A Proof of Giraud's Theorem

Recall

Thm (Giraud)

A category $\mathcal{E}$ is a Grothendieck topos if and only if

i) $\mathcal{E}$ is locally presentable
ii) colimits in $\mathcal{E}$ are universal
iii) coproducts in $\mathcal{E}$ are disjoint
iv) every eq'l relation in $\mathcal{E}$ is effective

We already saw that all Groth. topoi satisfy i)–iv).

Lemma Suppose $\mathcal{C}$ is a small left exact category, $\mathcal{E}$ satisfies i)–iv), and $f : \mathcal{C} \longrightarrow \mathcal{E}$ is any functor. Then $\mathrm{Lan}_y f : \mathrm{Set}^{\mathcal{C}^{op}} \longrightarrow \mathcal{E}$ is left exact $\Leftrightarrow$ f is.

Pf

$$\begin{tikzcd} \mathrm{Set}^{\mathcal{C}^{op}} \arrow[r, "\mathrm{Lan}_y f"] & \mathcal{E} \\ \mathcal{C} \arrow[u, "y"] \arrow[ur, "f"'] & \end{tikzcd}$$

; $\mathrm{Lan}_y f$ left exact $\Rightarrow$ $\mathrm{Lan}_y f \circ y \simeq f$ is, since y preserves all limits.

Conversely, suppose f is left exact. Let $F := \mathrm{Lan}_y f$, wts F is left exact.

F preserves 1: $F(1) = F(y(1)) \cong f(1) \cong 1$.

Suffices to show that F preserves p.b.s.

Def: $\alpha : Y \longrightarrow Z$ in $\mathrm{Set}^{\mathcal{C}^{op}}$ is good if all p.b. squares

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "\alpha"] \\ X \arrow[r] & Z \end{tikzcd}$$

are preserved by F.

Rmk Good morphisms are stable under composition.

Rmk Consider $\alpha : Y \to Z$

$$\rightsquigarrow \quad \mathrm{Set}^{\mathcal{C}^{op}}/Z \xrightarrow{\alpha^*} \mathrm{Set}^{\mathcal{C}^{op}}/Y \qquad \text{(pres. colimits)}$$

$$g : X \to Z \longmapsto (X \times_Z Y \to Y)$$

and we have α is good iff

$$\begin{tikzcd} \mathrm{Set}^{\mathcal{C}^{op}}/Z \arrow[r, "\alpha^*"] \arrow[d, "F/Z"'] & \mathrm{Set}^{\mathcal{C}^{op}}/Y \arrow[d, "F/Y"] \\ \mathcal{E}/F(Z) \arrow[r, "F(\alpha)^*"] & \mathcal{E}/F(Y) \end{tikzcd}$$

these preserve colimits since F does

pres. colimits since colimits are universal

commutes up to natural isomorphism. More precisely, we need the canonical map

$$\varepsilon_\alpha : F/Y \circ \alpha^* \Longrightarrow F(\alpha)^* \circ F/Z$$

to be an iso. This is an arrow in

colimit preserving

$$\mathrm{Fun}^{L}(\mathrm{Set}^{\mathcal{C}^{op}}/Z, \mathcal{E}/F(Y)) \simeq \mathrm{Fun}^{L}(\mathrm{Set}^{(\mathcal{C}/Z)^{op}}, \mathcal{E}/F(Y))$$

$$\begin{tikzcd} G \arrow[r, mapsto] & G \circ y/Z \in \mathrm{Fun}(\mathcal{C}/Z, \mathcal{E}/F(Y)) \end{tikzcd} \qquad \downarrow \wr$$

Yoneda.

where $y/Z : \mathcal{C}/Z \hookrightarrow \mathrm{Set}^{\mathcal{C}^{op}}/Z$

So α is good $\iff \varepsilon_\alpha$ is an iso $\iff \varepsilon_\alpha \circ (y/Z)$ is an iso

i.e. $\iff$ the components of ε_α along each object of $\mathrm{Set}^{\mathcal{C}^{op}}/Z$ of the form $\beta : y(c) \to Z$ is an iso.

Def: An object $Z \in \mathrm{Set}^{\mathcal{C}^{op}}$ is good $\iff \forall\ \alpha : Y \to Z$ is good

$(\iff \forall\ \beta : y(c) \to Z,\ \beta$ is good$)$

It suffices to show all objects in Set^{-} are good.

Obs Since $\mathcal{C}$ and F are left exact, any rep'l is good.

Sub-Lemma: The class of good objects is stable under coproducts:

Pf Suppose $Z = \coprod_i Z_i$ with each Z_i good.

WTS $\forall\ \alpha : y(c) \to Z$, α is good.

