- We have seen Giraud's theorem, which gives an axiomitization of when a category is a Grothendieck topos. Roughly speaking, one may interpret these axioms as saying a Grothendieck topos is a category that behaves like the category Set.

We have also seen that there is a deep connection between topoi and spaces:

$$\begin{tikzcd} \mathrm{Top} \arrow[r, shift left, "\mathcal{O}"] & \mathrm{Loc} \arrow[l, shift left, "pt"] \arrow[r, shift right, "Sh"'] & \mathcal{T}op \arrow[l, shift right, "Loc"'] \end{tikzcd}$$

(and $\mathrm{Loc} = 0\text{-}\mathcal{T}op = 0\text{-topoi}$), and every locale is a sublocale of a space.

We note in passing the following result:

<u>Thm</u> (Diaconescu's Cover Theorem)

For every Grothendieck topos $\mathcal{E}$ $\exists$ a localic grapoid $\mathcal{G} \in \mathrm{Gpd}(\mathrm{Loc})$

$$\begin{tikzcd} \mathcal{G}_2 := \mathcal{G}_1 {}^{s}\!\times^{t}_{\mathcal{G}_0} \mathcal{G}_1 \arrow[r, "m"] & \mathcal{G}_1 \arrow[loop above, "i"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathcal{G}_0 \arrow[l, bend left=60, "1"] \end{tikzcd}$$

$\mathcal{G}_0, \mathcal{G}_1 \in \mathrm{Loc}$

2-colimit in $\mathcal{T}op$.

s.t.

$$\mathcal{E} \simeq \varinjlim_{\text{hoco}} \Big(Sh(\mathcal{G}_1 \times_{\mathcal{G}_0} \rightrightarrows \mathcal{G}_1) \overset{pr_1}{\underset{pr_0}{\overset{m}{\rightrightarrows}}} Sh(\mathcal{G}_1) \rightrightarrows Sh(\mathcal{G}_0) \Big).$$

$$\text{''} \quad \text{“}\mathcal{G}_0 /\!/ \mathcal{G}_1\text{''}$$

- Morally: "Every topos is the homotopy quotient of a locale by a continuous grapoid."

Today we will see:

"Every topos is the embodiment of a logical theory."

Given a theory $\mathbb{T}$, models for $\mathbb{T}$ can take values in an appropriate category.

E.g. • If $\mathbb{T}$ is the theory of comm. unital rings, such a ring object in a category $\mathcal{C}$ is $R \in \mathcal{C}_0$ together with maps

$$m : R \times R \longrightarrow R \quad ; \quad u : \underset{\text{terminal}}{1} \longrightarrow R$$
$$a : R \times R \longrightarrow R$$

satisfying the associativity, distributive, and unital axioms (certain diagrams need to commute).

For this to make sense, we need finite products in $\mathcal{C}$,
"$\mathbb{T}$ is a finite product theory" (aka algebraic theory, aka Lawvere theory)

• If $\mathbb{T}$ is the theory of categories, a category object in $\mathcal{C}$ consists of a diagram of the form

$$\begin{tikzcd} C_1 \times^{s\,t}_{C_0} C_1 \arrow[r, "m"] & C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & C_0 \arrow[l, bend left=60, "1"] \end{tikzcd}$$

so $\mathbb{T}$ is a finite limit theory.

Since topoi behave a lot like Set, a large class of theories can have models in a topos, (in fact all first order theories can).

Geometric theories are precisely those theories $\mathbb{T}$ s.t. $\forall$ geometric morphisms $(f_*, f^*) : \mathcal{E} \longrightarrow \mathcal{F}$, f^* restricts to a functor

$$f^* : \mathbb{M}od_{\mathbb{T}}(\mathcal{F}) \longrightarrow \mathbb{M}od_{\mathbb{T}}(\mathcal{E}).$$

For such a theory, we get a 2-functor

$$\widetilde{\mathcal{J}op}^{\,op} \xrightarrow{\mathbb{M}od_{\mathbb{T}}(\cdot)} \mathcal{C}at.$$

Examples of Geometric Theories:

1) Monoids, groups, abelian groups, rings, comm. rings.

2) Categories

3) local rings, fields

Note: - A group object in $Sh(X)$ is the same thing as a sheaf of groups.

- A "local ring object" in $Sh(X)$ is the same as a sheaf of rings R whose stalks R_x are local $\forall\, x \in X$.

