)$ be any pseudo-functor. Let

$$\mu_S : F \Rightarrow \Delta_{\underrightarrow{\mathrm{plim}}\, F}$$

be a pseudo-colimiting cocone for F and

$$\mu_W : j \circ F \Rightarrow \Delta_{\underrightarrow{\mathrm{plim}}\, j \circ F}$$

be a pseudo-colimiting cocone for $j \circ F$. To simplify notation, let $S := \underrightarrow{\mathrm{plim}}\, F$ and $W := \underrightarrow{\mathrm{plim}}\, j \circ F$. Then

$$j\mu_S : j \circ F \Rightarrow \Delta j(S)$$

is a cocone for $j \circ F$ with vertex $j(S)$. Hence there exists a morphism

$$\phi : W \to S$$

such that

$$j\mu_S = \Delta_\phi \circ \mu_W.$$

We claim that ϕ is an isomorphism.

It suffices to show that for each $C \in \mathscr{C}_0$, the map $\phi(C) : W(C) \to S(C)$ is an isomorphism of groupoids. Consider the inclusion of the object C as a functor

$$* \xrightarrow{\imath} \mathscr{C}$$

from the terminal category. This induces two 2-functors, and by abuse of notation, we will denote both by $\imath^*$:

$$\imath^* : Psh(\mathscr{C}, Gpd) \to Gpd$$

$$\imath^* : Gpd^{\mathscr{C}^{op}} \to Gpd.$$

Both of these 2-functors are left pseudo-adjoints and are given by evaluation at the objet C, so clearly $\imath^* j = \imath^*$.

We want to show that $\phi(C) = \imath^*(\phi)$ is an isomorphism of groupoids. Since $\imath^*$ is a left pseudo-adjoint, it follows that

$$\imath^*\mu_W : \imath^* \circ j \circ F = F(C) \Rightarrow \imath^* \circ \Delta_W = \Delta_{W(C)}$$

is pseudo-colimiting. Now, since pseudo-colimits in $Psh(\mathscr{C}, Gpd)$ are computed point-wise, it follows that $\imath^*\mu S$ is pseudo-colimiting. So, there exists a functor

$$\psi : S(C) \to W(C)$$

such that

$$\Delta_\psi \circ \imath^* \mu_S = \imath^* \mu_W.$$

Notice that

$$\imath^* \mu_S = \imath^* \left(\Delta_\phi \circ \mu_W \right) = \Delta_{\phi(C)} \circ \imath^* \mu_W.$$

So

$$\Delta_\psi \circ \Delta_{\phi(C)} \circ \imath^* \mu_W = \imath^* \mu_W.$$

The left-hand side of this equation is equal to $\widehat{\imath^* \mu_W} \left(\psi \circ \phi (C) \right)$ whereas the right-hand side is equal to $\widehat{\imath^* \mu_W} \left(id_{W(C)} \right)$. But $\imath^* \mu_W$ is pseudo-colimiting, so $\widehat{\imath^* \mu_W}$ is an isomorphism of categories, hence $\psi \circ \phi (C) = id_{W(C)}$.

Notice further that

$$\begin{aligned} \widehat{\imath^* \mu_S} \left(\phi (C) \circ \psi \right) &= \Delta_{\phi(C)} \circ \Delta_\psi \circ \Delta_{\phi(C)} \circ \imath^* \mu_W \\ &= \Delta_{\phi(C)} \circ \imath^* \mu_W \\ &= \imath^* \mu_S \\ &= \widehat{\imath^* \mu_S} \left(id_{S(C)} \right). \end{aligned}$$

But $\imath^* \mu_S$ is pseudo-colimiting, so $\widehat{\imath^* \mu_S}$ is an isomorphism, hence

$$\phi (C) \circ \psi = id_{S(C)}.$$

Therefore, ϕ is an isomorphism. Since pseudo-colimits are stable under isomorphisms, it follows that $j(S)$ is a pseudo-colimit for $j \circ F$. □

Corollary A.2.1. *$j : Psh(\mathscr{C}, Gpd) \to Gpd^{\mathscr{C}^{op}}$ preserves weak colimits.*

Lemma A.2.2. *The 2-functor $a \circ j \circ i : Sh(\mathscr{C}, Gpd) \to \mathrm{St}(\mathscr{C})$ is weak colimit preserving.*

Proof. Let $\varphi : J \to Sh(\mathscr{C}, Gpd)$ be any weak 2-functor. Then the weak colimit is computed by $sh \left(\underset{\underset{J}{\longrightarrow}}{\operatorname{holim}}\, i \circ \varphi \right)$. By Proposition I.1.9,

$$a \circ j \circ i \left(sh \left(\underset{\underset{J}{\longrightarrow}}{\operatorname{holim}}\, i \circ \varphi \right) \right) \simeq a \circ j \left(\underset{\underset{J}{\longrightarrow}}{\operatorname{holim}}\, i \circ \varphi \right).$$

It follows from Theorem A.2.1 that j is weak colimit preserving. Hence:

$$a \circ j \circ i \circ sh \left(\underset{\underset{J}{\longrightarrow}}{\operatorname{holim}}\, \varphi \right) \simeq a \circ j \left(\underset{\underset{J}{\longrightarrow}}{\operatorname{holim}}\, i \circ \varphi \right) \simeq \underset{\underset{J}{\longrightarrow}}{\operatorname{holim}}\, a \circ j \circ i \circ \varphi.$$

□

A.3 Representing groupoids as weak colimits of their nerves

Let $\mathcal{H}$ be a (small) groupoid, and let

$$N(\mathcal{H}) : \mathbf{\Delta}^{op} \to \mathrm{Set}$$

denote its nerve.

Consider the (non-full) subcategory $\mathbf{\Delta}_{+}^{\leq 2}$ of the simplex category $\mathbf{\Delta}$ consisting of the three objects $[0]$, $[1]$, and $[2]$, and the strictly monotonic maps between them. Denote by $\mathcal{F}_{\mathcal{H}}$ the composite

$$\left(\mathbf{\Delta}_{+}^{\leq 2}\right)^{op} \to \mathbf{\Delta}^{op} \xrightarrow{N(\mathcal{H})} \mathrm{Set} \xrightarrow{(\,\cdot\,)^{id}} Gpd.$$

Lemma A.3.1. *$\mathcal{H}$ is a weak colimit of $\mathcal{F}_{\mathcal{H}}$.*

Proof. There is a natural way to construct a cocone for $\mathcal{F}_{\mathcal{H}}$, up to some choices. Throughout this proof, we will use simplicial notation for face maps. Let $p : \mathcal{H}_0 \to \mathcal{H}$ be the canonical map. One possible cocone

$$\sigma : \mathcal{F}_{\mathcal{H}} \Rightarrow \Delta_{\mathcal{H}}$$

can be described as follows:

$$\sigma_0 := p : \mathcal{H}_0 \to \mathcal{H}$$
$$\sigma_1 := pd_1 : \mathcal{H}_1 \to \mathcal{H}$$
$$\sigma_0 := pd_1 d_2 : \mathcal{H}_2 \to \mathcal{H},$$

together with the obvious 2-cells necessary to make this a well-defined cocone. For instance, the naturality square (actually a triangle in this case) for

$$d_1 : \mathcal{H}_1 \to \mathcal{H}_0$$

commutes on the nose, whereas the corresponding one for

$$d_0 : \mathcal{H}_1 \to \mathcal{H}_0$$

commutes up to the canonical natural transformation

$$\begin{aligned} \sigma(d_0) : pd_0 &\Rightarrow pd_1 \\ \sigma(d_0)(h) = h : d_0(h) &\to d_1(h). \end{aligned}$$

We claim that σ is colimiting. Let $\mathcal{G}$ be another groupoid, and consider the induced functor

$$\widehat{\sigma} : \mathrm{Hom}\,(\mathcal{H}, \mathcal{G}) \to \mathit{Cocone}\,(\mathcal{F}_{\mathcal{H}}, \mathcal{G})\,.$$

We wish to show that this functor is an equivalence.

We will first show that it is essentially surjective. Let

$$\rho : \mathcal{F}_{\mathcal{H}} \Rightarrow \Delta_{\mathcal{G}}$$

be a cocone on $\mathcal{G}$. Define the map

$$\varphi_0^\rho := \rho_0 : \mathcal{H}_0 \to \mathcal{G}_0,$$

where we have identified the map

$$\rho_0 : \mathcal{H}_0 \to \mathcal{G}$$

with the associated map into $\mathcal{G}_0$. Consider the composite of natural transformations

$$\rho_0 d_0 \xRightarrow{\rho(d_0)} \rho_1 \xRightarrow{\rho(d_1)^{-1}} \rho_0 d_1.$$

It corresponds to a map into $\mathcal{G}_1$ which we denote by

$$\varphi_1^\rho : \mathcal{H}_1 \to \mathcal{G}_1.$$

We claim that the two maps φ_0^ρ and φ_1^ρ determine a functor

$$\varphi^\rho : \mathcal{H} \to \mathcal{G}.$$

Notice that since ρ is a weak natural transformation, the following diagram must commute for each composable pair

$$\mathcal{H}_2 \xrightarrow{d_j} \mathcal{H}_1 \xrightarrow{d_i} \mathcal{H}_0 :$$

$$\begin{tikzcd} \rho_0 d_i d_j \arrow[rr, Rightarrow, "\rho(d_i)d_j"] \arrow[dr, Rightarrow, "\rho(d_i d_j)"'] & & \rho_1 d_j \arrow[dl, Rightarrow, "\rho(d_j)."] \\ & \rho_2 & \end{tikzcd}$$

