Frames and Locales

A quick review on posets:

<u>Def</u> Let $(P, \leq)$ be a poset and $\{p_\alpha \in P\}_{\alpha \in A}$.
The <u>meet</u> of $\{p_\alpha\}$, denoted by $\bigwedge_{\alpha \in A} p_\alpha$ if it exists, is the greatest element of P which is less than each p_α.
Dually, the <u>join</u> of $\{p_\alpha\}$, denoted by $\bigvee_{\alpha \in A} p_\alpha$ if it exists, is the smallest element of P which is greater than each p_α.

Regard $(P, \leq)$ as a category and let $F: I \longrightarrow P$ be a functor.

<u>Obs</u>: • $\varprojlim F = \bigwedge_{i \in I_0} F(i)$

• $\varinjlim F = \bigvee_{i \in I_0} F(i)$.

E.g. if

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

is a diagram in P, and $D \in P_0$,

then any pair of morphism $f: D \to A$, $g: D \to B$ (in fact, if such a pair exists at all, it is unique) must satisfy

$$\begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

by uniqueness of arrows $\Rightarrow$

$$A \times_C B = A \times B.$$

Moreover, f and g exist $\Leftrightarrow$ $D \leq A$ and $D \leq B \Leftrightarrow D \leq A \wedge B \Rightarrow$
$A \times B = A \wedge B$.

<u>Def</u> A <u>lattice</u> is a poset with binary meets and joins.

A lattice is <u>complete</u> if has all small joins and meets.

<u>Notation</u> 0 = initial object (smallest element), 1 = terminal object (largest element).

<u>Example</u> If X is any set, the power set $P(X)$ is a complete lattice, with joins given by $\bigcup$ and meets given by $\bigcap$.

$P(X)$ is also a <u>completely distributive lattice</u> in that

$$\star \qquad \bigcap_\beta \Big(\bigcup_\alpha A_{\alpha,\beta} \Big) = \bigcup_\alpha \Big(\bigcap_\beta A_{\alpha,\beta} \Big).$$

Suppose now that X carries a topology $\mathcal{T}$ and let $\mathcal{O}(X)$ denote the opens of $\mathcal{T}$. By definition of topology we have that:

1) $\mathcal{O}(X)$ is closed under arbitrary unions
2) $\mathcal{O}(X)$ is closed under finite intersections
3) Both X and $\emptyset$ are in $\mathcal{O}(X)$.

Let us consider these axioms categorically:

Let $i : \mathcal{O}(X) \hookrightarrow P(X)$ denote the inclusion.

1) + 3) $\Rightarrow$ i preserves all colimits $\Rightarrow$ $\exists$ a right adjoint; indeed $i \dashv \mathrm{Int}$, where $\mathrm{Int}(A)$ denotes the interior. $P(X)$ is complete and cocomplete $\Rightarrow$ $\mathcal{O}(X)$ is too:

- joins are computed by $\bigcup$ in $P(X)$
- meets are compute by $\mathrm{Int} \circ \bigcap$.

In particular 1) + 3) $\Rightarrow$ $\mathcal{O}(X)$ is a complete lattice.

2) + ~~$*$~~ $\Rightarrow$ arbitrary joins distribute over finite meets:

$$x \wedge \left(\bigvee_\alpha y_\alpha\right) = \bigvee_\alpha (x \wedge y_\alpha), \quad \text{i.e.}$$

colimits in $\mathcal{O}(X)$ are universal.

<u>Def</u> A <u>frame</u> is a complete lattice in which arbitrary joins distribute over finite meets.

Let $X \xrightarrow{f} Y$ be a map of sets.

$$\rightsquigarrow f^{-1} : P(Y) \longrightarrow P(X)$$
$$A \longmapsto f^{-1}(A)$$

and f^{-1} preserves arbitrary meets and joins.

