

A Proof of Giraud's Theorem

1.

Recall

Thm (Giraud)

A category \mathcal{E} is a Grothendieck topos if and only if

- i) \mathcal{E} is locally presentable
- ii) colimits in \mathcal{E} are universal
- iii) coproducts in \mathcal{E} are disjoint
- iv) every eq'd relation in \mathcal{E} is effective

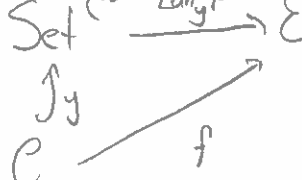
We already saw that all Groth. topos satisfy i)-iv).

Lemma Suppose \mathcal{C} is a small left exact category,

\mathcal{E} satisfies i)-iv), and $f: \mathcal{C} \rightarrow \mathcal{E}$ is any functor.

Then $\text{Lan}_y f: \text{Set}^{\text{cop}} \rightarrow \mathcal{E}$ is left exact $\Leftrightarrow f$ is.

Pf $\text{Set}^{\text{cop}} \xrightarrow{\text{Lan}_y f} \mathcal{E}$; $\text{Lan}_y f$ left exact $\rightarrow \text{Lan}_y f \circ y \cong f$
is, since y preserves all limits.



Conversely, suppose f is left exact. Let $F := \text{Lan}_y f$, wts F is left exact.

F preserves \perp : $F(\perp) = F(y(\perp)) \cong f(\perp) \cong \perp$.

Suffices to show that F preserves p.b.s.

Def: $\alpha: Y \rightarrow Z$ in Set^{cop} is good if all p.b. squares

$$\begin{array}{ccc} X \times_2 Y & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \alpha \\ X & \longrightarrow & Z \end{array}$$
 are preserved by F .

Rmk Good morphisms are stable under composition.

2.

Rmk Consider $\alpha: Y \rightarrow Z$

$$\rightsquigarrow \text{Set}^{\text{cop}}/Z \xrightarrow{\alpha^*} \text{Set}^{\text{cop}}/Y \quad (\text{pres. colimits})$$

$$g: X \rightarrow Z \longmapsto (X \times_Z Y \rightarrow Y)$$

and we have α is good iff

$$\text{Set}^{\text{cop}}/Z \xrightarrow{\alpha^*} \text{Set}^{\text{cop}}/Y$$

$$\downarrow F/Z$$

$$\downarrow F/Y$$

$$\boxed{E/F(Z) \xrightarrow{F(\alpha)^*} E/F(Y)}$$

these preserve colimits since F does

pres-colimits since colimits are universal

commutes up to natural isomorphism. More precisely, we need the canonical map

$$\epsilon_\alpha: F/Y \circ \alpha^* \implies F(\alpha)^* \circ F/Z$$

to be an iso. This is an arrow in $\text{colimit preserving}$

$$\text{Fun}^{\text{colim-pres}}(\text{Set}^{\text{cop}}/Z, E/F(Y)) \cong \text{Fun}^{\text{colim-pres}}(\text{Set}^{(E/Z)^{\text{op}}}, E/F(Y))$$

ω

G_1

Yoneda.

$$G_0 \circ \gamma/Z \in \text{Fun}(E/Z, E/F(Y))$$

where $(\gamma/Z): E/Z \hookrightarrow \text{Set}^{\text{cop}}/Z$

So α is good $\Leftrightarrow \epsilon_\alpha$ is an iso $\Leftrightarrow \epsilon_\alpha \circ (\gamma/Z)$ is an iso

i.e. \Leftrightarrow the components of ϵ_α along each object of $\text{Set}^{\text{cop}}/Z$ of the form $\beta: y(c) \rightarrow Z$ is an iso.

Def: An object $Z \in \text{Set}^{\text{cop}}$ is good $\Leftrightarrow \forall \alpha: Y \rightarrow Z$ is good
 $(\Leftrightarrow \forall \beta: y(c) \rightarrow Z, \beta$ is good)

It suffices to show all objects in Set^- are good. 3.