Putting all these together for each composable pair of face maps

$$\mathcal{H}_2 \xrightarrow{d_j} \mathcal{H}_1 \xrightarrow{d_1} \mathcal{H}_0,$$

it follows that for all composable arrows

$$x \xrightarrow{h} y \xrightarrow{g} z,$$

the following diagram commutes:

$$\begin{tikzcd}
\varphi_0(x) \arrow[rr, "\rho(d_0)d_0(g,h)"] \arrow[dddrr, dashed, "\varphi^\rho(h)"'] \arrow[dddddr, dashed, "\varphi^\rho(gh)"'] \arrow[ddrrr, "\rho(d_0)d_1(g,h)"] & & \rho_1(h) \arrow[drr, "\rho(d_0)(g,h)"] & & \\
& & & & \rho_2(g,h) \\
& & & \rho_1(gh) \arrow[ur, "\rho(d_1)(g,h)"] & \\
& & \varphi_0(y) \arrow[uuu, dotted, "\rho(d_1)d_0(g,h)"'] \arrow[dr, dotted, "\rho(d_0)d_0(g,h)"] \arrow[ddl, dashed, "\varphi^\rho(g)"'] & & \\
& & & \rho_1(g) \arrow[uuur, "\rho(d_0)(g,h)"'] & \\
& \varphi_0(z) \arrow[uuurr, "\rho(d_1)d_1(g,h)"'] \arrow[urr, "\rho(d_1)d_2(g,h)"'] & & &
\end{tikzcd}.$$

Hence φ^ρ respects composition. It remains to show it respects units. However, using the same diagram for the pair $(\mathbb{1}_x, \mathbb{1}_x)$, one can deduce that units are preserved as well. Hence φ^ρ is a functor. Consider now the cocone $\widehat{\sigma}(\varphi^\rho)$. By construction, its component along $[0]$ is equal to that of ρ. Its component along $[1]$ is given by $\rho_0 d_1$ and $\rho(d_1)$ furnishes a natural transformation

$$\rho(d_1) : \rho_0 d_1 \Rightarrow \rho_1.$$

Similarly, its component along $[2]$ is given by

$$\rho_0 d_1 d_2$$

and we have

$$\rho(d_1 d_2) : \rho_0 d_1 d_2 \Rightarrow \rho_2.$$

It is easy to check that these natural transformations are the components of a modification

$$\widehat{\sigma}(\varphi^\rho) \Rightarrow \rho.$$

Hence $\widehat{\sigma}$ is essentially surjective.

We now aim to show that $\widehat{\sigma}$ is faithful. Suppose that we have two functors

$$\varphi, \varphi' : \mathcal{H} \to \mathcal{G}$$

and a natural transformation

$$\alpha : \varphi \Rightarrow \varphi'.$$

Note that

$$\widehat{\sigma}\left(\alpha\right)_0\left(x\right) = \left(\Delta_\alpha \sigma\right)_0\left(x\right) = \alpha\left(x\right),$$

for all $x \in \mathcal{H}_0$. It follows that $\widehat{\sigma}$ is faithful.

It remains to show that it is full. Suppose that

$$\omega : \widehat{\sigma}\left(\varphi\right) \Rightarrow \widehat{\sigma}\left(\varphi'\right)$$

is a modification. That data of ω consists of natural transformations

$$\omega_i : \widehat{\sigma}\left(\varphi\right)_i \Rightarrow \widehat{\sigma}\left(\varphi'\right)_i$$

for each object $[0]$, $[1]$, and $[2]$, satisfying certain coherence relations. These coherence relations (Definition I.1.25) in this case reduce to the fact that for each face map $d_i : \mathcal{H}_k \to \mathcal{H}_{k-1}$,

$$\widehat{\sigma}\left(\varphi'\right)\left(d_i\right) \circ \omega_{k-1} d_i = \omega_k \widehat{\sigma}\left(\varphi\right)\left(d_i\right).$$

The component of ω along $[0]$ is a natural transformation

$$\omega_0 : \varphi \circ p \Rightarrow \varphi' \circ p,$$

that is, it is a function

$$\omega_0 : \mathcal{H}_0 \to \mathcal{G}$$

such that for all $x \in \mathcal{H}_0$,

$$\omega_0\left(x\right) : \varphi_0\left(x\right) \to \varphi'_0\left(x\right).$$

The coherence relations for $d_0 : \mathcal{H}_1 \to \mathcal{H}_0$ imply that the following diagram commutes for each $h \in \mathcal{H}_1$:

$$\begin{tikzcd}
\varphi_0\left(d_0\left(h\right)\right) \arrow[r, "\omega_0(d_0(h))"] \arrow[d, "\varphi(h)"'] & \varphi'_0\left(d_0\left(h\right)\right) \arrow[d, "\varphi'(h)"] \\
\varphi_0\left(d_1\left(h\right)\right) \arrow[r, "\omega_1(h)"'] & \varphi'_0\left(d_1\left(h\right)\right).
\end{tikzcd}$$

The coherency relations for $d_1 : \mathcal{H}_1 \to \mathcal{H}_0$ simply imply that $\omega_1 = \omega_0 d_1$. Therefore,

$$\omega_0 : \mathcal{H}_0 \to \mathcal{G}_1$$

actually encodes a natural transformation

$$\tilde{\omega}_0 : \varphi \Rightarrow \varphi'.$$

Moreover, the coherency relations for

$$d_2 : \mathcal{H}_2 \to \mathcal{H}_1$$

imply that

$$\omega_2 = \omega_1 d_2 = \omega_0 d_1 d_2.$$

Therefore

$$\omega = \widehat{\sigma}\left(\tilde{\omega}_0\right)$$

showing $\widehat{\sigma}$ is full. Hence, $\widehat{\sigma}$ is an equivalence. $\square$

Corollary A.3.1. *Let $(\mathscr{C}, J)$ be a Grothendieck site and $\mathcal{H}$ a groupoid object in $\mathrm{Set}^{\mathscr{C}^{op}}$. Denote by $[\mathcal{H}]_J$ the stackification of the associated (strict) presheaf of groupoids. Denote by $N(\mathcal{H})$ the presheaves-enriched nerve of $\mathcal{H}$,*

$$N(\mathcal{H}) : \mathbf{\Delta}^{op} \to \mathrm{Set}^{\mathscr{C}^{op}},$$

and by $\mathcal{F}_{\mathcal{H}}$ the composite

$$\left(\mathbf{\Delta}_{+}^{\leq 2}\right)^{op} \to \mathbf{\Delta}^{op} \xrightarrow{N(\mathcal{H})} \mathrm{Set}^{\mathscr{C}^{op}} \xrightarrow{(\,\cdot\,)^{id}} Gpd^{\mathscr{C}^{op}} \xrightarrow{a_J} St_J(\mathscr{C}).$$

Then $[\mathcal{H}]_J$ is a weak colimit of $\mathcal{F}_{\mathcal{H}}$.

Proof. First consider the functor $\tilde{\mathcal{F}}_{\mathcal{H}}$:

$$\left(\mathbf{\Delta}_{+}^{\leq 2}\right)^{op} \to \mathbf{\Delta}^{op} \xrightarrow{N(\mathcal{H})} \mathrm{Set}^{\mathscr{C}^{op}} \xrightarrow{(\,\cdot\,)^{id}} Gpd^{\mathscr{C}^{op}}.$$

Since weak colimits in $Gpd^{\mathscr{C}^{op}}$ are computed point-wise, it follows from Lemma A.3.1 that $\tilde{y}(\mathcal{H})$, the associated strict presheaf of groupoids, is a weak colimit in $Gpd^{\mathscr{C}^{op}}$ of $\tilde{\mathcal{F}}_{\mathcal{H}}$. The result now follows since the stackification functor a_J is a left 2-adjoint and hence preserves weak colimits. $\square$

Definition A.3.1. Let $\mathcal{H}$ and $(\mathscr{C}, J)$ be as in the previous corollary. Then for all stacks $\mathscr{X}$, let $\mathit{Cocycle}(\mathcal{H}, \mathscr{X})$ denote the following groupoid:

The objects are pairs (θ, α) with $\theta \in \mathscr{X}(\mathcal{H}_0)$ and

$$\alpha : \mathscr{X}(d_1)(\theta) \to \mathscr{X}(d_0)(\theta)$$

morphisms which make the following diagram commute:

$$\begin{tikzcd}
\mathscr{X}(d_2)\mathscr{X}(d_1)(\theta) \arrow[r, "\mathscr{X}(d_2)(\alpha)"] \arrow[d, "\mathscr{X}(d_1,d_2)(\theta)"'] & \mathscr{X}(d_2)\mathscr{X}(d_0)(\theta) \arrow[r, "\mathscr{X}(d_0,d_2)(\theta)"] & \mathscr{X}(d_0d_2)(\theta) \arrow[r, equal] & \mathscr{X}(d_1d_0)(\theta) \arrow[d, "\mathscr{X}(d_0,d_1)^{-1}(\theta)"] \\
\mathscr{X}(d_1d_2)(\theta) \arrow[d, equal] & & & \mathscr{X}(d_0)\mathscr{X}(d_1)(\theta) \arrow[d, "\mathscr{X}(d_0)(\alpha)"] \\
\mathscr{X}(d_1d_1)(\theta) \arrow[d, "\mathscr{X}(d_1,d_1)^{-1}(\theta)"'] & & & \mathscr{X}(d_0)\mathscr{X}(d_0)(\theta) \arrow[d, "\mathscr{X}(d_0,d_0)(\theta)"] \\
\mathscr{X}(d_1)\mathscr{X}(d_1)(\theta) \arrow[r, "\mathscr{X}(d_1)(\alpha)"'] & \mathscr{X}(d_1)\mathscr{X}(d_0)(\theta) \arrow[r, "\mathscr{X}(d_0,d_1)(\theta)"'] & \mathscr{X}(d_0d_1)(\theta) \arrow[r, equal] & \mathscr{X}(d_0d_0)(\theta).
\end{tikzcd}$$