Now suppose that X and Y have topologies s.t. f is cont.
Then f^{-1} descends to a functor

$$f^{-1} : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X).$$

f^{-1} still preserves arbitrary joins, but only needs to preserve <u>finite</u> meets, since arbitrary meets may not be given by intersection.

<u>Def</u> A <u>map of frames</u> $P \longrightarrow Q$ is a functor which preserves finite meets and arbitrary joins.

Denote by Frm the category of frames.

<u>Def</u> The category $\mathrm{Loc} := \mathrm{Frm}^{op}$ is the category of locales.

There is an evident functor

$$\mathcal{O} : \mathrm{Top} \longrightarrow \mathrm{Loc}.$$

Def A locale $\mathbb{L}$ is said to be spatial if it is in the essential image of $\mathcal{O}$.

Let X be a top'l space. A point $x \in X$ is the same as a morphism $* \xrightarrow{x} X$.

Note: $\mathcal{O}(*) = \{\phi, *\} = \{0,1\}$.

Def A point of a locale $\mathbb{L}$ is a morphism $\mathcal{O}(*) \longrightarrow \mathbb{L}$.
Denote the set $\mathrm{Hom}_{\mathrm{Loc}}(\mathcal{O}(*), \mathbb{L})$ of points of $\mathbb{L}$ by $pt(\mathbb{L})$.

Rmk For X an arbitrary space, the induced map

$$\underline{X} = \mathrm{Hom}_{\mathrm{Top}}(*, X) \longrightarrow \mathrm{Hom}_{\mathrm{Loc}}(\mathcal{O}(*), \mathcal{O}(X)) = pt(\mathcal{O}(X))$$

may not be a bijection.
(We will see that this holds $\Leftrightarrow$ X is sober).

Let $\mathbb{L}$ be a locale and $\ell \in \mathbb{L}$. Let $U_\ell \subset pt(\mathbb{L})$ be the following subset:

$$U_\ell = \{ p : \mathcal{O}(*) \longrightarrow \mathbb{L} \mid p^*(\ell) = 1 \}$$

(Note: For $U \subset \mathcal{O}(X)$ and $x \in X$, $x^{-1} : \mathcal{O}(X) \longrightarrow \mathcal{O}(*)$, $x^{-1}(U) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$
$\Rightarrow U = \{ x \in X \mid x^{-1}(U) = 1 \}$)

where $p^* : \mathbb{L} \longrightarrow \mathcal{O}(*)$ is the map in Frm corresponding to p.

Prop The assignment $U_{(\cdot)} : \mathbb{L} \longrightarrow P(pt(\mathbb{L}))$, $\ell \longmapsto U_\ell$ is a map of frames.

Pf
- $U_0 = \phi$ since any $p^\#: \mathbb{L} \longrightarrow \mathcal{O}(*)$ has $p^\#(0) = 0$
- $U_1 = pt(\mathbb{L})$ since ↗ has $p^\#(1) = 1$

- Let $\bigvee_i l_i \in \mathbb{L}$. Then:

$$U_{\bigvee_i l_i} = \{ p^\#: \mathbb{L} \to \mathcal{O}(*) \mid p^\#(\bigvee_i l_i) = 1 \}$$
$$= \{ p^\# \mid \bigvee_i p^\# l_i = 1 \}.$$

Suppose that $\forall i,\ p^\# l_i = 0 \Rightarrow p^\# \bigvee_i l_i = \bigvee_i 0 = 0$ so:

$$U_{\bigvee_i l_i} = \{ p^\# \mid \exists\, i \text{ s.t. } p^\# l_i = 1 \}$$
$$= \bigcup_i \{ p^\# \mid p^\#(l_i) = 1 \} = \bigcup_i U_{l_i}.$$

- Let $a, b \in \mathbb{L}$:

$$U_{a \wedge b} = \{ p^\# \mid p^\#(a \wedge b) = p^\#(a) \wedge p^\#(b) = 1 \}$$

But in $\mathcal{O}(*)$: $1 \wedge 1 = 1$, $1 \wedge 0 = 0$ and $0 \wedge 0 = 0$, so

$$U_{a \wedge b} = \{ p^\# \mid p^\#(a) = p^\#(b) = 1 \}$$
$$= U_a \cap U_b. \ \boxtimes$$

Cor The subsets of $pt(\mathbb{L})$ of the form U_l for $l \in \mathbb{L}$ constitute a topology on $pt(\mathbb{L})$.