Obs Since \mathcal{C} and \mathcal{F} are left exact, any rep'l is good.

sub-lemma: The class of good objects is stable under coproducts!

PF Suppose $Z = \coprod_i Z_i$ with each Z_i good.

WTS $\forall \alpha: y(\mathcal{C}) \rightarrow Z$, α is good.

Colimits computed object-wise $\Rightarrow \text{Hom}(y(\mathcal{C}), \coprod_i Z_i) \cong \prod_i \text{Hom}(y(\mathcal{C}), Z_i)$,

so α factors as $y(\mathcal{C}) \xrightarrow{\alpha'} Z_j \xrightarrow{\phi_j} \coprod_i Z_i$ for some j .

α' is good by assumption, so suffices to show each ϕ_j is.

Same argument again reduces to showing p.b.'s of the form

$Z_k \times_Z Z_j \rightarrow Z_j$ are preserved by \mathcal{F} .

$$\begin{array}{ccc} Z_k \times_Z Z_j & \rightarrow & Z_j \\ \downarrow & & \downarrow \phi_j \\ Z_k & \xrightarrow{\phi_k} & \coprod_i Z_i \end{array}$$

But colimits in $\left\{ \begin{array}{l} \text{Set}^{\text{cop}} \\ \mathcal{E} \end{array} \right.$ are universal $\Rightarrow Z_k \times_Z Z_j \cong \begin{cases} \phi & j \neq k \\ Z_j & j = k \end{cases}$

\hookrightarrow since $\mathcal{F}(\coprod_i Z_i) \cong \prod_i \mathcal{F}(Z_i)$. $\square \hookrightarrow$ sub-lemma.

Now, let $X \in \text{Set}^{\text{cop}}$.

$$X \cong \underset{y(\mathcal{C}) \rightarrow X}{\text{colim}} y(\mathcal{C}) \Rightarrow X \cong \text{coeq} \left(\begin{array}{ccc} \prod y(\mathcal{C}) & \rightrightarrows & \prod y(\mathcal{C}) \\ y(\mathcal{C}) \rightarrow y(\mathcal{C}) & & y(\mathcal{C}) \rightarrow X \\ \downarrow & & \downarrow \\ & X & \end{array} \right).$$

\therefore it suffices to show that if

$$Z_1 \rightrightarrows Z_0 \xrightarrow{s} Z_{-1} \quad \text{is a coequalizer diagram}$$

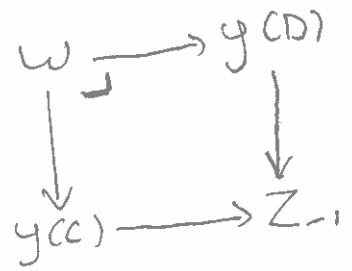
with Z_1 and Z_0 coproducts of reps, then Z_{-1} is good.

Note sub-lemma $\Rightarrow Z_0$ and Z_1 are good.

Also $Z_0 \times Z_0 \cong \coprod_{\alpha \in A} y(\alpha) \times \coprod_{\alpha \in A} y(\alpha) \cong \coprod_{\alpha \in A} y(\alpha \times \alpha)$

$\Rightarrow Z_0 \times Z_0$ is good.

Consider a p.b. diagram



Similar argument to sub-lemma shows, to show it is preserved by F , can reduce to showing $Z_0 \times_{Z_{-1}} Z_0 \rightarrow Z_0$ is preserved.