An arrow between a pair (θ, α) and (π, β) is a morphism

$$f : \theta \to \pi$$

in $\mathscr{X}(\mathcal{H}_0)$ such that the following diagram in $\mathscr{X}(\mathcal{H}_1)$ commutes:

$$\begin{tikzcd} \mathscr{X}(d_1)(\theta) \arrow[r, "\alpha"] \arrow[d, "\mathscr{X}(d_1)(f)"'] & \mathscr{X}(d_0)(\theta) \arrow[d, "\mathscr{X}(d_0)(f)"] \\ \mathscr{X}(d_1)(\pi) \arrow[r, "\beta"] & \mathscr{X}(d_0)(\pi). \end{tikzcd}$$

Lemma A.3.2. *Let $\mathcal{H}$ and $(\mathscr{C}, J)$ be as in the previous corollary. Then for all stacks $\mathscr{X}$, there is a functorial equivalence of groupoids*

$$\mathrm{Hom}\left([\mathcal{H}]_J, \mathscr{X}\right) \xrightarrow{\sim} \mathit{Cocycle}(\mathcal{H}, \mathscr{X}).$$

Proof. It suffices to show that there is a functorial equivalence

$$\mathit{Cocone}(\mathcal{F}_{\mathcal{H}}, \mathscr{X}) \xrightarrow{\sim} \mathit{Cocycle}(\mathcal{H}, \mathscr{X})$$

since, provided this, by composition with the functorial equivalence

$$\mathrm{Hom}\left([\mathcal{H}]_J, \mathscr{X}\right) \xrightarrow{\sim} \mathit{Cocone}(\mathcal{F}_{\mathcal{H}}, \mathscr{X})$$

induced by the colimiting cocone, we would be done.

Define a functor

$$\Theta : \mathit{Cocone}(\mathcal{F}_{\mathcal{H}}, \mathscr{X}) \to \mathit{Cocycle}(\mathcal{H}, \mathscr{X})$$

as follows. Given a cocone

$$\rho : \mathcal{F}_{\mathcal{H}} \Rightarrow \Delta_{\mathscr{X}},$$

consider the data of

$$\rho_0 : \mathcal{H}_0 \to \mathscr{X}$$

and the 2-cell

$$\rho_0 d_1 \xRightarrow{\rho(d_1)} \rho_1 \xRightarrow{\rho(d_0)^{-1}} \rho_0 d_0.$$

Under Yoneda, this corresponds to an object $x(\rho) \in \mathscr{X}(\mathcal{H}_0)$ (corresponding to ρ_0) and a map

$$m(\rho) : \mathscr{X}(d_1)(x(\rho)) \to \mathscr{X}(d_0)(x(\rho))$$

in $\mathscr{X}(\mathcal{H}_1)$ (corresponding to $\rho(d_0)^{-1}\rho(d_1)$). We claim that the pair

$$(x(\rho), m(\rho))$$

is an object in $\mathcal{C}ocycle\left(\mathcal{H},\mathscr{X}\right)$. The proof is analogous to the proof that the maps φ^{ρ} (used in the proof of Lemma A.3.1) preserve composition, so we leave it to the reader. We define Θ on objects by

$$\Theta\left(\rho\right)=\left(x\left(\rho\right),m\left(\rho\right)\right).$$

Suppose that $\omega:\rho\Rightarrow\rho'$ is a modification between two cocones. Taking the component along [0], we have

$$\omega_0:\rho_0\Rightarrow\rho_0',$$

which under Yoneda corresponds to a morphism

$$f\left(\omega\right):x\left(\rho\right)\rightarrow x\left(\rho'\right)$$

in $\mathscr{X}\left(\mathcal{H}_1\right)$. We claim that this is a morphism in $\mathcal{C}ocycle\left(\mathcal{H},\mathscr{X}\right)$. It suffices to prove that the following diagram commutes:

$$\begin{tikzcd}
\rho_0 d_1 \arrow[r, Rightarrow, "\rho(d_0)^{-1}\rho(d_1)"] \arrow[d, Rightarrow, "\omega_0 d_1"'] & \rho_0 d_0 \arrow[d, Rightarrow, "\omega_0 d_0"] \\
\rho_0' d_1 \arrow[r, Rightarrow, "\rho'(d_0)^{-1}\rho'(d_1)"] & \rho_0' d_0.
\end{tikzcd}$$

This follows since, because of the coherency conditions making ω a modification,

$$\omega_1=\rho'\left(d_i\right)\left(\omega_0 d_i\right)\rho\left(d_i\right)^{-1} \tag{A.1}$$

for all i. We therefore define Θ on arrows by $\Theta\left(\omega\right)=f\left(\omega\right)$. We will now show that Θ is essentially surjective. Given

$$\left(\theta,\alpha\right)\in\mathcal{C}ocycle\left(\mathcal{H},\mathscr{X}\right)_0$$

denote by

$$\tilde{\theta}:\mathcal{H}_0\rightarrow\mathscr{X}$$

and

$$\tilde{\alpha}:\tilde{\theta}d_1\Rightarrow\tilde{\theta}d_0$$

their images under Yoneda. Define a cocone $\gamma:\mathcal{F}_{\mathcal{H}}\rightarrow\mathscr{X}$ as follows. Let the components be given by:

$$\gamma_0:=\tilde{\theta}$$
$$\gamma_1:=\tilde{\theta}d_1$$
$$\gamma_0:=\tilde{\theta}d_1d_2.$$

For $d_0:\mathcal{H}_1\rightarrow\mathcal{H}_0$, let $\gamma\left(d_0\right)$ be given by $\tilde{\alpha}^{-1}$ and for $d_0:\mathcal{H}_2\rightarrow\mathcal{H}_1$ let $\gamma\left(d_0\right)$ be given by

$$\tilde{\theta}d_1d_0=\tilde{\theta}d_0d_2\xRightarrow{\tilde{\alpha}^{-1}d_2}\tilde{\theta}d_1d_2.$$

The naturality squares (triangles) for the remaining face maps commute by definition. The coherency conditions for a weak natural transformation now define what the rest of the structure must be on composites. Clearly

$$\Theta\left(\gamma\right)=\left(\theta,\alpha\right).$$

Hence Θ is essentially surjective.

Given $\omega:\rho\to\rho'$ a modification between two cocones, equation (A.1) determines ω_1 in terms of ω_0 and similarly

$$\omega_2=\rho'\left(d_i\right)\left(\omega_1 d_i\right)\rho\left(d_i\right)^{-1}$$

holds as well. It follows that Θ is faithful. It suffices to show it is full.

For this, suppose that

$$\tilde{f}:\Theta\left(\rho\right)\Rightarrow\Theta\left(\rho'\right)$$

is a morphism in $\mathcal{C}ocycle\left(\mathcal{H},\mathscr{X}\right)$. It corresponds to a 2-cell $f:\rho_0\Rightarrow\rho_0'$, such that the following diagram commutes:

$$\begin{tikzcd}
\rho_0 d_1 \arrow[r, Rightarrow, "\rho(d_1)^{-1}\rho(d_0)"] \arrow[d, Rightarrow, "fd_1"'] & \rho_0 d_0 \arrow[d, Rightarrow, "fd_0"] \\
\rho_0' d_1 \arrow[r, Rightarrow, "\rho'(d_1)^{-1}\rho'(d_0)"] & \rho_0' d_0.
\end{tikzcd}$$

We let $\omega_0:=f$ and we define ω_1 to be the map

$$\omega_1=\rho'\left(d_0\right)\left(fd_0\right)\rho\left(d_0\right)^{-1}=\rho'\left(d_1\right)\left(fd_1\right)\rho\left(d_1\right)^{-1}.$$

This implies that ω_0 and ω_1 satisfy the necessary coherency relations. Finally, let ω_2 be the dashed arrow in the following half of a commuting hypercube:

$$
\begin{tikzcd}
 & & & \rho_0 d_1 d_0 \arrow[ddl, dotted, "f d_1 d_0"] & & & & \\
 & \rho_1 d_0 \arrow[urr] \arrow[rrrr] \arrow[ddl, dotted, "\omega_1 d_0"'] & & & & \rho_0 d_0 d_1 \arrow[ddl, dotted, "f d_0 d_1"'] \arrow[dddd, leftrightarrow] & & \\
 & & \rho_0' d_1 d_0 \arrow[dll] \arrow[dddd] & & & & & \\
\rho_1' d_0 \arrow[rrrr] \arrow[dddd] & & & & \rho_0' d_0 d_1 \arrow[dddd] & & & \\
 & & & \rho_1 d_2 \arrow[uuuu] \arrow[rrrr] \arrow[ddl, dotted, "\omega_1 d_2"] & & & & \rho_0 d_1 d_2 \arrow[ddl, dotted, "f d_1 d_2"] \\
 & \rho_2 \arrow[uuuu] \arrow[urr] \arrow[rrrr] \arrow[ddl, dashed, "\omega_2"] & & & & \rho_1 d_1 \arrow[urr] \arrow[ddl, dotted, "\omega_1 d_1"] & & \\
 & & \rho_1' d_2 \arrow[dll] & & & & \rho_0' d_1 d_2 \arrow[llll] \arrow[dll] & \\
\rho_2' & & & & \rho_1' d_1 \arrow[llll] & & &
\end{tikzcd}
$$

where the unlabeled maps are the obvious structure maps of ρ and ρ'. Since this diagram commutes, it follows that ω is a well-defined modification, and since $\omega_0 = f$, we are done. $\square$

A.4 Internal Fibrations of Groupoids

The goal of this section is to show that the 2-functor

$$\mathcal{H}\ltimes : Gpd\,(\mathcal{H} - spaces) \to (S - Gpd)\,/\mathcal{H}$$

preserves weak colimits. To accomplish this, we shall factor it as a composite of several other 2-functors each of which will be shown to preserve weak colimits.

