Weekly Homework 9

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Topos Theory

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Problem 1. Prime elements.

Definition 1. An element $u \in \mathbb{L}$ of a locale is **prime** if

i) $u \neq 1$ and

ii) For all a and $b \in \mathbb{L}$, if $a \wedge b \leq u$ then either $a \leq u$ or $b \leq u$.

1. a) Show that for any locale $\mathbb{L}$, there is a natural bijection between points of $\mathbb{L}$ and prime elements of $\mathbb{L}$.

1. b) Show that for a topological space X, an element $U \in \mathscr{O}(X)$ is prime if and only if it is the compliment of an irreducible closed subset.

Problem 2. A "pointless" locale

Let $\mathbb{L}$ be the sublattice of $\mathscr{O}(\mathbb{R})$ consisting of those open subsets U such that

$$\overline{\operatorname{Int}(U)} = U.$$

(These are called **regular** open subsets.) Show that:

(a) $\mathbb{L}$ is a locale

(b) $\mathbb{L}$ has no points.

Problem 3. Stone Duality

(a) Show that the functor

$$\mathscr{O} : \mathbb{TOP} \to \mathbb{LOC}$$

is left adjoint to

$$pt : \mathbb{LOC} \to \mathbb{TOP}.$$

(b) Show that η_X is an isomorphism if and only if X is sober, where η is the unit of the adjunction.

(c) Show that $\epsilon_{\mathbb{L}}$ is an isomorphism if and only if $\mathbb{L}$ is spatial, where ϵ is the co-unit of the adjunction.

(d) Deduce that $\mathscr{O} \dashv pt$ restricts to an equivalence between sober topological spaces and spatial locales.

(e) Deduce that sober spaces are reflective in $\mathbb{TOP}$.

Problem 4. The Sierpiński space

Definition 2. The **Sierpiński space** Sp is the set $\{0, 1\}$ equipped with the topology

$$\mathscr{O}(Sp) = \{\{1\}, \emptyset, \{0, 1\}\}.$$

(a) Show that the Sierpiński space is sober.

(b) Show that there is a functor, defined on objects as

$$\begin{aligned} \mathrm{Fr}(\,\cdot\,) : \mathbf{Set} &\to \mathrm{Frm} \\ A &\mapsto \mathscr{O}\left(Sp^A\right), \end{aligned}$$

which is left adjoint to the forgetful functor from frames to sets. (In particular, one sees that $\mathscr{O}(Sp)$ is the free frame on one generator.)

Remark. The functor $\mathrm{Fr}(\,\cdot\,)$ sends a set A to the free frame on the set A. From this exercise we see that free frames are spatial. Since every frame is the quotient frame of a free frame, it follows dually that any locale is a sublocale of a spatial locale.

Problem 5. Locales from a topos.

Let $\mathcal{E}$ be a topos and E and object in $\mathcal{E}$. Show that the poset of subobjects of E naturally form a locale.