Colimits computed object-wise $\Rightarrow$ $Hom(y(c), \coprod_i Z_i) \cong \coprod_i Hom(y(c), Z_i)$,

so α factors as $y(c) \xrightarrow{\alpha'} Z_j \xrightarrow{\phi_j} \coprod_i Z_i$ for some j.

α' is good by assumption, so suffices to show each ϕ_j is.

Same argument again reduces to showing p.b.'s of the form

$$\begin{tikzcd} Z_K \times_Z Z_j \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z_j \arrow[d, "\phi_j"] \\ Z_K \arrow[r, "\phi_K"] & \coprod_i Z_i \end{tikzcd}$$

are preserved by F.

But colimits in $\begin{cases} Set^{\mathcal{C}^{op}} \text{ are universal} \Rightarrow Z_K \times_Z Z_j \cong \begin{cases} \phi & j \neq K \\ Z_j & j = K \end{cases} \\ \mathcal{E} \quad " \quad " \Rightarrow F(Z_K) \times_{F(Z)} F(Z_j) \cong \begin{cases} \phi & j \neq K \\ F(Z_j) & j = K \end{cases} \end{cases}$

since $F(\coprod_i Z_i) \cong \coprod_i F(Z_i)$. $\boxtimes$ sub-lemma.

Now, let $X \in Set^{\mathcal{C}^{op}}$.

$$X \cong \varinjlim_{y(c) \to X} y(c) \Rightarrow X \cong coeq\Big(\coprod_{\substack{y(c) \to y(D) \\ \searrow \swarrow \\ X}} y(D) \rightrightarrows \coprod_{y(c) \to X} y(c) \Big).$$

$\therefore$ it suffices to show that if

$$Z_1 \rightrightarrows Z_0 \xrightarrow{s} Z_{-1} \quad \text{is a coequalizer diagram}$$

with Z_1 and Z_0 coproducts of rep'ls, then Z_{-1} is good.

Note sub-lemma $\Rightarrow$ Z_0 and Z_1 are good.

Also $Z_0 \times Z_0 \cong \coprod_{\alpha \in A} y(C_\alpha) \times \coprod_{\alpha' \in A} y(C_{\alpha'}) \cong \coprod_{\alpha\alpha' \in A^2} y(C_\alpha \times C_{\alpha'})$

$\Rightarrow$ $Z_0 \times Z_0$ is good.

Consider a p.b. diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & y(D) \arrow[d] \\ y(C) \arrow[r] & Z_{-1} \end{tikzcd}$$

Similar argument to sub-lemma shows, to show it is preserved by F, can reduce to showing

$$\begin{tikzcd} Z_0 \times_{Z_{-1}} Z_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z_0 \arrow[d, "s"] \\ Z_0 \arrow[r, "s"'] & Z_{-1} \end{tikzcd}$$

is preserved.

Now $\overset{R}{\overset{\shortparallel}{Z_0 \times_{Z_{-1}} Z_0}} \rightarrowtail Z_0 \times Z_0$ is an eq'l rel

and "$Z_0 \times Z_0$ is good $\Rightarrow$ F is left exact at $Z_0 \times Z_0$"

$\Rightarrow$ $\overset{F(R)}{\overset{\shortparallel}{F(Z_0 \times_{Z_{-1}} Z_0)}} \rightarrowtail F(Z_0 \times Z_0) \cong F(Z_0) \times F(Z_0)$

uses $1 = y(1)$ is good

is an eq'l relation on $F(Z_0)$ in $\mathcal{E}$.

F pres colims $\Rightarrow$ $F(Z_0)/F(R) := \mathrm{coeq}\left(F(Z_0 \times_{Z_{-1}} Z_0) \rightrightarrows F(Z_0)\right)$

$\cong F(\mathrm{coeq}(Z_0 \times_{Z_{-1}} Z_0 \rightrightarrows Z_0))$

$\cong F(Z_{-1})$.

Eq'l relations in $\mathcal{E}$ are effective $\Rightarrow$

$$\begin{tikzcd} F(Z_0 \times_{Z_{-1}} Z_0) \arrow[r] \arrow[d] & F(Z_0) \arrow[d] \\ F(Z_0) \arrow[r] & F(Z_0)/F(R) \cong F(Z_{-1}) \end{tikzcd}$$

is a p.b. diagram.

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Giraud's thm:

Recall $\mathcal{E}$ locally presentable means: $\mathcal{E}$ is cocomplete and $\exists$ a regular cardinal κ and a small subcategory $\mathcal{C} \overset{\ell}{\hookrightarrow} \mathcal{E}$ s.t.,

a) $\mathrm{Lan}_\ell \ell \cong id_{\mathcal{E}}$ (ℓ strongly generates $\mathcal{E}$)

b) Each $C \in \mathcal{C}$ has $\ell(C)$ κ-compact:

$$\mathrm{Hom}(\ell C, \cdot) : \mathcal{E} \longrightarrow \mathrm{Set}$$

preserves κ-filtered colimits.

a) means $\forall\ E \in \mathcal{E}$, $E = \varinjlim_{\ell(C) \to E} \ell(C)$.

