CL 1.

Let $F \in \mathrm{Set}^{\mathcal{C}^{op}}$ and consider the functor

$$\mathcal{C}/F \xrightarrow{\pi_F} \mathcal{C} \underset{y}{\hookrightarrow} \mathrm{Set}^{\mathcal{C}^{op}}.$$

Note: There is a canonical cocone

$$\rho_F : y \circ \pi_F \Longrightarrow \Delta_F$$

whose component on $(f : y(c) \to F) \in \mathcal{C}/F$ is :

$$f : y(c) \to F.$$

Prop ρ_F is colimiting.

Pf WTS $\forall\, G \in \mathrm{Set}^{\mathcal{C}^{op}}$, the induced map

$$\begin{aligned} \mathrm{Hom}(F,G) &\xrightarrow{\hat{\rho}_F} \mathrm{Cocone}(y\circ\pi_F, G) = \mathrm{Hom}(y\circ\pi_F, \Delta_G) \\ F \overset{\lambda}{\Longrightarrow} G &\longmapsto y\circ\pi_F \overset{\rho}{\Longrightarrow} \Delta_F \overset{\Delta_\lambda}{\Longrightarrow} \Delta_G \end{aligned}$$

is an isomorphism.

Notice we can factor it as

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}^{op}}}(F,G) \arrow[rr, "\hat{\rho}_F"] \arrow[dr, "\pi_{F,G}"'] & & \mathrm{Hom}(y\circ\pi_F, \Delta_G) \\ & \mathrm{Hom}_{\mathrm{DFib}(\mathcal{C})}(\mathcal{C}/F, \mathcal{C}/G) \arrow[ur] & \end{tikzcd}$$

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$\lambda : F \Rightarrow G$

since π is an eq'l.

$$\begin{tikzcd} \mathcal{C}/F \arrow[rr, "\pi\lambda"] \arrow[dr, "\pi_F"'] & & \mathcal{C}/G \arrow[dl, "\pi_G"] \\ & \mathcal{C} & \end{tikzcd}$$

$$\pi\lambda\left(y(c) \xrightarrow{f} F\right) = y(c) \xrightarrow{\lambda\circ f} G$$

ctd.

where the map $\mathrm{Hom}_{\mathrm{DFib}(\mathcal{C})}(\mathcal{C}/F, \mathcal{C}/G) \longrightarrow \mathrm{Hom}(y\circ\pi_F, \Delta_G)$

$$\begin{tikzcd} \mathcal{C}/F \arrow[rr, "\theta"] \arrow[dr, "\pi_F"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathcal{C}/G \arrow[dl, "\pi_G"] \\ & \mathcal{C} & \end{tikzcd} \longmapsto \quad \tilde{\theta} : y\circ\pi_F \Longrightarrow \Delta_G$$

$$\tilde{\theta}(y(c) \xrightarrow{f} F) = \theta(f) : y(c) \longrightarrow G$$

has an inverse

$$\begin{tikzcd} \mathcal{C}/F \arrow[rr, "\tilde{\alpha}"] \arrow[dr, "\pi_F"'] & & \mathcal{C}/G \arrow[dl, "\pi_G"] \\ & \mathcal{C} & \end{tikzcd} \longleftarrow\!\shortmid \quad \alpha : y\circ\pi_F \Longrightarrow \Delta_G$$

objects: $\tilde{\alpha}(y(c) \xrightarrow{f} F) = \alpha(f) : y(c) \longrightarrow G$

arrows:
$$\begin{tikzcd} y(c) \arrow[rr, "y(h)"] \arrow[dr, "f"'] & & y(d) \arrow[dl, "g"] \\ & F & \end{tikzcd} \longmapsto \begin{tikzcd} y(c) \arrow[rr, "y(h)"] \arrow[dr, "\alpha(f)"'] & \arrow[d, phantom, "\circlearrowleft"] & y(d) \arrow[dl, "\alpha(g)"] \\ & G & \end{tikzcd}$$ since α is natural.

$\therefore$ ρ_F is colimiting.

We usually write this as

$$F = \varinjlim_{y(c)\to F} y(c) \quad \left(\text{or } F = \varinjlim_{c\to F} c\right)$$

and say: "every presheaf is canonically a colimit of representables."