Definition A.4.1. A **split fibration** over an S-groupoid $\mathcal{H}$ is a morphism $\varphi : \mathcal{G} \to \mathcal{H}$ in $S - Gpd$ together with a map of spaces

$$
\begin{array}{rcl}
c : \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 & \to & \mathcal{G}_1 \\
(h, x) & \mapsto & c_{h,x}
\end{array}
$$

such that

i) $\varphi(c_{h,x}) = h$,

ii) $c_{h'h,x} = c_{h_*x,h'} \circ c_{h,x}$, and

iii) $c_{h,\mathbb{1}_{\varphi(x)}} = \mathbb{1}_x$,

where $h_*x := t \circ c_{h,x}$.

Such a c is called a **splitting** of φ.

A **morphism** between two split fibrations (φ, c) and (φ', c') is a morphism π of underlying S-groupoids such that

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "\pi"] \arrow[dr, "\varphi"'] & & \mathcal{G}' \arrow[dl, "\varphi'"] \\ & \mathcal{H} & \end{tikzcd}$$

strictly commutes and such that $c'_{h,\pi(x)} = \pi(c_{h,x})$.

A 2-**cell** between two such π and π' is an internal natural transformation

$$\alpha : \pi \Rightarrow \pi'$$

such that $\varphi'\alpha = id_{\varphi}$. We denote the 2-category of split fibrations over $\mathcal{H}$ by $SpFib/\mathcal{H}$.

Remark. Technically speaking, what we have defined should be called a split *op*fibration, however, as $\mathcal{H} \cong \mathcal{H}^{op}$ for any groupoid, we make no such distinction.

Remark. For $\mathcal{K}$ in $Gpd(\mathcal{H} - spaces)$, $\theta_{\mathcal{K}} : \mathcal{H} \ltimes \mathcal{K} \to \mathcal{H}$ comes with a canonical splitting given by $c^{\mathcal{K}}_{h,x} := (h, \mathbb{1}_{hx})$, where $\theta_{\mathcal{K}}$ is as in Section III.3.1.

Recall that if $\psi : \mathcal{K} \to \mathcal{L}$ is a morphism in $Gpd(\mathcal{H} - spaces)$, then $\mathcal{H} \ltimes (\psi)$ strictly commutes over $\mathcal{H}$, hence it may be seen as a morphism in $SpFib/\mathcal{H}$. Similarly, suppose $\varphi, \psi : \mathcal{K} \to \mathcal{L}$ and $\alpha : \psi \Rightarrow \varphi$. Then it can be easily checked that $\theta_{\mathcal{L}}\mathcal{H} \ltimes (\alpha) = id_{\theta_{\mathcal{K}}}$, hence is a 2-cell in $SpFib/\mathcal{H}$. So, we get strict 2-functor:

$$\int : Gpd(\mathcal{H} - spaces) \to SpFib/\mathcal{H},$$

where $\int \mathcal{K} := \left(\mathcal{H} \ltimes \mathcal{K}, c^{\mathcal{K}}\right)$.

Theorem A.4.1. *$\int : Gpd(\mathcal{H} - spaces) \to SpFib/\mathcal{H}$, is an equivalence of 2-categories.*

Proof. Define $\mathscr{S} : SpFib/\mathcal{H} \to Gpd\,(\mathcal{H} - spaces)$ as follows:

On objects:

Let (φ, c), with $\varphi : \mathcal{G} \to \mathcal{H}$ be a split fibration. Let the **object space** of $\mathscr{S}(\varphi)$ be $\mathcal{G}_0$ together with the left $\mathcal{H}$-action along φ_0 given by

$$h \cdot x := h_*(x).$$

Let the **arrow space** be the fibered product

$$\begin{tikzcd} \mathcal{H}_0 \times_{\mathcal{H}_1} \mathcal{G}_1 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathcal{G}_1 \arrow[d, "\varphi_1"] \\ \mathcal{H}_0 \arrow[r, "\mathbb{1}"] & \mathcal{H}_1 \end{tikzcd}$$

with left $\mathcal{H}$-action along pr_1 defined by

$$h \cdot (x, f) := (t(h), h_*(f)),$$

where

$$h_*(f) := c_{h,s(f)} \circ f \circ \left(c_{h,t(f)}\right)^{-1}.$$

The **source** and **target** maps are given by

$$s(z, g) = s(g),$$

$$t(z, g) = t(g).$$

Composition is defined by

$$(g', a)(g, a) = (g'g, a).$$

The **unit** map is defined by

$$\mathbb{1}_x = \left(\mathbb{1}_{\varphi(x)}, \mathbb{1}_x\right).$$

Finally, **inverses** are computed as

$$(g, a)^{-1} = \left(g^{-1}, a\right).$$

It is easy to check that this is a well-defined groupoid object in $\mathcal{H}$-spaces.

On arrows:

Let $\pi : (\varphi, c) \to (\varphi', c')$ be a morphism of split fibrations. On objects, let

$$\mathscr{S}(\pi)_0 = \pi_0.$$

On arrows, let $\mathscr{S}(\pi)$ be defined by

$$\mathscr{S}(\pi)_1(x, g) = (x, \pi_1(g)).$$

On 2-cells:
Suppose $\alpha : \pi \Rightarrow \pi'$. Define $\mathscr{S}(\alpha)(x) := (\varphi(x), \alpha(x))$.
Given $\mathcal{K}$ a groupoid object in $\mathcal{H}$-spaces, define a morphism

$$\eta_{\mathcal{K}} : \mathcal{K} \to \mathscr{S}\left(\int \mathcal{K}\right)$$

on objects as $id_{\mathcal{K}_0}$ and on arrows by

$$\eta_{\mathcal{K}}(k) = \left(\mu_1(k), \left(\mathbb{1}_{\mu_1(k)}, k\right)\right).$$

It is easy to check that $\eta_{\mathcal{K}}$ is an isomorphism of groupoid objects in $\mathcal{H}$-spaces, and that η is a strict natural isomorphism $\eta : id_{Gpd(\mathcal{H}-spaces)} \Rightarrow \mathscr{S} \circ \int$.

Conversely, given (φ, c) a split fibration $\varphi : \mathcal{G} \to \mathcal{H}$, define

$$\Theta_{(\varphi,c)} : \int \mathscr{S}(\varphi, c) \to (\varphi, c)$$

on objects as $id_{\mathcal{G}_0}$ and on arrows by

$$\Theta_{(\varphi,c)}(h, (x, g)) = gc_{h, h_*^{-1}(s(g))}.$$

Then $\Theta_{(\varphi,c)}$ is an isomorphism of groupoids with the inverse given on arrows by

$$g \mapsto \left(\varphi(g), \varphi(t(g)), gc_{\varphi(g)_*(s(g)), \varphi(g)^{-1}}\right).$$

It is easily checked that this is a map of fibrations and that this assembles into a natural isomorphism $\Theta : \int \circ \mathscr{S} \Rightarrow id_{SpFib/\mathcal{H}}$. $\square$

Remark. Suppose that $\mathcal{H}$ is étale. Denote by $SpFib^{et}/\mathcal{H}$ the 2-category of split fibrations of groupoid objects internal to S^{et} (spaces and local homeomorphisms). (Equivalently, this is the full sub-2-category of $SpFib/\mathcal{H}$ consisting of only those 1 and 2 morphisms which are component-wise local homeomorphisms) Then the equivalence restricts to an equivalence of 2-categories

$$\int : Gpd(\mathcal{B}\mathcal{H}) \to SpFib^{et}/\mathcal{H}$$

Definition A.4.2. An **internal fibration** [58] over $\mathcal{H}$ in the 2-category of $\mathcal{H}$-spaces is a morphism $\varphi : \mathcal{G} \to \mathcal{H}$ such that for any $\mathcal{L}$, the induced morphism

$$\mathrm{Hom}(\mathcal{L}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{L}, \mathcal{H})$$

is a Grothendieck fibration in groupoids. A morphism between two such fibrations is any morphism in $(S - Gpd)/\mathcal{H}$ which strictly commutes over the base, and the 2-cells are defined the same way as for split fibrations. We denote the 2-category of internal fibrations over $\mathcal{H}$ by $Fib/\mathcal{H}$.