Lemma The assignment $\mathbb{L} \longmapsto pt(\mathbb{L})$ extends to a functor $\mathrm{Loc} \xrightarrow{pt} \mathrm{Top}$.

Pf: Suppose $f : \mathbb{L} \to \mathbb{M}$ is a map of locales,

let $pt(f) = \mathrm{Hom}_{\mathrm{Loc}}(\mathcal{O}(*), f) : \mathrm{Hom}_{\mathrm{Loc}}(\mathcal{O}(*), \mathbb{L}) \longrightarrow \mathrm{Hom}_{\mathrm{Loc}}(\mathcal{O}(*), \mathbb{M})$.

Just need to show $pt(f)$ is continuous. Let $m \in \mathbb{M}$, then

$$pt(f)^{-1}(U_m) = \{ p^* : \mathbb{L} \to \mathcal{O}(*) \mid p^* \circ f^* \in U_m \}$$

$$= \{ p^* : \mathbb{L} \to \mathcal{O}(*) \mid p^*(f^*(m)) = 1 \}$$

$$= U_{f^*(m)} \text{ is open. } \boxtimes$$

Sober Spaces

Def: A closed subset $C \subset X$ of a top'l space is irreducible if

1) $C \neq \emptyset$
2) $\forall\ C_1, C_2 \subset X$ closed

$$C \subset C_1 \cup C_2 \Rightarrow C \subset C_1 \text{ or } C \subset C_2.$$

Example: Let $x \in X$, then $\overline{\{x\}}$ is irreducible:

$$\overline{\{x\}} \subset C_1 \cup C_2 \Rightarrow x \in C_1 \text{ or } C_2 \Rightarrow \overline{\{x\}} \subset C_1 \text{ or } \overline{\{x\}} \subset C_2.$$

Def A top'l space X is called sober if $\forall\ C \subset X$ irreducible and closed, $\exists!\ x \in X$ s.t. $C = \overline{\{x\}}$.

<u>Example</u> Every Hausdorff space is sober.

The converse is not true, but every sober space is T_0.

(The converse of <u>this</u> ↗ is not true either).

<u>Theorem</u> (Stone Duality)

The functor $\mathcal{O}: \mathbf{Top} \longrightarrow \mathbf{Loc}$ is left adjoint to

$$\mathrm{pt}: \mathbf{Loc} \longrightarrow \mathbf{Top}.$$

Moreover, this adjunction restricts to an equivalence between sober spaces and spatial locales.

(You will prove this in the homework.)

<u>Sheaves over a Locale</u>

Warm up: Let $\mathbb{L} = \mathcal{O}(X)$ for X a top'l space.

Then $\bigvee = \bigcup$.

<u>Recall</u>: The open-cover pretopology on $\mathcal{O}(X)$ is given by declaring a collection of morphisms $(U_\alpha \leq U)_\alpha$ in $\mathcal{O}(X)$ to be a covering family $\iff \bigcup_\alpha U_\alpha = U$.

<u>Def</u> Let $\mathbb{L}$ be an arbitrary locale. Define a collection of morphisms $(l_\alpha \leq l)_\alpha$ in $\mathbb{L}$ to be a covering family $\iff \bigvee_\alpha l_\alpha = l$.
This is the <u>open-cover</u> pre-topology on $\mathbb{L}$.

<u>Def</u> Denote by $Sh(\mathbb{L})$ the topos of sheaves on $\mathbb{L}$ wrt open covers.

Note that if $\mathbb{L} = \mathcal{O}(X)$, $Sh(\mathcal{O}(X)) = Sh(X)$.