Thm Given a geometric theory $\mathbb{T}$, the functor $\mathbb{M}od_{\mathbb{T}}(\cdot)$ is representable, i.e. $\exists$ (a necessarily unique) Grothendieck topos $B\mathbb{T}$ (called the classifying topos) s.t. there is an eq'l of categories $\forall\, \mathcal{E}$

$$Hom_{\mathcal{T}op}(\mathcal{E}, B\mathbb{T}) \simeq \mathbb{M}od_{\mathbb{T}}(\mathcal{E})$$

which is natural in $\mathcal{E}$.

Conversely, given a Grothendieck topos $\mathcal{F}$, $\exists!$ a geometric theory $\mathbb{T}$ s.t. $y(\mathcal{F}) \simeq \mathbb{M}od_{\mathbb{T}}$. * (equivalently $\mathcal{F} \simeq B\mathbb{T}$)

In fact the assignment $\mathbb{T} \longmapsto B\mathbb{T}$ is part of an equivalence of 2-categories

$$\mathrm{GeomThys}^{op} \xrightarrow{\sim} \mathcal{T}op.$$

* Note There is a 2-categorical Yoneda embedding

For $\mathbb{C}$ a 2-category, write $y: \mathbb{C} \hookrightarrow 2\text{-}Fun(\mathbb{C}^{op}, Cat)$.

$$C \longmapsto Hom_{\mathbb{C}}(\cdot, C)$$

<u>Def</u>: Let $\mathbb{T}$ be a geometric theory, then $\exists$ an eq'l

$$\alpha : y(B\mathbb{T}) \xrightarrow{\sim} \mathbb{M}\mathrm{od}_{\mathbb{T}}(\cdot).$$

In particular $U_{\mathbb{T}} := \alpha_{B\mathbb{T}}(id_{B\mathbb{T}}) \in \mathbb{M}\mathrm{od}_{\mathbb{T}}(B\mathbb{T})_0$ is a $\mathbb{T}$-model in $B\mathbb{T}$. $U_{\mathbb{T}}$ is called the <u>universal</u> $\mathbb{T}$-model.

The following justifies the name:

Let $f : \mathcal{E} \longrightarrow B\mathbb{T}$ be a geometric morphism

$f \in y(B\mathbb{T})(\mathcal{E}) \xrightarrow[\sim]{\alpha_{\mathcal{E}}} \mathbb{M}\mathrm{od}_{\mathbb{T}}(\mathcal{E})$ so f classifies the $\mathbb{T}$-model $\alpha_{\mathcal{E}}(f)$.

Note that the following diagram commutes up to natural iso:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{Top}}(B\mathbb{T}, B\mathbb{T}) \arrow[r, "\alpha_{B\mathbb{T}}", "\sim"'] \arrow[d, "(\cdot)\circ f"'] & \mathbb{M}\mathrm{od}_{\mathbb{T}}(B\mathbb{T}) \arrow[d, "f^*"] \\
\mathrm{Hom}_{\mathfrak{Top}}(\mathcal{E}, B\mathbb{T}) \arrow[r, "\alpha_{\mathcal{E}}", "\sim"'] & \mathbb{M}\mathrm{od}_{\mathbb{T}}(\mathcal{E})
\end{tikzcd}$$

$\Rightarrow$ $\alpha_{\mathcal{E}}(f) \cong f^*(U_{\mathbb{T}})$. So every model is a pullback of the universal model.

<u>Baby-example</u> Let Fin denote the category of finite sets.

<u>Claim</u> Just as Set is obtained from $*$ by freely adjoining arbitrary colimits, Fin is obtained by freely adjoining <u>finite</u> colimits: i.e. $\forall$ B w/ fin colims, if $1 : * \longrightarrow \mathrm{Fin}$,

$$\begin{tikzcd}
\mathrm{Fun}^{REX}(\mathrm{Fin}, B) \arrow[r, "1^*", "\sim"'] \arrow[rr, bend left, "cv_*"] & \mathrm{Fun}(*, B) \arrow[r, "\sim"] & B .
\end{tikzcd}$$

$\varphi \longmapsto \varphi(*)$

(inverse $b \in B \mapsto \varphi_b$, $\varphi_b(A) = \coprod_{a \in A} b$)

Fun^{REX} = fin colim. pres.