Now $Z_0 \times_{Z_{-1}} Z_0 \xrightarrow{R} Z_0 \times Z_0$ is an eq'l rel

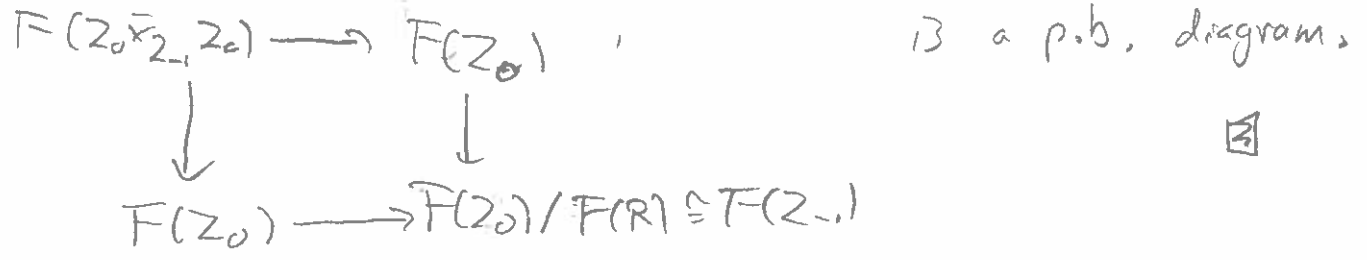
and " $Z_0 \times Z_0$ is good $\Rightarrow F$ is left exact at $Z_0 \times Z_0$ "

$\Rightarrow F(Z_0 \times_{Z_{-1}} Z_0) \xrightarrow{F(R)} F(Z_0 \times Z_0) \cong F(Z_0) \times F(Z_0)$
 \uparrow uses $1=y(1)$ is good

is an eq'l relation on $F(Z_0)$ in \mathcal{E} .

F pres colims $\Rightarrow F(Z_0)/F(R) := \text{coeq}(F(Z_0 \times_{Z_{-1}} Z_0) \rightrightarrows F(Z_0))$
 $\cong F(\text{coeq}(Z_0 \times_{Z_{-1}} Z_0 \rightrightarrows Z_0))$
 $\cong F(Z_{-1})$

Eq'l relations in \mathcal{E} are effective \Rightarrow



Giraud's Thm:

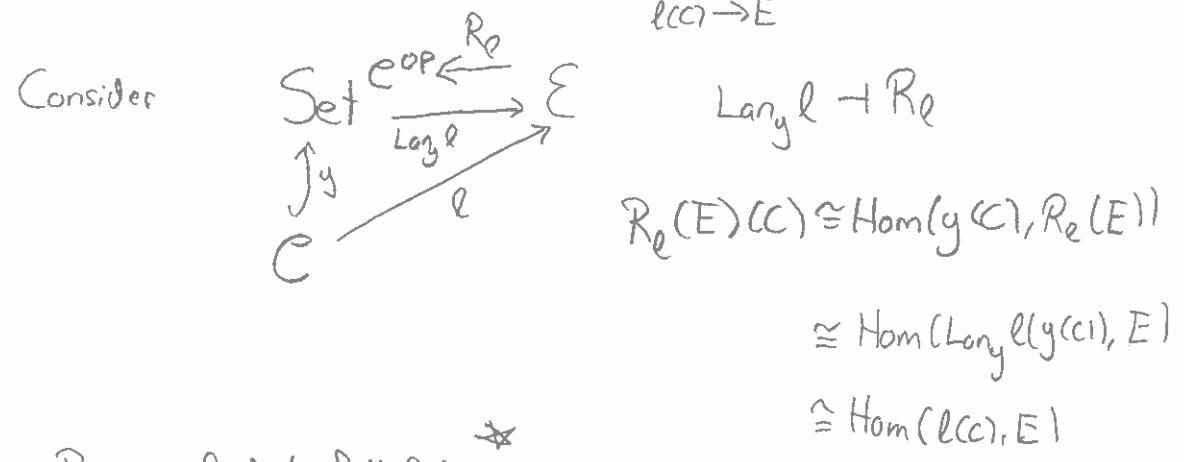
Recall \mathcal{E} locally presentable means: \mathcal{E} is cocomplete and
 } a regular cardinal κ
 and a small subcategory $\mathcal{C} \xrightarrow{l} \mathcal{E}$ s.t.