Remark. Any split fibration is an internal fibration.

If $\mathcal{H}$ is étale, we similarly denote by $Fib^{et}/\mathcal{H}$ the internal fibrations over $\mathcal{H}$ in $Gpd\left(S^{et}\right)$.

There is a canonical 2-functor $l : SpFib^{et}/\mathcal{H} \to Fib^{et}/\mathcal{H}$ which is the identity on objects, arrows and 2-cells, but the functor is not full on arrows. We aim to show that l preserves pseudo-colimits. First, we will need a series of lemmas.

Remark. It suffices to prove this for when S is topological spaces, since, if $\mathcal{H}$ is an étale Lie groupoid and $\tilde{\mathcal{H}}$ is its underlying topological groupoid, there are canonical isomorphisms of 2-categories

$$SpFib^{et}/\mathcal{H} \cong SpFib^{et}/\tilde{\mathcal{H}}$$

and

$$Fib^{et}/\mathcal{H} \cong Fib^{et}/\tilde{\mathcal{H}}$$

which commute over j.

Consider the category $Arr\,(Top)$ whose objects are continuous maps

$$f : X \to Y$$

of topological spaces, and whose morphisms are commutative squares. The map sending an arrow $f : X \to Y$ to its target Y turns

$$Arr\,(Top) \to Top$$

into a fibered category of Top. It is equivalent to the Grothendieck construction of the pseudo-functor

$$Top/\left(\,\cdot\,\right) : Top \to Cat.$$

Let Top^{et} denote the category of topological spaces and local homeomorphisms. Consider also the category $Arr^{et}\,(Top)$ whose objects are local homeomorphisms $f : X \to Y$, and whose morphisms are commutative squares which need not be local homeomorphisms. This is similarly a fibered category over Top equivalent to the Grothendieck construction of the pseudo-functor

$$Top^{et}/\left(\,\cdot\,\right) : Top \to Cat.$$

Under the equivalence

$$Top^{et}/X \simeq Sh\,(X)\,,$$

the étalé space construction gives us fiber-wise adjunctions

$$Top^{et}/X \underset{i_{et}}{\overset{\Gamma}{\rightleftarrows}} Top/X\ .$$

By [12], 8.4.2, these assemble into a fibered adjunction

$$Arr^{et}\,(Top) \underset{i_{et}}{\overset{\Gamma}{\rightleftarrows}} Arr\,(Top)\ ,$$

i.e. an internal adjunction in the 2-category of categories fibered over Top.

Lemma A.4.2. *Let $X_\bullet$ be a simplicial object in Top^{et}. Then the inclusion functor*
$$i_{et} : \left(Top^{et}\right)^{\Delta^{op}}/X_\bullet \hookrightarrow Top^{\Delta^{op}}/X_\bullet$$
has a right adjoint Γ.

Proof. Given any object $g : T \to X_n$ in Top^{et}/X_n, we denote its underlying space by $\underline{g} = T$ so that $g : \underline{g} \to X_n$. Given a simplicial space $f : Z_\bullet \to X_\bullet$ over $X_\bullet$, define a sequence of spaces

$$\Gamma(Z)_n := \underline{\Gamma(f_n)}.$$

Since the square

$$\begin{tikzcd} Z_n \arrow[r, "d_i"] \arrow[d, "f_n"'] & Z_{n-1} \arrow[d, "f_{n-1}"] \\ X_n \arrow[r, "d_i"] & X_{n-1} \end{tikzcd}$$

commutes, the pair (d_i, d_i) may be considered as a morphism from

$$f_n : Z_n \to X_n$$

to

$$f_{n-1} : Z_{n-1} \to X_{n-1}$$

in $Arr(Top)$. By applying the functor

$$\Gamma : Arr(Top) \to Arr^{et}(Top),$$

we get a morphism

$$\begin{tikzcd} \underline{\Gamma(f_n)} \arrow[r] \arrow[d, "\Gamma(f_n)"'] & \underline{\Gamma(f_{n-1})} \arrow[d, "\Gamma(f_{n-1})"] \\ X_n \arrow[r, "d_i"] & X_{n-1}, \end{tikzcd}$$

in $Arr^{et}(Top)$ from $\Gamma(f_n)$ to $\Gamma(f_{n-1})$. We define the i^{th} face map d_i be the map

$$\underline{\Gamma(f_n)} \to \underline{\Gamma(f_{n-1})}$$

that is the top arrow of this diagram. Degeneracy maps are defined similarly. It is easy to check that this gives the sequence $\Gamma(Z)_n$ the structure of a simplicial object in Top^{et}, and moreover the maps

$$\Gamma(f_n) : \Gamma(Z)_n \to X_n$$

assemble into a map

$$\Gamma(f) : \Gamma(Z)_\bullet \to X_\bullet.$$

This describes Γ on objects. Let us now describe what it does on arrows. Suppose

$$\begin{tikzcd} Z_\bullet \arrow[rr, "\pi"] \arrow[dr, "f"'] & & Y_\bullet \arrow[dl, "g"] \\ & X_\bullet & \end{tikzcd},$$

is an arrow in $Top^{\Delta^{op}}/X_\bullet$. Let

$$\Gamma(\pi)_n := \Gamma(\pi_n).$$

Denote by η_n and ε_n the unit and co-unit of the adjunction

$$Top^{et}/X_n \underset{i_{et}}{\overset{\Gamma}{\rightleftarrows}} Top/X_n .$$

(Note: η_n is a natural isomorphism) These assemble into natural transformations

$$\eta : id_{(Top^{et})^{\Delta^{op}}/X_\bullet} \Rightarrow \Gamma \circ i_{et}$$
$$\varepsilon : i_{et} \circ \Gamma \Rightarrow id_{Top^{\Delta^{op}}/X_\bullet},$$

with η a natural isomorphism, and since for all n, η_n and ε_n satisfy the triangle equations, so do η and ε. Hence Γ is right adjoint to i_{et}.

$\square$

Proposition A.4.1. *Suppose $X_\bullet$ in the previous lemma is the enriched nerve $N(\mathscr{C})$ of a category $\mathscr{C}$ internal to Top^{et}, and suppose $F : \mathscr{D} \to \mathscr{C}$ is a continuous functor of topological categories. Then there exists a unique internal functor*

$$\Gamma(F) : \underline{\Gamma(F)} \to \mathscr{C}$$

from a topological category $\underline{\Gamma(F)}$, such that

$$N(\Gamma(F)) = \Gamma(N(F)),$$

where N denotes the enriched nerve construction. Furthermore, if $\mathscr{D}$ is a topological groupoid, so is $\underline{\Gamma(F)}$.

Proof. For the first part, since N is fully faithful, it suffices to show that the underlying simplicial space of $\Gamma(N(F))$ is the nerve of a category internal to Top^{et}. This follows immediately from the fact each $\Gamma : Top/X_n \to Top^{et}/X_n$ preserves limits. Now, suppose that $\mathscr{D}$ is a groupoid. We know the diagram

$$\begin{tikzcd} \underline{\Gamma(F)}_2 = \underline{\Gamma(F)}_1 \times_{\underline{\Gamma(F)}_0} \underline{\Gamma(F)}_1 \arrow[r, "d_1"] & \underline{\Gamma(F)}_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \underline{\Gamma(F)}_0 \arrow[l, bend left, "s_0"] \end{tikzcd}$$

is a internal category in Top^{et}, $\underline{\Gamma(F)}$, with each

$$\underline{\Gamma(F)}_i = \underline{\Gamma(F_i)}.$$

Let $i : \mathscr{D}_1 \to \mathscr{D}_1$ be the inverse map. The obvious identity $t \circ F_1 \circ i = s \circ F_1$ induces a map

$$\underline{\Gamma(i)} : \underline{\Gamma(d_0 \circ F_1)} \to \underline{\Gamma(d_1 \circ F_1)}.$$

Note that for all i, there are canonical isomorphisms

$$\delta_i : \underline{\Gamma(F_1)} \to \underline{\Gamma(d_i \circ F_1)}.$$

The composite

$$\delta_1{}^{-1} \circ \underline{\Gamma(i)} \circ \delta_0$$

is then map $\underline{\Gamma(F)}_1 \to \underline{\Gamma(F)}_1$ sending each arrow to its inverse. □

Remark. If

$$\begin{tikzcd} \mathscr{D} \arrow[rr, "\pi"] \arrow[dr, "F"'] & & \mathscr{D}' \arrow[dl, "G"] \\ & \mathscr{C} & \end{tikzcd}$$

strictly commutes, then by taking nerves, get a map $N(\pi)$ in $Top^{\Delta^{op}}/N(\mathscr{C})$, and by applying Γ we get a map

$$\Gamma(F) \to \Gamma(G).$$

In particular, we get a map

$$N(\Gamma(F)) = \underline{\Gamma(N(F))} \to \underline{\Gamma((G))} = N(\Gamma(G)),$$

and since N is fully faithful, this corresponds to a unique continuous functor

$$\Gamma(\pi) : \Gamma(F) \to \Gamma(G).$$

Lemma A.4.3. *Suppose (φ, c) with $\varphi : \mathcal{G} \to \mathcal{H}$ is a split fibration, with $\mathcal{H}$ étale. Then $\Gamma(\varphi)$ has a canonically induced splitting $\Gamma(c)$ such that the co-unit map*

$$\varepsilon_\phi : \Gamma(\varphi) \to \phi$$

becomes a morphism of split fibrations. Moreover, this assignment is functorial: if $\pi : \mathcal{G} \to \mathcal{G}'$ is a morphism of split fibrations (φ, c) to (φ', c'), then

$$\Gamma(\pi) : \Gamma(\varphi) \to \Gamma(\varphi')$$

is a morphism of split fibrations.