Dually
$\Longrightarrow$ $\forall\ \mathcal{D}$ with finite limits,

$$\underset{\substack{\shortparallel \\ \text{left exact}}}{\mathrm{Fun}^{LEX}}(\mathrm{Fin}^{op}, \mathcal{D}) \xrightarrow[\sim]{ev_*} \mathcal{D}.$$

<u>Note</u> If $\mathcal{D} = \mathcal{E}$ a Groth. topos, since Fin^{op} is left exact, we get an eq'l

$$\mathrm{Hom}_{\mathcal{T}op}(\mathcal{E}, \mathrm{Set}^{\mathrm{Fin}}) \simeq \underbrace{\mathrm{Fun}^{L,+LEX}}_{\substack{\text{colimit pres} \\ +\text{ left exact}}}(\mathrm{Set}^{\mathrm{Fin}}, \mathcal{E}) \xrightarrow{y^*} \mathrm{Fun}^{LEX}(\mathrm{Fin}^{op}, \mathcal{E}) \xrightarrow[\sim]{ev_1} \mathcal{E}$$

So $\mathrm{Set}^{\mathrm{Fin}}$ is the classifying topos for <u>objects</u>.
The <u>universal object</u> is the terminal object $1 \in \mathrm{Set}^{\mathrm{Fin}}$:
$\forall\ E \in \mathcal{E}\ \exists!\ \chi_E : \mathcal{E} \to \mathrm{Set}^{\mathrm{Fin}}$ s.t. $\chi_E^*(1) \cong E$.

<u>Rmk</u> For $\mathbb{L}$ a locale $l \in \mathbb{L}$, $\exists!\ \chi_L : \mathbb{L} \to S$ = Sierpinski space s.t. $\chi_L^*(1) = l$.
In fact $S = \mathrm{Loc}(\mathrm{Set}^{\mathrm{Fin}})$ - the localic reflection.

<u>The Classifying Topos for (comm. unital) Rings.</u>

<u>Def</u> A comm. unital ring R is <u>finitely presented</u> if it is isomorphic to the quotient of a finitely generated free ring by a finitely generated ideal, i.e. $R \cong \mathbb{Z}[X_1, \ldots, X_n]/(P_1, \ldots, P_k)$.

Denote the category of finitely presented rings by <u>f.p. Ring</u>.

Thm $\mathrm{Set}^{\text{f.p. Ring}}$ is the classifying topos for (comm. unital) rings.

<u>Sketch</u>: Denote by FRing the category of finitely generated free rings. Denote by $\mathrm{Comm} := \mathrm{FRing}^{op}$.

1 <u>Claim</u> If $\mathcal{C}$ has finite products $\exists$ an eq'l of categories
$$\mathrm{Ring}(\mathcal{C}) \simeq \mathrm{Fun}^{\pi}(\mathrm{Comm}, \mathcal{C})$$
$\hookrightarrow$ finite product preserving

Check when $\mathcal{C} = \text{Set}$:

Given $R \in \text{Ring}$, $\tilde{R} : \underset{\text{Comm}}{\overset{\shortparallel}{\text{FRing}^{op}}} \longrightarrow \text{Set}$ preserves finite limits.

$$B \longmapsto \text{Hom}(B, R)$$

Show $\text{Ring} \longrightarrow \text{Fun}^{\pi}(\text{Comm}, \text{Set})$ is an eq'l,

$$R \longmapsto \tilde{R}$$

<u>Note</u> $\text{Comm} \cong$ category of affine planes $\mathbb{A}^n_{\mathbb{Z}}$ + regular morphisms,

$$\begin{array}{ccc} \mathbb{A}^n_{\mathbb{Z}} & \longrightarrow & \mathbb{A}^m_{\mathbb{Z}} \\ \updownarrow & & \updownarrow \\ \mathbb{Z}[X_1, \ldots, X_n] & \xleftarrow{p} & \mathbb{Z}[Y_1, \ldots, Y_m] \end{array} \quad p \leadsto p_1, \ldots, p_m \quad p_i = p(Y_i).$$

$\mathbb{A}_{\mathbb{Z}}$ is a ring object:

$$\begin{array}{ccc} m : \mathbb{A}_{\mathbb{Z}} \times \mathbb{A}_{\mathbb{Z}} & \longrightarrow & \mathbb{A}_{\mathbb{Z}} \\ & \updownarrow & \\ \mathbb{Z}[X, Y] & \xleftarrow{m} & \mathbb{Z}[X] \end{array}, \quad m(X, Y) = X \cdot Y$$

$$\begin{array}{ccc} a : \mathbb{A}_{\mathbb{Z}} \times \mathbb{A}_{\mathbb{Z}} & \longrightarrow & \mathbb{A}_{\mathbb{Z}} \\ \mathbb{Z}[X, Y] & \xleftarrow{a} & \mathbb{Z}[X] \end{array} \quad a(X, Y) = X + Y$$

$$\begin{array}{ccc} u : \mathbb{A}^0_{\mathbb{Z}} & \longrightarrow & \mathbb{A}_{\mathbb{Z}} \\ 1 & \longleftarrow\!\shortmid & X \\ \mathbb{Z} & \longleftarrow & \mathbb{Z}[X] \end{array}$$

Given $\Theta : \text{Comm} \longrightarrow \text{Set}$ fin. prod. pres., $\Theta(\mathbb{A}_{\mathbb{Z}}, a, m, u)$ is a ring and $\widetilde{\Theta(\mathbb{A}_{\mathbb{Z}})} \cong \Theta$.