a) $\text{Lan}_l l \cong \text{id}_{\mathcal{E}}$ (l strongly generates \mathcal{E})

b) Each $C \in \mathcal{C}$ has $l(C)$ κ -compact:

$\text{Hom}(l(C), \cdot) : \mathcal{E} \rightarrow \text{Set}$
 preserves κ -filtered colimits.

a) means $\forall E \in \mathcal{E}, E = \underset{l(C) \rightarrow E}{\text{colim}} l(C).$



R_l is full & faithful $\Leftrightarrow \epsilon : \text{Lan}_l l \circ R_l \xrightarrow{\cong} \text{id}_{\mathcal{E}}$.

The co-unit is given by:

$$\text{Lan}_l l \circ R_l(E) \cong \underset{\gamma(C) \rightarrow R_l(E)}{\text{colim}} l(C) \cong \underset{l(C) \rightarrow E}{\text{colim}} l(C) \xrightarrow{\sim} E$$

\uparrow Yoneda \uparrow from i)

So $\mathcal{E} \xleftarrow{\text{Lan}_l l} \text{Set}^{e_{op}} \xrightarrow{R_l} \mathcal{E}$ is reflective.

and cocompleteness

Note: This only used a). In fact, by \star i) holds \Leftrightarrow

$$\mathcal{E} \xleftrightarrow{\quad} \text{Set}^{e_{op}} \text{ for some small } \mathcal{C}.$$

Suppose i) - iv) hold for \mathcal{E} . i) $\Rightarrow \mathcal{E} \xleftarrow{\text{Lan}_y l} \text{Set}^{\text{op}} \xrightarrow{\text{Re}} \mathcal{E}$ (6)

If \mathcal{C} does not have finite limits, replace it with $\mathcal{C}' =$ smallest subcat of \mathcal{E} wrt fin. limits containing \mathcal{C} .
 \mathcal{C}' is small since every finite limit is a subobject of a finite product, and $\text{Sub}_{\mathcal{E}}(\mathcal{E}) \subseteq \text{Sub}_{\text{Set}^{\text{op}}}(\text{Re}(\mathcal{E}))$ - a set. s.t. $\mathcal{C}' \hookrightarrow \mathcal{E}$ pres. fin. limits.

So we have arranged \mathcal{C} to have fin. limits and for $l: \mathcal{C} \hookrightarrow \mathcal{E}$ to pres. them. By the lemma \Rightarrow the left adjoint $\text{Lan}_y l \dashv \text{Re}$ is left exact. So $\mathcal{E} \xleftarrow{\text{Lan}_y l} \text{Set}^{\text{op}} \xrightarrow{\text{Re}} \mathcal{E}$ is a left exact localization,

and by the homework $\Rightarrow \exists!$ Groth. topology \mathcal{J} on \mathcal{C} and an eq'l $\mathcal{E} \cong \text{Sh}_{\mathcal{J}}(\mathcal{C})$ under which $\text{Lan}_y l$ is a sheafification.

□

Rmk In i) we only need i') \mathcal{E} is cocomplete and strongly generated by a small subcategory. The above proof shows i') + ii) - iv) $\Rightarrow \mathcal{E}$ is a Groth. tops. We have seen however that Groth. tops are locally presentable \therefore i') + ii) - iv) = i) - iv).

Rmk Since $\text{Lan}_y l \circ y \cong l: \mathcal{C} \hookrightarrow \mathcal{E} \Rightarrow$ the Groth. topology \mathcal{J}

is subcanonical.

\Rightarrow Cor. If \mathcal{E} is a Groth. tops, $\mathcal{E} \cong \text{Sh}_{\mathcal{J}}(\mathcal{C})$ for a subcanonical site $(\mathcal{C}, \mathcal{J})$ with finite limits.