Proof. The construction is as follows. Since c is a splitting, the following diagram commutes:

$$\begin{tikzcd} \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \arrow[rr, "c"] \arrow[dr, "pr_1"'] & & \mathcal{G}_1 \arrow[dl, "\varphi_1"] \\ & \mathcal{H}_1. & \end{tikzcd}$$

Hence, we get an induced map

$$\tilde{c} : \underline{\Gamma(pr_1)} \to \underline{\Gamma(\varphi_1)} = \Gamma(\varphi)_1,$$

such that

$$\Gamma(\varphi_1) \circ \tilde{c} = \Gamma(\pi).$$

We now defined $\Gamma(c)$ to be the composite

$$\mathcal{H}_1 \times_{\mathcal{H}_0} \Gamma(\varphi)_0 \xrightarrow{\sim} \underline{\Gamma(s)} \times_{\mathcal{H}_0} \underline{\Gamma(\varphi_0)} \xrightarrow{\sim} \underline{\Gamma(pr_1)} \xrightarrow{\tilde{c}} \underline{\Gamma(\varphi_1)} = \Gamma(\varphi)_1.$$

□

Theorem A.4.4. *If $\mathcal{H}$ is an étale topological groupoid, the canonical 2-functor*

$$i_{et} : SpFib^{et}/\mathcal{H} \to SpFib/\mathcal{H}$$

has a right pseudo-adjoint Γ.

Proof. Define Γ on objects as

$$\Gamma((\varphi, c)) := ((\Gamma(\varphi)), \Gamma(c)),$$

and on arrows by $\Gamma(\pi)$, as in the remark proceeding Proposition A.4.1. Finally, suppose that $\alpha : \pi \Rightarrow \pi'$ is a 2-cell in $SpFib/\mathcal{H}$ between two arrows π and π' between (φ, c) and (φ', c'). In particular, the following diagram commutes

$$\begin{tikzcd} \mathcal{G}_0 \arrow[rr, "\alpha"] \arrow[dr, "\varphi_0"'] & & \mathcal{G}_1 \arrow[dl, "\varphi_0' \circ s."] \\ & \mathcal{H}_0 & \end{tikzcd}$$

Hence, there is an induced map

$$\tilde{\alpha} : \underline{\Gamma(\varphi_0)} \to \underline{\Gamma(s \circ \varphi_0')} = \underline{\Gamma(d_0 \circ \varphi_0')}.$$

We define

$$\Gamma(\alpha) = \delta_0^{-1} \circ \tilde{\alpha}.$$

We leave it to the reader to verify that this indeed encodes a 2-cell in $SpFib^{et}/\mathcal{H}$.

On one hand, from Lemma A.4.3, we have that the co-unit ε of the adjunction $i_{et} \dashv \Gamma$ from Lemma A.4.2 restricts to a strict 2-natural transformation

$$\varepsilon : \Gamma \circ i_{et} \Rightarrow id_{SpFib/\mathcal{H}}.$$

On the other hand, it is easy to check that the unit of this adjunction η restricts to strict 2-natural *isomorphism*

$$\eta : id_{SpFib^{et}/\mathcal{H}} \Rightarrow i_{et} \circ \Gamma.$$

Since η and ε from Lemma $A.4.3$ satisfy the triangle equations, it follows that their restrictions do as well. $\square$

Corollary A.4.1. *The* 2*-functor* $i_{et} : SpFib^{et}/\mathcal{H} \to SpFib/\mathcal{H}$ *preserves pseudo-colimits.*

Lemma A.4.5. *Let* $u : SpFib/\mathcal{H} :\to SpFib/\underline{\mathcal{H}}$ *be the* 2*-functor which sends a split fibration over* $\mathcal{H}$ *to its underlying split fibration over the underlying set-theoretical groupoid* $\underline{\mathcal{H}}$*. Suppose that* $F : J \to SpFib/\mathcal{H}$ *is any pseudo-functor from a* 2*-category* J*, and denote*

$$F(j) = \left(\varphi_j : \widehat{F(j)} \to \mathcal{H}, c_j\right).$$

Let

$$\rho : u \circ F \Rightarrow \Delta_{\underrightarrow{\mathrm{plim}}\, u \circ F}$$

be a pseudo-colimiting cocone for $u \circ F$*. Write*

$$\underrightarrow{\mathrm{plim}}\, u \circ F = \left(\underline{\varphi} : \widehat{\underrightarrow{\mathrm{plim}}\, u \circ} F \to \underline{\mathcal{H}}, \underline{c}\right)$$

and denote by $\widehat{\rho(j)}$ *the underlying map of groupoids associated to the* j^{th}*-component of* ρ*. Then for* $i = 0, 1$*, we have a family of maps*

$$\left\{ F(j)_i \xrightarrow{\widehat{\rho(j)}_i} \left(\underrightarrow{\mathrm{plim}}\, u \circ F\right)_i |j \in J_0 \right\},$$

with each $F(j)_i$ *a topological space. This equips the sets* $\left(\underrightarrow{\mathrm{plim}}\, u \circ F\right)_i$ *with the final topology with respect to these maps; this is the finest topology on* $\left(\underrightarrow{\mathrm{plim}}\, u \circ F\right)_i$ *which makes each* $\widehat{\rho(j)}_i$ *continuous. The groupoid* $\widehat{\underrightarrow{\mathrm{plim}}\, u \circ} F$ *equipped with this topology becomes a topological groupoid* $\mathcal{G}$*, and the maps* $\underline{\varphi}$ *and* $\underline{c}$ *become continuous maps* φ*,* c*, as do the structure maps of the cocone* ρ*. Moreover, the induced cocone*

$$\tilde{\rho} : F \to \Delta_{(\mathcal{G},\varphi,c)}$$

is pseudo-colimiting.

Proof. The fact that $\mathcal{G}$ becomes a topological groupoid is automatic by the definition of the final topology. For example, to see that the source map $s : \mathcal{G}_1 \to \mathcal{G}_0$ is continuous, note that for all j we have a commutative diagram:

$$\begin{tikzcd} F(j)_1 \arrow[r, "\widehat{\rho(j)}_1"] \arrow[d, "s"'] & \mathcal{G}_1 \arrow[d, "s"] \\ F(j)_0 \arrow[r, "\widehat{\rho(j)}_0"] & \mathcal{G}_0. \end{tikzcd}$$

So $s \circ \widehat{\rho(j)}_1$ is continuous for all j, hence s is continuous. A similar argument works for the other structure maps, and for $\underline{\varphi}$. To see that the splitting c is continuous, note that $\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0$ has the final topology with respect to the maps

$$\begin{array}{rcl} \mathcal{H}_1 \times_{\mathcal{H}_0} F(j)_0 & \to & \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \\ (h, x) & \mapsto & \left(h, \widehat{\rho(j)}_0(x)\right) \end{array}$$

and moreover, since each $\widehat{\rho(j)}$ is a map of split fibrations, the following diagram commutes for all j:

$$\begin{tikzcd} \mathcal{H}_1 \times_{\mathcal{H}_0} F(j)_0 \arrow[r] \arrow[d, "c_j"'] & \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{G}_0 \arrow[d, "c"] \\ F(j)_1 \arrow[r, "\widehat{\rho(j)}_1"] & \mathcal{G}_1. \end{tikzcd}$$

The underlying groupoid maps of ρ are continuous by definition, so indeed we get an induced cocone

$$\tilde{\rho} : F \to \Delta_{(\mathcal{G},\varphi,c)}.$$

It suffices to show that it is pseudo-colimiting. We need to show that for all split fibrations $(\psi : \mathcal{L} \to \mathcal{H}, c')$, the induced functor

$$\widehat{\tilde{\rho}} : \operatorname{Hom}\left((\mathcal{G}, \varphi, c), (\psi : \mathcal{L} \to \mathcal{H}, c')\right) \to \mathit{Cocone}\left(F, (\psi : \mathcal{L} \to \mathcal{H}, c')\right)$$

is an isomorphism of categories. Consider the canonical functors

$$\imath : \operatorname{Hom}\left((\varphi : \mathcal{G} \to \mathcal{H}, c), (\psi : \mathcal{L} \to \mathcal{H}, c')\right) \to \operatorname{Hom}\left(\left(\underline{\varphi} : \widehat{\underset{\longrightarrow}{\operatorname{plim}}\, u \circ} F \to \underline{\mathcal{H}}, \underline{c}\right), \left(\underline{\psi} : \underline{\mathcal{L}} \to \underline{\mathcal{H}}, \underline{c'}\right)\right),$$

$$\jmath : \mathit{Cocone}\left(F, (\psi : \mathcal{L} \to \mathcal{H}, c')\right) \to \mathit{Cocone}\left(u \circ F, \left(\underline{\psi} : \underline{\mathcal{L}} \to \underline{\mathcal{H}}, \underline{c'}\right)\right).$$

