Weekly Homework 6

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Topos Theory

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Problem 1. The Grothendieck topology associated to a basis

(a) Using notation from class, if B is a basis for Grothendieck on a category $\mathscr{C}$, show that the assignment to each object C of $\mathscr{C}$ the set of sieves $Cov_{J(B)}(C)$ defines a Grothendieck topology $J(B)$ on $\mathscr{C}$.

Definition 1. Let $\mathscr{C}$ be a category and

$$\mathcal{U} = (f_i : C_i \to C)_{i \in I}$$

a set of arrows. A **refinement** of $\mathcal{U}$ is a set of arrows

$$(g_\alpha : V_\alpha \to C)_{\alpha \in A}$$

such that there exists a function

$$\lambda : A \to I$$

such that each g_α factors through $f_{\lambda(\alpha)}$. We say that a basis B' on $\mathscr{C}$ is **subordinate** to B, which we write as $B' \leq B$, if every covering family $(C_i \to C)_i$ in B' has a refinement by a covering family in B. B and B' are **equivalent** if and only if $B \leq B'$ and $B' \leq B$.

(b) Show that if B and B' are two Grothendieck pretopologies on $\mathscr{C}$, then

$$J(B) = J(B')$$

if and only if B and B' are equivalent.

(c) Show that the global local homeomorphism pretopology on **Top** is equivalent to the open cover pretopology.

Problem 2. The Étale Topology

Definition 2. A ring homomorphism between (commutative unital) rings $f : A \to B$ is said to be étale if

a) B is a finitely generated A-algebra

b) For any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & C/N, \end{array}$$

where N is a nilpotent ideal, and q the canonical projection, there is a unique lift

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & \overset{k}{\nearrow} & \downarrow{\scriptstyle q} \\ B & \xrightarrow{h} & C/N. \end{array}$$

Prove that

(a) if

$$A \xrightarrow{f} B \xrightarrow{g} C$$

has f étale, then g is étale if and only if gf is.

(b) a pushout of an étale morphism is étale.

(c) if S is a finitely-generated multiplicative submonoid of A, then the canonical map

$$A \to A[S^{-1}]$$

is étale.

Definition 3. Let $\mathscr{C}$ be the opposite category of commutative K-algebras for some commutative ring K. Say a collection of morphisms in $\mathscr{C}$,

$$(f_i : D_i \to D)$$

corresponding to ring homomorphisms

$$(f_i : D \to D_i)$$

is an **étale covering family** if every prime ideal of D is of the form $f_i^* p$ for some i and some prime ideal p of D_i.

(d) Show that the assignment to each K-algebra its étale covering families defines a Grothendieck pretopology. The associated Grothendieck topology is called the **étale topology.**

(e) **Bonus:** Show the étale topology is subcanonical.