Consider

$$\begin{tikzcd} \mathrm{Set}^{\mathcal{C}^{op}} \arrow[r, shift left, "\mathrm{Lan}_y \ell"'] & \mathcal{E} \arrow[l, shift right, "R_\ell"'] \\ \mathcal{C} \arrow[u, "y"] \arrow[ur, "\ell"'] & \end{tikzcd}$$

$\mathrm{Lan}_y \ell \dashv R_\ell$

$$\begin{aligned} R_\ell(E)(C) &\cong \mathrm{Hom}(y(C), R_\ell(E)) \\ &\cong \mathrm{Hom}(\mathrm{Lan}_y \ell(y(C)), E) \\ &\cong \mathrm{Hom}(\ell(C), E) \end{aligned}$$

R_ℓ is full & faithful $\overset{☆}{\Leftrightarrow}$ $\epsilon : \mathrm{Lan}_y \ell \circ R_\ell \overset{\sim}{\Longrightarrow} id_{\mathcal{E}}$.

The co-unit is given by:

$$\mathrm{Lan}_y \ell \circ R_\ell(E) \cong \varinjlim_{y(C) \to R_\ell(E)} \ell(C) \underset{\text{Yoneda}}{\cong} \varinjlim_{\ell(C) \to E} \ell(C) \xrightarrow[\text{from i)}]{\sim} E$$

So $\mathcal{E} \underset{R_\ell}{\overset{\mathrm{Lan}_y \ell}{\leftrightarrows}} \mathrm{Set}^{\mathcal{C}^{op}}$ is reflective.

Note: This only used a) and cocompleteness. In fact, by ☆ i) holds $\Leftrightarrow$ $\mathcal{E} \leftrightarrows \mathrm{Set}^{\mathcal{C}^{op}}$ for some small $\mathcal{C}$.

Suppose i) – iv) hold for $\mathcal{E}$. i) $\Rightarrow$ $\mathcal{E} \underset{R_\ell}{\overset{\mathrm{Lan}_y \ell}{\rightleftarrows}} \mathrm{Set}^{\mathcal{C}^{op}}$.

If $\mathcal{C}$ does not have finite limits, replace it with $\mathcal{C}'$ = smallest subcat of $\mathcal{E}$ w/ fin. limits containing $\mathcal{C}$. s.t. $\mathcal{C}' \hookrightarrow \mathcal{E}$ pres. fin. limits.

$\mathcal{C}'$ is small since every finite limit is a subobject of a finite product, and $\mathrm{Sub}_{\mathcal{E}}(E) \subseteq \mathrm{Sub}_{\mathrm{Set}^{\mathcal{C}^{op}}}(R_\ell(E))$ - a set.

So we have arranged $\mathcal{C}$ to have fin. limits and for $\ell : \mathcal{C} \hookrightarrow \mathcal{E}$ to pres. them. By the lemma $\Rightarrow$ the left adjoint $\mathrm{Lan}_y \ell \dashv R_\ell$ is left exact. So $\mathcal{E} \underset{R_\ell}{\overset{\mathrm{Lan}_y \ell}{\rightleftarrows}} \mathrm{Set}^{\mathcal{C}^{op}}$, is a left exact localization, and by the homework $\Rightarrow$ $\exists!$ Groth. topology J on $\mathcal{C}$ and an eq'e $\mathcal{E} \simeq Sh_J(\mathcal{C})$ under which $\mathrm{Lan}_y \ell \rightsquigarrow a$ = sheafification

$\boxed{}$

Rmk In i) we only need i)': $\mathcal{E}$ is cocomplete and strongly generated by a small subcategory. The above proof shows i)' + ii) – iv) $\Rightarrow$ $\mathcal{E}$ is a Groth. topos. We have seen however that Groth. topoi are locally presentable $\therefore$ i)' + ii) – iv) = i) – iv).

Rmk Since $\underset{a \circ y}{\overset{\shortparallel}{\mathrm{Lan}_y \ell \circ y}} \cong \ell : \mathcal{C} \hookrightarrow \mathcal{E}$ $\Rightarrow$ the Groth. topology J is subcanonical.

$\rightsquigarrow$ Cor: If $\mathcal{E}$ is a Groth. topos, $\mathcal{E} \simeq Sh_J(\mathcal{C})$ for a subcanonical site $(\mathcal{C}, J)$ with finite limits.