It suffices to show that $\hat{\rho}^{-1} \circ \jmath$ lies in the image of $\imath$. We will show this on objects. The argument for arrows is completely analogous. Let

$$\mu : F \Rightarrow \Delta_{(\psi : \mathcal{L} \to \mathcal{H}, c')}$$

be a cocone for F. Then $\lambda := \hat{\rho}^{-1}(\jmath(\mu))$ is the unique map

$$\lambda : \left(\underline{\varphi} : \widehat{\underrightarrow{\text{plim}}\, u \circ F} \to \underline{\mathcal{H}}, \underline{c}\right) \to \left(\underline{\psi} : \underline{\mathcal{L}} \to \underline{\mathcal{H}}, \underline{c'}\right)$$

such that

$$\jmath(\mu) = \Delta_\lambda \circ \rho.$$

(Note: This is precisely why we need *pseudo*-colimits, since otherwise, this equality may only be an isomorphism of cocones.) It remains to show that $\hat{\lambda} : \mathcal{G} \to \mathcal{L}$ is continuous. However, this follows since for $i = 0, 1$ and for all $j \in J_0$, the following diagram commutes:

$$\begin{tikzcd} F(j)_i \arrow[r, "\widehat{\rho(j)}_i"] \arrow[dr, "\mu(j)_i"'] & \mathcal{G}_i \arrow[d, "\widehat{\lambda}_i"] \\ & \mathcal{L}_i. \end{tikzcd}$$

□

Theorem A.4.6. *If $\mathcal{H}$ is an étale topological groupoid, then*

$$l : SpFib^{et}/\mathcal{H} \to Fib^{et}/\mathcal{H}$$

preserves pseudo-colimits.

Proof. Let $F : J \to SpFib^{et}/\mathcal{H}$ be a pseudo-functor from a 2-category J. From Corollary A.4.1, we can calculate

$$\underrightarrow{\text{plim}}\, i_{et} \circ F$$

and the result will actually be an object of $SpFib^{et}/\mathcal{H}$ and represent the pseudo-colimit of F. From Lemma A.4.5, we have an explicit description for this pseudo-colimit. From this description, we see that $l\left(\underrightarrow{\text{plim}}\, F\right)$ is $(\varphi : \mathcal{G} \to \mathcal{H})$ where $\mathcal{G}$ has underlying groupoid $\widehat{\underrightarrow{\text{plim}}\, u \circ F}$, and is topologized by the final topology with respect to the components of its colimiting cocone. It is standard that the 2-category of split fibrations over $\underline{\mathcal{H}}$ is equivalent (as a strict 2-category) to $Psh(\underline{\mathcal{H}}, Gpd)$ and $Fib/\underline{\mathcal{H}}$ is similarly strictly equivalent to $Gpd^{\underline{\mathcal{H}}^{op}}$, in such a way that, under these identifications,

$$\underline{l} : SpFib/\underline{\mathcal{H}} \to Fib/\underline{\mathcal{H}}$$

is the 2-functor j of Theorem A.2.1, hence preserves pseudo-colimits. Therefore, $\left(\underline{\varphi} : \widehat{\underrightarrow{\text{plim}}\, u \circ F} \to \underline{\mathcal{H}}\right)$ is the vertex of a pseudo-colimiting cocone for

$$\underline{l} \circ u \circ F = u \circ l \circ F,$$

where the cocone

$$\rho' : \underline{l} \circ u \circ F \Rightarrow \Delta_{\left(\underline{\varphi}:\ \widehat{\underrightarrow{\mathrm{plim}}\ u \circ F} \to \underline{\mathcal{H}}\right)}$$

has the same components as the pseudo-colimiting cocone

$$\rho : u \circ F \Rightarrow \Delta_{\underrightarrow{\mathrm{plim}}\ u \circ F}.$$

The rest of the argument is the same as in the previous lemma. □

Corollary A.4.2. *If $\mathcal{H}$ is an étale Lie groupoid, then*

$$l : SpFib^{et}/\mathcal{H} \to Fib^{et}/\mathcal{H}$$

preserves pseudo-colimits.

Theorem A.4.7. *Let $\mathcal{H}$ be an S-groupoid, and consider the canonical 2-functor*

$$v : Fib/\mathcal{H} \to (S - Gpd)/\mathcal{H}.$$

There is a strict 2-adjunction $v \dashv l \circ \int \circ P$, where we have, by abuse of notation, used l to denote the canonical 2-functor $l : SpFib/\mathcal{H} \to Fib/\mathcal{H}$ and where $\int$ is as in Theorem A.4.1 and P is as in Section A.4.1.

Proof. Given φ an internal fibration over $\mathcal{H}$ and $\psi : \mathcal{L} \to \mathcal{H}$ a map of S-groupoids, define a functor

$$m_{\varphi,\psi} : Hom_{Fib/\mathcal{H}}\left(\varphi, l \circ \int P(\psi)\right) \to Hom_{(S-Gpd)/\mathcal{H}}(v\varphi, \psi)$$

by composition with εv, where ε is the natural transformation of Lemma III.3.1, i.e.:

$$\varphi \xrightarrow{G} l \circ \int P(\psi) \mapsto v\varphi \xrightarrow{\quad v(G) \quad} v \circ l \circ \int P(\psi) = \mathcal{H} \times P(\psi) \xrightarrow{\varepsilon_\psi} \psi,$$

and similarly on arrows. We will show that $m_{\varphi,\psi}$ is an isomorphism of categories by constructing an explicit inverse.

Suppose $(F, \alpha) : v\varphi \to \psi$, then $l \circ \int P(\psi)$ is the fibration

$$\tilde{\theta}_\psi : (\mathcal{H} \times \mathcal{L}) \ltimes (\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{L}_0) \to \mathcal{H}.$$

Suppose $\varphi : \mathcal{G} \to \mathcal{H}$. Define

$$n_{\varphi,\psi}(F)_0 : \mathcal{G}_0 \to \mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{L}_0$$

by

$$n_{\varphi,\psi}(F)_0(x) = (\alpha(x), F_0(x)).$$

Define

$$n_{\varphi,\psi}(F)_1(g) = ((\varphi(g), F_1(g)), (\alpha(s(g)), F(s(g)))).$$

It is easy to check that

$$n_{\varphi,\psi}(F) : \mathcal{G} \to (\mathcal{H} \times \mathcal{L}) \ltimes (\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{L}_0),$$

is a morphism of fibrations

$$\varphi \to l \circ \int P(\psi).$$

This defines the object part of a functor

$$n_{\varphi,\psi} : Hom_{(S-Gpd)/\mathcal{H}}(v\varphi, \psi) \to Hom_{Fib/\mathcal{H}}\left(\varphi, l \circ \int P(\psi)\right).$$

Now suppose that (F, α) and (F', α') are morphisms $v\varphi \to \psi$ and that

$$\beta : F \Rightarrow F'$$

encodes a 2-cell between them. Define a 2-cell

$$n_{\varphi,\psi}(\beta) : n_{\varphi,\psi}(F) \Rightarrow n_{\varphi,\psi}(F'),$$

by

$$n_{\varphi,\psi}(\beta)(x) = \left(\left(\mathbb{1}_{\varphi(x)}, \beta(x)\right), (\alpha(x), F_0(x))\right).$$

It is easy to check that $n_{\varphi,\psi}$ defines an inverse functor to $m_{\varphi,\psi}$ and together they form a strict 2-adjunction $v \dashv l \circ \int \circ P$. □

Remark. The adjunction of the previous theorem restricts to an adjunction $v_{et} \dashv l \circ \int \circ P$ between

$$v_{et} : Fib^{et}/\mathcal{H} \to \left(S^{et} - Gpd\right)/\mathcal{H},$$

and

$$l \circ \int \circ P : \left(S^{et} - Gpd\right)/\mathcal{H} \to Fib^{et}/\mathcal{H}.$$

Hence, both v and v_{et} preserve pseudo-colimits.

Theorem A.4.8. *Denote by $Y : (S^{et} - Gpd)/\mathcal{H} \to \mathrm{St}(S)/\mathscr{X}$, the canonical functor, where $\mathscr{X} = [\mathcal{H}]$. The composite*

$$Y \circ \mathcal{H}\ltimes : Gpd(\mathcal{BH}) \to \mathrm{St}(S)/\mathscr{X}$$

preserves weak colimits.

Proof. We can factor this 2-functor as

$$Gpd\,(\mathcal{BH}) \xrightarrow{\int} SpFib^{et}/\mathcal{H} \xrightarrow{l} Fib^{et}/\mathcal{H} \xrightarrow{v_{et}} \left(S^{et} - Gpd\right)/\mathcal{H} \xrightarrow{Y} \mathrm{St}\,(S)\,/\mathscr{X}.$$

The composite $v_{et} \circ l \circ \int$ is pseudo-colimit preserving, so it suffices to show that Y is as well. Note that Y preserves arbitrary co-products, as covers of a disjoint unions of spaces correspond to covers of each space individually. Since both $\left(S^{et} - Gpd\right)/\mathcal{H}$ and $\mathrm{St}\,(S)\,/\mathscr{X}$ are $(2,1)$-categories, it suffices to show that Y preserves pseudo-co-equalizers. Suppose that

$$\begin{tikzcd} \mathscr{A} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, "a"'] & \mathscr{B} \arrow[r, "h"] \arrow[d, "b"] & \mathscr{C} \arrow[dl, "c"] \\ & \mathcal{H} & \end{tikzcd}$$

is a pseudo-co-equalizing diagram in $\left(S^{et} - Gpd\right)/\mathcal{H}$. To be rigorous, this diagram is actually a pseudo-functor

$$F : 2 \to \left(S^{et} - Gpd\right)/\mathcal{H},$$

where 2 is the category with objects 0 and 1 and two non-identity arrows $0 \rightrightarrows 1$, together with a pseudo-colimiting cocone

$$\rho : F \Rightarrow \Delta_c.$$

Hence we have that $\rho_b = h$, and we can assume without loss of generality that $\rho_a = h \circ f$ so that

$$\rho\,(g) : h \circ g \Rightarrow h \circ f.$$

We wish to show that

$$Y \circ v_{et}\,(a) \underset{Y \circ v_{et}(g)}{\overset{Y \circ v_{et}(f)}{\rightrightarrows}} Y \circ v_{et}\,(b) \xrightarrow{Y \circ v_{et}(h)} Y \circ v_{et}\,(c)$$

is a weak-co-equalizer, i.e.

$$Y v_{et} \rho : Y v_{et} F \Rightarrow \Delta_{Y v_{et}(c)}$$

is weak-colimiting.