For $\mathcal{C} \neq \text{Set}$, given $\Theta : \text{Comm} \longrightarrow \mathcal{C}$ fin. prod. pres,

$y \circ \Theta : \text{Comm} \longrightarrow \text{Set}^{\mathcal{C}^{op}}$ is fin. prod. pres.

$\widehat{y \circ \Theta} : \mathcal{C}^{op} \longrightarrow \text{Set}^{\text{Comm}}$ s.t. $\widehat{y \circ \Theta}(C)$ is fin. prod. pres

$\updownarrow$

$\widehat{y \circ \Theta} : \mathcal{C}^{op} \longrightarrow \text{Ring}$, but representable by $\Theta(\mathbb{A}_{\mathbb{Z}})$.

② $i : \mathrm{FRing} \hookrightarrow \text{f.p. Ring}$, $\mathcal{D}$ - finite colimits,

finite coproduct pres.

$$i^* : \mathrm{Fun}^{\mathrm{REX}}(\text{f.p.Ring}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^{\amalg}(\mathrm{FRing}, \mathcal{D}).$$

<u>Idea</u>:

$$\begin{tikzcd} \mathrm{FRing} \arrow[r, hook, "i"] \arrow[dr, "\varphi"'] & \text{f.p. Ring} \arrow[d, dashed, "\tilde{\varphi}"] \\ & \mathcal{D} \end{tikzcd}$$

φ pres. finite coproducts

$\ni R \cong \mathbb{Z}[X_1, \dots, X_n]/(P_1, \dots, P_k)$

$\Rightarrow R = \underrightarrow{\mathrm{colim}}\left(\mathbb{Z}[Y_1, \dots, Y_k] \overset{P}{\underset{0}{\rightrightarrows}} \mathbb{Z}[X_1, \dots, X_n]\right)$

$$\tilde{\varphi}(R) = \underrightarrow{\mathrm{colim}}\left(\underset{\underset{\coprod_{j=1}^{k} \varphi(\mathbb{Z}[X])}{\wr\wr}}{\varphi(\mathbb{Z}[Y_1, \dots, Y_k])} \overset{\varphi(P)}{\underset{\varphi(0)}{\rightrightarrows}} \varphi(\mathbb{Z}[X_1, \dots, X_n])\right).$$

Dually $\Rightarrow$ ②' $\mathcal{D}'$ - finite limits

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{LEX}}((\text{f.p.Ring})^{op}, \mathcal{D}') \arrow[r, "(i^{op})^*", "\sim"'] & \mathrm{Fun}^{\Pi}(\underset{\underset{\mathrm{FRing}^{op}}{\shortparallel}}{\mathrm{Comm}}, \mathcal{D}') \arrow[d, "\mathrm{ev}_{A_{\mathbb{Z}}}", "\wr"'] \\ & \mathrm{Ring}(\mathcal{D}') \end{tikzcd}$$

If $\mathcal{E} = \mathcal{D}'$ is a topos,

$$\mathrm{Fun}^{L+\mathrm{LEX}}(\mathrm{Set}^{\text{f.p. Ring}}, \mathcal{E}) \xrightarrow[\sim]{y^*} \mathrm{Fun}^{\mathrm{LEX}}((\text{f.p.Ring})^{op}, \mathcal{E}) \xrightarrow{\sim} \mathrm{Ring}(\mathcal{E})$$

$\boxed{\text{uses } (\text{f.p.Ring})^{op} \text{ has finite limits}}$

$$\wr\wr$$

$$\mathrm{Hom}_{\mathfrak{Top}}(\mathcal{E}, \mathrm{Set}^{\text{f.p. Ring}})$$

The universal ring object is the image of $A_{\mathbb{Z}}$ under

$$\mathrm{Comm} \xhookrightarrow{i^{op}} (\text{f.p. Ring})^{op} \xhookrightarrow{y} \mathrm{Set}^{\text{f.p. Ring}}.$$

<u>Rmk</u> $(\text{f.p. Ring})^{op} \simeq \left\{ \begin{matrix} \text{Affine schemes} \\ \text{of finite type} \end{matrix} \right\} = \mathrm{Aff\,Sch\,FT}$

$\exists$ a natural Grothendieck topology: The Zariski topology Zar

$\mathrm{Sh}((\text{f.p.Ring})^{op}, \mathrm{Zar})$ is the classifying topos for <u>local</u> rings