Since

$$i_{et} : Fib^{et}/\mathcal{H} \to Fib/\mathcal{H}$$

preserves pseudo-colimits, and since

$$v : Fib/\mathcal{H} \to (S - Gpd)\,/\mathcal{H},$$

and the forgetful functor

$$(S - Gpd)\,/\mathcal{H} \to (S - Gpd)$$

do as well, it follows that

$$\begin{tikzcd} \mathscr{A} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathscr{B} \arrow[r, "h"] & \mathscr{C} \end{tikzcd}$$

is a pseudo-co-equalizing diagram of S-groupoids. In particular, the underlying diagram of set-theoretic groupoids is also a pseudo-co-equalizer. Hence

$$\underline{h} : \underline{\mathscr{B}} \to \underline{\mathscr{C}}$$

is essentially surjective, and so is

$$\underline{q} : \underline{\mathscr{A}} \to \underline{\mathscr{B}} \times_{\underline{\mathscr{C}}} \underline{\mathscr{B}},$$

where the latter is induced by the obvious cone over $\mathscr{A}$. Consider the pullback diagram

$$\begin{tikzcd} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{B}_0 \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & \mathscr{B}_0 \arrow[d, "h_0"] \\ \mathscr{C}_1 \arrow[r, "s"] & \mathscr{C}_0. \end{tikzcd}$$

Since h_0 is a local homeomorphism, it follows that pr_1 is as well. Hence

$$t \circ pr_1 : \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{B}_0 \to \mathscr{C}_0$$

is also a local homeomorphism. However, as $\underline{h}$ is essentially surjective, it follows that $t \circ pr_1$ is both surjective and a local homeomorphism, hence a cover. This in turn implies that the induced map $[\mathscr{B}] \to [\mathscr{C}]$ is an epimorphism in $\mathrm{St}(S)$. Similarly for the induced map

$$[\mathscr{A}] \to [\mathscr{B} \times_{\mathscr{C}} \mathscr{B}] = [\mathscr{B}] \times_{[\mathscr{C}]} [\mathscr{B}].$$

It follows that

$$Yv_{et}(h) : Yv_{et}(b) \to Yv_{et}(c)$$

is an epimorphism in $\mathrm{St}(S)/\mathscr{X}$ as is

$$Yv_{et}(q) : Yv_{et}(a) \to Yv_{et}(b) \times_{Yv_{et}(c)} Yv_{et}(b).$$

Let

$$c' := \underrightarrow{\mathrm{plim}}\ (Yv_{et}(a) \rightrightarrows Y \circ v_{et}(b)),$$

and let

$$\mu : Yv_{et}F \Rightarrow \Delta_{c'}$$

be its pseudo-colimiting cocone. Now,

$$Yv_{et}\rho : Yv_{et}F \Rightarrow \Delta_{Yv_{et}(c)}$$

is a cocone on $Yv_{et}(c)$, so it corresponds to a unique map

$$\varphi : c' \to Yv_{et}(c)$$

such that

$$\Delta_\varphi \circ \mu = Yv_{et}\rho.$$

We will exhibit a weak inverse for this map.

We claim that the map

$$\mu_b : Yv_{et}(b) \to c'$$

is descent data for the cover

$$Yv_{et}(h) : Yv_{et}(b) \to Yv_{et}(c).$$

Consider the 2-commutative diagram:

$$\begin{tikzcd}
Yv_{et}(a) \arrow[dr, "Yv_{et}(q)"] \arrow[drr, "Yv_{et}(f)"] \arrow[ddr, "Yv_{et}(g)"'] & & \\
& Yv_{et}(b) \times_{Yv_{et}(c)} Yv_{et}(b) \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & Yv_{et}(b) \arrow[d, two heads, "Yv_{et}(h)"] \\
& Yv_{et}(b) \arrow[r, two heads, "Yv_{et}(h)"] \arrow[ur, Rightarrow, "\rho(g)"] & Yv_{et}(c).
\end{tikzcd}$$

Notice that

$$\mu_b \circ pr_1 \circ Yv_{et}(q) = \mu_b \circ Yv_{et}(g)$$

and similarly

$$\mu_b \circ pr_2 \circ Yv_{et}(q) = \mu_b \circ Yv_{et}(f).$$

Now

$$\mu(g) : \mu_b \circ Yv_{et}(g) \Rightarrow \mu_b \circ Yv_{et}(f)$$

is an isomorphism, and $Yv_{et}(q)$ is an epimorphism, hence

$$\mu_b \circ pr_1 \cong \mu_b \circ pr_2.$$

We leave it to the reader to check that this isomorphism assembles into descent data for the cover. This implies that there exists a morphism

$$\psi : Yv_{et}(c) \to c'$$

such that $\psi \circ h \cong \mu_b$. It is easy to check that such a ψ must be a weak inverse for φ.

We have shown that any pseudo-colimit is weakly isomorphic to the corresponding pseudo-colimit in $\mathrm{St}(S)/\mathscr{X}$, hence is a weak-colimit. This in turn implies that $Y \circ \mathcal{H}\ltimes$ preserves weak colimits. $\square$

Samenvatting

Dit proefschrift richt zich op de theorie van topologische en differentieerbare *stacks*. Topologische en differentieerbare stacks gedragen zich als ruimten waarvan de punten zelf intrinsieke symmetrie hebben. Er is een analogie met algebraïsche stacks, welke een belangrijke rol spelen in de algebraïsche meetkunde. Er zijn veel recente toepassingen van topologische en differentieerbare stacks in diverse gebieden van de zuivere wiskunde, en ze spelen ook een belangrijke rol in de quantumveldentheorie en de snaartheorie.

Het eerste hoofdresultaat van dit proefschrift is de constructie van een geschikte theorie van topologische stacks, die we "compact voortgebrachte stacks" noemen, en waarin veel natuurlijke constructies die niet opgaan voor alle topologische stacks, direct kunnen worden uitgevoerd. Voor twee compact voortgebrachte stacks vormt de verzameling afbeeldingen ertussen weer een compact voortgebrachte stack, iets wat niet opgaat voor willekeurige topologische stacks. Dit probleem bestaat al op het niveau van topologische ruimten, waar men vaak werkt met compact voortgebrachte Hausdorffruimten die populair werden door een artikel uit 1967 van Norman Steenrod. We laten zien dat compact voortgebrachte topologische stacks dezelfde rol spelen als compact voortgebrachte Hausdorffruimten in de context van topologische ruimten.

De rest van het proefschrift richt zich op de ontwikkeling van een abstract kader voor de studie van schoven en stacks over *étale stacks*. Étale stacks zijn stacks die bijna ruimten zijn, in de zin dat elk punt slechts een discrete collectie symmetrieën heeft. Vervolgens gebruiken we deze machinerie om een verrassend verband aan te tonen tussen de manier waarop deze symmetrieën werken, en *gerbes*.

Curriculum Vitae

The author of this thesis was born on March 20$^{\text{th}}$, 1982 in Quincy, Massachusetts, USA. He attended public school in his home town of Walpole, Massachusetts until the age of fourteen, at which time he enrolled at Xaverian Brothers High School. During his time at Xaverian, he discovered his love for mathematics. In the year 2000, he began his studies at Worcester Polytechnic Institute (WPI) with a double major in mathematics and physics. During the first semester of 2002, he took temporary leave from WPI to participate in the “Math in Moscow” program hosted by the Independent University of Moscow (IUM) in Moscow, Russia. He then returned to WPI, graduating in 2004 with cum laude. He also received the senior thesis award from both the mathematics and the physics department, as well as the senior math award. In 2004, the author started his formal studies as a mathematics PhD student at Purdue University, in Indiana, as a Vigor fellow. In 2006, he took a one year leave of absence from Purdue to participate in the Mathematics Research Institute’s (MRI) “Masterclass in symplectic geometry and beyond.” funded by the Netherlands Organization for Scientific Research (NWO). During this masterclass, he was exposed to new areas of mathematics in which is subsequently started research. Because of this, the author transferred from Purdue to Utrecht University, accepting a PhD position under the guidance of Professor Ieke Moerdijk. The contents of this thesis is a product of the research conducted during this appointment. He received his master’s degree for his work completed during his studies at Purdue from 2004 to 2006, in 2008. During the summers of 2009 and 2010, he was a visiting scholar of the topology group at the Massachusetts Institute of Technology (MIT). In October, 2011, he will be a postdoctoral research fellow at the Max Planck Institute for Mathematics in Bonn, Germany.

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